Coupled KPZ equation from multi-species zero-range processes

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Bernardin-F-Sethuraman, arXiv:1908.07863, Ann. Appl. Probab., in press

Plan of the course (10 lectures)

1 Introduction

2 Supplementary materials

   Brownian motion, Space-time Gaussian white noise, (Additive) linear SPDEs, (Finite-dimensional) SDEs, Martingale problem, Invariant/reversible measures for SDEs, Martingales

3 Invariant measures of KPZ equation (F-Quastel, 2015)

4 Coupled KPZ equation by paracontrolled calculus (F-Hoshino, 2017)

5 Coupled KPZ equation from interacting particle systems (Bernardin-F-Sethuraman, 2020+)

   5.1 Independent particle systems

   5.2 Single species zero-range process

   5.3 n-species zero-range process

   5.4 Hydrodynamic limit, Linear fluctuation

   5.5 KPZ limit=Nonlinear fluctuation

Plan of this lecture

0 Coupled KPZ equation (Brief recall of Lecture No 4)
(pathwise theory, strong solution)

3 Microscopic Model: n-species zero-range processes
(=Interacting Random Walks of n types' particles)

4 Hydrodynamic limit (LLN), Linear fluctuation (CLT)

5 Nonlinear fluctuation leading to coupled KPZ equation
- (2nd order) Boltzmann-Gibbs principle
- martingale problem approach (called energy solution)
- trilinear condition
- We derive KPZ-Burgers equation (equation for $Y := \partial_u h$) for particle density. In particular, renormalization is unnecessary (heuristically, $\partial_u(\delta_u(u)) = 0$).

0. Multi-component coupled KPZ equation

- $\mathbb{R}^n$-valued KPZ equation for $h(t,u) = (h^i(t,u))_{i=1}^n$ on $\mathbb{T} = [0,1)$ (or $\mathbb{R}$):

$$\partial_t h^i = \frac{1}{2}\partial_u^2 h^i + \frac{1}{2}\Gamma^i_{jk}\partial_u h^j \partial_u h^k + \dot{W}^i, \quad 1 \le i \le n.$$

- We write i, j, k instead of α, β, γ in Lecture No 4 and macroscopic spatial variable u.
- We use Einstein's convention for sum.
- $\dot{W}(t,u) = (\dot{W}^i(t,u))_{i=1}^n$ is an $\mathbb{R}^n$-valued space-time Gaussian white noise with covariance structure

$$E[\dot{W}^i(t,u)\dot{W}^j(s,v)] = \delta^{ij}\delta(u-v)\delta(t-s).$$

- Coupling constants Γ^i_{jk}
  bilinear condition: $\Gamma^i_{jk} = \Gamma^i_{kj}$ for all i, j, k,
  trilinear condition **(T)**: $\Gamma^i_{jk} = \Gamma^i_{kj} = \Gamma^j_{ik}$ for all i, j, k.
- We also consider the coupled KPZ eq with constant drifts:

$$\partial_t h^i = \frac{1}{2}\partial_u^2 h^i + \frac{1}{2}\Gamma^i_{jk}\partial_u h^j \partial_u h^k + c^i \partial_u h^i + \dot{W}^i, \quad 1 \le i \le n.$$

Recall: Results on coupled KPZ eq (Lecture No 4, on $\mathbb{T}$)

- We may assume $c^i = 0$ by considering $\tilde{h}^i(t,u) := h^i(t, u - c^i t)$.
- Local well-posedness by applying paracontrolled calculus due to Gubinelli-Imkeller-Perkowski 2015.
- Under the trilinear condition (T),
  - (unique) invariant measure = Wiener measure
  - Global well-posedness (existence, uniqueness for all initial values in Besov space $\mathcal{C}^\alpha = (\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}))^n, \alpha \in (0, \frac{1}{2})$)
  - Strong Feller property (Hairer-Mattingly 2016)
  - cancellation in log-renormalization (for 4th order terms)
  - two types of approximations, difference of two limits (cf. F-Quastel 2015 when $n = 1$)
- (Conjecture) "Inv meas=Wiener meas" $\Leftrightarrow$ Condition (T)
  This holds, for example, in discrete setting.
  We have a heuristic proof, F 2019 (Proc IHP)

Motivation to study coupled KPZ equation:

- Nonlinear fluctuating hydrodynamics (Spohn), KPZ universality
- Component-wise different drifts $c^i \partial_u h^i$ play a role.

Our goal: Derivation of coupled KPZ equation from microscopic systems.

$n = 1$ (single component scalar-valued case)

- Bertini-Giacomin (WASEP, microscopic Cole-Hopf transf)
- Gonçalves-Jara (WAEP with speed change, gradient type)
- Gonçalves-Jara-Sethuraman (WA zero-range process, gradient type $\to$ Lecture No 5A)
- Gonçalves-Perkowski-Simon (WASEP+Dirichlet bdy cond)
- K. Yang (WASEP with boundary condition $\to \partial_u h = c$ at boundary 2020; non-stationary energy solution 2020)

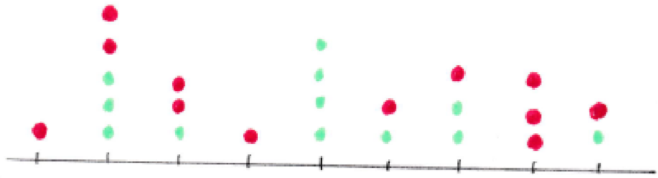
$n = 2$

- Chen-de Gier-Hiki-Sasamoto (Two-species EP, 2018)
- Ahmed-Bernardin-Gonçalves-Simon (Hamilton systems with conservative noises)

3. n-species zero-range processes on $\mathbb{T}_N$

- To derive n-component system in the limit, we need to consider a system with n-conserved quantities at microscopic level.
- $\mathbb{T}_N = \{1, 2, \dots, N\}$ with periodic boundary condition. This is a microscopic space corresponding to macroscopic $\mathbb{T} = [0, 1)$.
- Our model: Particles of n types, which perform Random Walks on $\mathbb{T}_N$ and interact only at the same sites.
- Configuration space of particles: $\alpha = (\alpha^i)_{i=1}^n \in \mathcal{X}_N^n$, $\mathcal{X}_N = \mathbb{Z}_+^{\mathbb{T}_N}$.
- $\alpha^i = (\alpha^i(x))_{x \in \mathbb{T}_N}$; $\alpha^i(x) \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$, $x \in \mathbb{T}_N$, $1 \le i \le n$: number of ith species particles at x.
- Instead of $\eta_x, \eta_x(t)$ in Lecture No 5-A, we write $\alpha(x), \alpha_t(x)$.

- Weak asymmetry: Once jump happens, the probabilities of jump of ith particles to right/left are $p_i(\pm 1) = \frac{1}{2} \pm c^{i,N}$ with small $c^{i,N}$.
- $c^{i,N} = \frac{c^i}{N}$ i.e., $O(\frac{1}{N})$ for HD limit and linear fluctuation.
- $c^{i,N} = \frac{c}{\sqrt{N}} + \frac{c^i}{N}$, i.e., $O(\frac{1}{\sqrt{N}})$ for KPZ fluctuation. Note that the constant c in leading order is common in i.
- We introduce a diffusive time change $t \mapsto N^2 t$ for the microscopic process.
- The process is denoted by $\alpha_t^N = (\alpha_t^{N,i}(x))$.



- The generator of α_t^N is given, for functions f on $\mathcal{X}_N^n$, by

$$L_N f(\alpha) = N^2 \sum_{x \in \mathbb{T}_N, 1 \le i \le n, e = \pm 1} p_i(e) g_i(\alpha(x)) \left\{ f(\alpha^{x,x+e;i}) - f(\alpha) \right\}.$$

- $\alpha^{x,y;i}$ = the configuration α after one ith particle jumps from x to y (which is possible only when $\alpha^i(x) \ge 1$).
- Zero-range property: Jump rate g_i of ith particles is a function on $\mathbb{Z}_+^n$ (=configuration space at a single site):
  $$g_i = g_i(\mathbf{k}) \text{ for } \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n.$$
  In particular, interaction occurs only at the same sites.

Conditions on jump rates $\{g_i(\mathbf{k})\}_{1\le i\le n, \mathbf{k}\in\mathbb{Z}_+^n}$ (Grosskinsky-Spohn)

(1) (Non-degeneracy) For every $1 \le i \le n$,
$g_i(\mathbf{k}) = 0 \Leftrightarrow k_i = 0$ and $\inf_{\mathbf{k}:k_i>0} g_i(\mathbf{k}) > 0$ hold.

(2) (Linear growth)
$\max_{1\le i,j\le n} \sup_{\mathbf{k}\in\mathbb{Z}_+^n} |g_i(\mathbf{k}^j, k_j+1) - g_i(\mathbf{k})| < \infty$.

(3) (Detailed balance w.r.t. product measures)
$\frac{g_i(\mathbf{k})}{g_i(\mathbf{k}^j, k_j-1)} = \frac{g_j(\mathbf{k})}{g_j(\mathbf{k}^i, k_i-1)}$, for all $i \neq j$ and
$\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ with $k_i, k_j \ge 1$, where
$(\mathbf{k}^j, k_j - 1) = (k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_n)$.

(4) (Non-triviality of $Dom_Z := \{\varphi \in (0,\infty)^n; Z_\varphi < \infty\}$ to contain a neighborhood of $(0,\ldots,0)$)
$\varphi_* := \liminf_{|\mathbf{k}|\to\infty} \mathbf{g}!(\mathbf{k})^{\frac{1}{|\mathbf{k}|}} > 0.$ ($\mathbf{g}!(\mathbf{k}) \to$ next page)

Example. n-color zero-range process: Jump rate of color-blind particles $g : \mathbb{Z}_+ \to (0,\infty)$, $g(0) = 0$, is given and
$g_i(\alpha(x)) = g(\eta(x))\frac{\alpha^i(x)}{\eta(x)}$, where $\eta(x) := \sum_{i=1}^n \alpha^i(x)$ is number of color-blind particles at x.

Invariant measures (Equilibrium states)

- Product measures $\{\bar{\nu}_{\boldsymbol{\varphi}} := p_{\boldsymbol{\varphi}}^{\otimes \mathbb{T}_N}\}$ with one site marginal

$$p_{\boldsymbol{\varphi}}(\mathbf{k}) = \frac{1}{Z_{\boldsymbol{\varphi}}} \frac{\boldsymbol{\varphi}^{\mathbf{k}}}{\mathbf{g}!(\mathbf{k})}$$

- Here $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ are non-negative parameters called fugacity, $\boldsymbol{\varphi}^{\mathbf{k}} := \varphi_1^{k_1} \cdots \varphi_n^{k_n}$,

$$\mathbf{g}!(\mathbf{k}) := \prod_{\ell=1}^{|\mathbf{k}|} g_{i(\ell)}(\mathbf{k}_\ell),$$

with $|\mathbf{k}| = k_1 + \cdots + k_n$, is a product along an increasing path $\mathbf{k}_0 = \mathbf{0} \to \cdots \to \mathbf{k}_\ell \to \cdots \to \mathbf{k}_{|\mathbf{k}|} = \mathbf{k}$ connecting $\mathbf{0}$ and $\mathbf{k}$ in $\mathbb{Z}_+^n$ such that $|\mathbf{k}_\ell| = \ell$, $0 \leq \ell \leq |\mathbf{k}|$, and

$$Z_{\boldsymbol{\varphi}} := \sum_{\mathbf{k} \in \mathbb{Z}_+^n} \frac{\boldsymbol{\varphi}^{\mathbf{k}}}{\mathbf{g}!(\mathbf{k})}.$$

- Note that, by the condition (3), $\mathbf{g}!(\mathbf{k})$ does not depend on the choice of the increasing path $\{\mathbf{k}_\ell\}$, so is well-defined.

Change of the parameter $\varphi \mapsto \mathbf{a} = (a^1, \ldots, a^n)$: density

- $\bar{\nu}_\varphi$ is well-defined for $\varphi \in (0,\infty)^n$ s.t. $Z_\varphi < \infty$.
- Change the parametrization in terms of density:
  For $\mathbf{a}$, choose φ so that the mean is given by $\mathbf{a}$, i.e.,
  $$a^i \equiv a^i(\varphi) := E^{\bar{\nu}_\varphi}[\alpha_i(0)], \quad i = 1, \ldots, n \tag{1}$$
  holds and denote $\nu_{\mathbf{a}} := \bar{\nu}_\varphi$.
- Denote the map $R : \varphi \to \mathbf{a}$, taking fugacity to its associated density, defined on
  $$Dom_R := \{\varphi \in (0,\infty)^n; Z_\varphi < \infty, a^i(\varphi) < \infty, \; i = 1, \ldots, n\}.$$
- The correspondence $\varphi \leftrightarrow \mathbf{a}$ is $1:1$.
  Denote, by $\Phi : \mathbf{a} \to \varphi$, the inverse map of R.
- We accordingly have a family of invariant measures $\{\nu_{\mathbf{a}}\}_{\mathbf{a}}$ parametrized by densities $\mathbf{a} = (a^1, \ldots, a^n) \in [0,\infty)^n$.

4. Hydrodynamic limit (LLN) and Linear fluctuation (CLT)

4.1 Hydrodynamic limit

- Weak asymmetry is $O(\frac{1}{N})$, i.e., $p_i(\pm 1) = \frac{1}{2} \pm \frac{c^i}{N}$ and c^i may be different for different species.
- Similarly to single species case (Lecture No 5-A), we consider an $\mathbb{R}^n$-valued macroscopically scaled empirical measure $X_t^N = (X_t^{N,i})_{i=1}^n$ on $\mathbb{T}$ defined by

$$X_t^{N,i}(du) := \frac{1}{N} \sum_x \alpha_t^{N,i}(x) \delta_{\frac{x}{N}}(du), \quad u \in \mathbb{T}.$$

- Recall $\alpha_t^N = (\alpha_t^{N,i}(x))_{x \in \mathbb{T}_N}$ is the n-species zero-range process generated by $N^2 L_N$.
- As a straightforward extension of Theorem 4 (HDL) of Lecture No 5-A for the single species case, we can show the following.

- HD limit for n-species system: X_t^N converges to $\mathbf{a}(t,u)du = (a^i(t,u)du)_{i=1}^n$ and the limit density $a^i(t,u)$ is the solution of the system of nonlinear PDEs:

$$\partial_t a^i = \frac{1}{2}\partial_u^2 \varphi_i(\mathbf{a}) - 2c^i \partial_u \varphi_i(\mathbf{a}), \quad 1 \le i \le n.$$

where

$$\varphi_i(\mathbf{a}) \equiv \langle g_i \rangle(\mathbf{a}) := E^{\nu_{\mathbf{a}}}[g_i(\alpha(0))].$$

- Indeed, $\varphi_i(\mathbf{a}) = \langle g_i \rangle(\mathbf{a})$ is shown as

$$\langle g_i \rangle(\mathbf{a}) = \frac{1}{Z_\varphi} \sum_{\mathbf{k}} g_i(\mathbf{k}) \frac{\boldsymbol{\varphi}^{\mathbf{k}}}{\mathbf{g}!(\mathbf{k})} = \frac{1}{Z_\varphi} \sum_{\mathbf{k}} \frac{\boldsymbol{\varphi}^{\mathbf{k}}}{\mathbf{g}!(\mathbf{k}-e_i)}$$
$$= \frac{1}{Z_\varphi} \sum_{\mathbf{k}} \frac{\varphi_i \boldsymbol{\varphi}^{\mathbf{k}-e_i}}{\mathbf{g}!(\mathbf{k}-e_i)} = \varphi_i.$$

- The diffusion matrix is given by $(\frac{\partial \varphi_i}{\partial a_j}) = \left(\varphi_i (\operatorname{cov} \nu_{\mathbf{a}})_{ij}^{-1}\right)$ (cf. [BFS, Lemma 2.1]) and parabolic in the sense $\sum_{ij} \frac{\partial \varphi_i}{\partial a_j} \xi_i \xi_j \ge 0$ for any $\xi = (\xi_i) \in \mathbb{R}^n$.

Heuristic derivation of HD equation

- Take a test function $G \in C^\infty(\mathbb{T})$. Then, exactly in the same way as Lecture No 5-A, in Dynkin's formula, we have

$$L_N X^{N,i}(G) = \frac{1}{2N} \sum_x g_i(\alpha(x)) N^2 \Delta G(\tfrac{x}{N}) + \frac{c^i}{N} \sum_x g_i(\alpha(x)) \left\{ N\nabla G(\tfrac{x}{N}) + N\nabla G(\tfrac{x-1}{N}) \right\},$$

where $\nabla G(\frac{x}{N}) := G(\frac{x+1}{N}) - G(\frac{x}{N})$ and Δ is the discrete Laplacian.

- For martingale terms, $\lim_{N\to\infty} E[M_t^{N,i}(G)^2] = 0$ hold.
- Local ergodicity (local equilibrium): One can replace $g_i(\alpha(x))$ by its local average $\langle g_i \rangle(\mathbf{a}(t, \frac{x}{N}))$ and obtain the Hydrodynamic equation for $a^i(t,u)$ in the limit.

4.2 Linear fluctuation

- Keep weak asymmetry $O(\frac{1}{N})$, i.e., $p_i(\pm 1) = \frac{1}{2} \pm \frac{c^i}{N}$.
- We discuss equilibrium fluctuation (CLT),
  i.e. assume $\alpha_0^N \overset{\text{law}}{=} \nu_{\mathbf{a}_0}$ for any fixed $\mathbf{a}_0 \in (0, \infty)^n$.
- Consider the fluctuation field: $Y_t^N = (Y_t^{N,i})_{i=1}^n$

$$Y_t^{N,i}(du) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} \left(\alpha_t^{N,i}(x) - a_0^i\right) \delta_{\frac{x}{N}}(du), \quad u \in \mathbb{T}.$$

- $Y_t^N = (Y_t^{N,i})_{i=1}^n$ converges to Ornstein-Uhlenbeck process Y_t in law ($\rightarrow$ next page).
- This class of models having OU scaling limit is sometimes called Edwards-Wilkinson university class.

- The limit $Y_t = (Y_t^i)_{i=1}^n$ is the solution (unique in law) of linear SPDE:

$$\partial_t Y = \frac{1}{2} Q(\mathbf{a}_0)\partial_u^2 Y - 2\mathbf{C} Q(\mathbf{a}_0)\partial_u Y + q(\mathbf{a}_0)\partial_u \dot{W},$$

where $\dot{W} = (\dot{W}^i)_{i=1}^n$ is $\mathbb{R}^n$-valued space-time Gaussian white noise, and $\mathbf{C}$, $Q(\mathbf{a})$ and $q(\mathbf{a})$ are $d \times d$ matrices such that

$$\begin{aligned} \mathbf{C} &= \mathrm{diag}(c^i)_{1\le i\le n}, \\ Q(\mathbf{a}) &= \big(Q_{ij}(\mathbf{a})\big)_{1\le i,j\le n} = (\partial_{a^j}\varphi_i(\mathbf{a})))_{1\le i,j\le n}, \\ q(\mathbf{a}) &= \mathrm{diag}\big(q^i(\mathbf{a})\big)_{1\le i\le n} = \mathrm{diag}\left(\sqrt{\varphi_i(\mathbf{a})}\right)_{1\le i\le n}. \end{aligned}$$

- The matrix $Q(\mathbf{a}_0)$ arises as a linearization of $\varphi_i(\mathbf{a})$ in the HD equation around $\mathbf{a}_0$:

$$\varphi_i(\mathbf{a}) = \varphi_i(\mathbf{a}_0) + \sum_{j=1}^n \partial_{a_j}\varphi_i(\mathbf{a}_0)(a_j - a_{0,j}) + \cdots$$

Reason to have the limit noise $\left(\sqrt{\varphi_i(\mathbf{a})}\partial_u \dot{W}^i\right)_i$:

- Compute quadratic and cross variations of the martingale term $M_t^{N,i}(G)$ of $Y_t^{N,i}(G)$.
- Indeed, similar to Lecture No 5-A, we have

$$\begin{aligned}\frac{d}{dt}\langle M^{N,i}(G)\rangle_t &= N\left(L_N\langle \alpha_t^{N,i}, G\rangle^2 - 2\langle \alpha_t^{N,i}, G\rangle L_N\langle \alpha_t^{N,i}, G\rangle\right)\\ &= \frac{1}{N}\sum_x g_i(\alpha_t^N(x))\left(N\nabla G(\tfrac{x}{N})\right)^2 + O(\tfrac{1}{N})\\ &\underset{N\to\infty}{\longrightarrow} \varphi_i(\mathbf{a}_0)\int_{\mathbb{T}}(G'(u))^2 du,\end{aligned}$$

since $\alpha_t^N \overset{\text{law}}{=} \nu_{a_0}$ for all $t \geq 0$.
- For $i \neq j$, we have

$$\begin{aligned}&\frac{d}{dt}\langle M^{N,i}(G_1), M^{N,j}(G_2)\rangle_t\\ &= N\Big(L_N(\langle \alpha_t^{N,i}, G_1\rangle\langle \alpha_t^{N,j}, G_2\rangle) - \langle \alpha_t^{N,i}, G_1\rangle L_N\langle \alpha_t^{N,j}, G_2\rangle\\ &\qquad - \langle \alpha_t^{N,j}, G_2\rangle L_N\langle \alpha_t^{N,i}, G_1\rangle\Big)\\ &= 0.\end{aligned}$$

Heuristic reason to have the drift term in the limit

- Make Taylor expansion in the HD equation:

$$a^i \,(= a^i(t,u)) = a_0^i + \frac{1}{\sqrt{N}} Y^i + \cdots$$

$$\varphi_i(\mathbf{a}) = \varphi_i(\mathbf{a}_0) + \frac{1}{\sqrt{N}} \sum_{j=1}^{n} \partial_{a^j}\varphi_i(\mathbf{a}_0) \cdot Y^j + \cdots .$$

- Insert these into the HD equation with noise error term:

$$\partial_t a^i = \tfrac{1}{2}\partial_u^2 \varphi_i(\mathbf{a}) - 2c^i \partial_u \varphi_i(\mathbf{a}) + \frac{1}{\sqrt{N}}(\text{noise})$$

- For example, since $\mathbf{a}_0$ is a constant,

$$\partial_t a^i = \frac{1}{\sqrt{N}} \partial_t Y^i + \cdots .$$

- Multiplying the both sides by $\sqrt{N}$, we obtain the limit SPDE.
- For the proof, 1st order Boltzmann-Gibbs principle is needed.

5. Nonlinear fluctuation leading to coupled KPZ equation

- Now weak asymmetry is $O(\frac{1}{\sqrt{N}})$, i.e.,
$$p_i(\pm 1) = \tfrac{1}{2} \pm \tfrac{c}{\sqrt{N}} \pm \tfrac{c^i}{N},$$
which is larger than HD limit and linear fluctuation.
- Note that the leading constant c is common, to have the common moving frame ($\to$ see below).
- In other words, c^i are replaced by $c\sqrt{N} + c^i$ so that the HD equation for ith particles would look like

$$\partial_t a^i = \tfrac{1}{2}\partial_u^2 \varphi_i(\mathbf{a}) - 2(c\sqrt{N} + c^i)\partial_u \varphi_i(\mathbf{a}) + \frac{1}{\sqrt{N}}(\text{noise})$$

- We consider the fluctuation field under equilibrium, i.e. $\alpha_0^N \overset{\text{law}}{=} \nu_{\mathbf{a}_0}$ for some $\mathbf{a_0}$, this time chosen properly.

- To cancel the diverging factor $2c\sqrt{N}$, we introduce the moving frame with speed $2c\lambda\sqrt{N}$ at macroscopic level with suitably chosen $\lambda = \lambda(\mathbf{a}_0)$.

$$Y_t^{N,i}(du) := \frac{1}{\sqrt{N}} \sum_x \left(\alpha_t^{N,i}(x) - a_0^i\right)\delta_{\frac{x}{N} - 2c\lambda\sqrt{N}t}(du)$$

- The frame should have common speed for all i.
  $\to$ This gives a restriction to the choice of $\mathbf{a}_0$.
- We choose $\mathbf{a}_0$ and $\lambda(\mathbf{a}_0)$ properly.
- Especially we need to assume the Frame Condition:
  $Q(\mathbf{a}_0) = -\lambda I$ for $\mathbf{a}_0$ and λ ($\to$ see below).

Main result (coupled KPZ limit = nonlinear fluctuation)

Theorem 1

Assume the frame condition. Then, $Y^N_t = (Y^{N,i}_t)_{i=1}^n$ converges to $Y_t = (Y^i_t)_{i=1}^n$ in law in the space $D([0,T], \mathcal{S}'(\mathbb{T})^n)$. The limit Y_t is the (unique) stationary martingale solution of coupled KPZ-Burgers equation:

$$\begin{aligned}\partial_t Y^i =& \tfrac{1}{2} Q^i(\mathbf{a}_0)\partial_u^2 Y^i + \Gamma^i_{jk}(\mathbf{a}_0)\partial_u(Y^j Y^k) \\ & -2c^i Q^i(\mathbf{a}_0)\partial_u Y^i + q^i(\mathbf{a}_0)\partial_u \dot{W}^i, \quad u \in \mathbb{T}.\end{aligned}$$

- $(\dot{W}^i)_{i=1}^n$ is $\mathbb{R}^n$-valued space-time Gaussian white noise.
- $Q^i(\mathbf{a}_0)$, $\Gamma^i_{jk}(\mathbf{a}_0)$ and $q^i(\mathbf{a}_0)$ are given by

$$\begin{aligned} Q^i(\mathbf{a}_0) &= \partial_{a^i}\varphi_i(\mathbf{a}_0), \\ \Gamma^i_{jk}(\mathbf{a}_0) &= -c\partial_{a^j}\partial_{a^k}\varphi_i(\mathbf{a}_0), \\ q^i(\mathbf{a}_0) &= \sqrt{\varphi_i(\mathbf{a}_0)}. \end{aligned}$$

- The reason to have the limit noise $q^i(\mathbf{a}_0)\partial_u \dot{W}^i$ is the same as the linear fluctuation.

Heuristic reason to have the nonlinear drift term in the limit

- Combine averaging due to ergodicity and Taylor expansion, now up to the second order terms:

$$
\begin{aligned}
a^i &= a_0^i + \frac{1}{\sqrt{N}} Y^i + \cdots \\
\partial_t a^i &= \frac{1}{\sqrt{N}} \partial_t Y^i + 2c\lambda\sqrt{N}\partial_u a^i + \cdots \\
&= \frac{1}{\sqrt{N}} \partial_t Y^i + 2c\lambda \partial_u Y^i + \cdots \\
\varphi_i(\mathbf{a}) &= \varphi_i(\mathbf{a}_0) + \frac{1}{\sqrt{N}} \sum_{j=1}^{n} \partial_{a^j} \varphi_i(\mathbf{a}_0) \cdot Y^j \\
&\quad + \frac{1}{2N} \sum_{j,k=1}^{n} \partial_{a^j}\partial_{a^k} \varphi_i(\mathbf{a}_0) \cdot Y^j Y^k + \cdots .
\end{aligned}
$$

- Noting $\partial_u \varphi_i(\mathbf{a}_0) = 0$, putting these expansions to the HD equation and multiplying both sides by $\sqrt{N}$, we obtain:

$$
\begin{aligned}
\partial_t Y^i =& \frac{1}{2} \sum_{j=1}^{n} \partial_{a^j} \varphi_i(\mathbf{a}_0) \cdot \partial_u^2 Y^j \\
&- 2(c\sqrt{N} + c^i) \sum_{j=1}^{n} \partial_{a^j} \varphi_i(\mathbf{a}_0) \cdot \partial_u Y^j - 2c\lambda\sqrt{N}\partial_u Y^i \\
&- c \sum_{j,k=1}^{n} \partial_{a^j} \partial_{a^k} \varphi_i(\mathbf{a}_0) \cdot \partial_u (Y^j Y^k) + q^i(\mathbf{a}_0)\partial_u \dot{W}^i + o(1).
\end{aligned}
$$

- Note that the noise term $q^i(\mathbf{a}_0)\partial_u \dot{W}^i$ has the same distribution under the shift by moving frame.
- The second line (except c^i) is a diverging term.
  (If $c = 0$, the above eq is same as linear fluctuation.)

- This line vanishes, if one can choose $\mathbf{a}_0$ (and λ) such that

  [Frame Condition] $\lambda = -\partial_{a^i}\varphi_i(\mathbf{a}_0),\ \partial_{a^j}\varphi_i(\mathbf{a}_0) = 0$ if $i \neq j$.

- This condition is equivalent to "$V_{ij} = 0$ $(i \neq j)$ and φ_i/V_{ii} is constant in i", where $V \equiv \big(V_{ij}(\mathbf{a}_0)\big) :=\mathrm{cov}(\nu_{\mathbf{a}_0})$ ($\to$ Prop 3.3 of [BFS]).
- Thus, we obtain the KPZ-Burgers equation in the limit:

$$\partial_t Y^i = \tfrac{1}{2}\partial_{a^i}\varphi_i(\mathbf{a}_0)\, \partial_u^2 Y^i - c \sum_{j,k=1}^{n} \partial_{a^j}\partial_{a^k}\varphi_i(\mathbf{a}_0)\, \partial_u (Y^j Y^k) \\ - 2c^i \partial_{a^i}\varphi_i(\mathbf{a}_0)\, \partial_u Y^i + q^i(\mathbf{a}_0)\partial_u \dot{W}^i.$$

(End of heuristic argument) □

Proof of Theorem 1

- For the proof, we need to establish the Boltzmann-Gibbs principle, i.e., replacement under space-time average of nonlinear function f of α s.t. $\langle f\rangle(\mathbf{a}_0) = \partial_{a^i}\langle f\rangle(\mathbf{a}_0) = 0$ $({}^\forall i)$ by quadratic function of $\alpha^i - a^i$, in equilibrium $\nu_{\mathbf{a}_0}$.
- For identification of the limit, we use the uniqueness of stationary coupled martingale solutions due to Gubinelli-Perkowski, PTRF 2020.
- In the limit SPDE, drift term with c^i can be killed by the spatial shift:

$$\tilde{Y}^i_t(u) := Y^i_t(u + 2c^i Q^i(\mathbf{a}_0)t).$$

- So we assume $c^i = 0$ below for simplicity.
- We also show the tightness of $\{Y^N_t\}_N$ in the uniform topology in $D([0,T], \mathcal{S}'(\mathbb{T})^n)$.

Boltzmann-Gibbs principle

- For $\zeta = (\zeta(x))$, the average of ζ around x in size $\ell \geq 1$ is defined by $\zeta^{(\ell)}(x) := \frac{1}{2\ell+1} \sum_{|y| \leq \ell} \zeta(x+y)$.

Theorem 2

Let $f = f(\alpha) \in L^5(\nu_{\mathbf{a}_0})$ be a local function supported on sites $|y| \leq \ell_0$ s.t. $\langle f \rangle(\mathbf{a}_0) = 0$ and $\nabla \langle f \rangle(\mathbf{a}_0) = 0$. Then, ${}^{\exists}C = C(\ell_0) > 0$ s.t. for $T > 0$, $\ell \geq \ell_0$ and $\phi : \mathbb{T}_N \to \mathbb{R}$,

$$E^{\nu_{\mathbf{a}_0}}\left[\sup_{0 \leq t \leq T} \left(\int_0^t ds \sum_{x \in \mathbb{T}_N} \phi(x - [cs]) \Big(f(\tau_x \alpha_s^N) - \frac{1}{2} \sum_{j,k=1}^{n} \partial_{a^j} \partial_{a^k} \langle f \rangle (\mathbf{a}_0) \right.\right.$$

$$\left.\left. \times \left\{ \left(\left(\alpha_s^{j,N}\right)^{(\ell)}(x) - a_0^j \right) \left(\left(\alpha_s^{k,N}\right)^{(\ell)}(x) - a_0^k \right) - \frac{V_{jk}(\mathbf{a}_0)}{2\ell + 1} \right\} \Big) \right)^2 \right]$$

$$\leq C \|f\|^2_{L^5(\nu_{\mathbf{a}_0})} \left(\frac{T\ell}{N} \|\phi\|^2_{L^2(\mathbb{T}_N)} + \frac{T^2 N^2}{\ell^3} \|\phi\|_{L^1(\mathbb{T}_N)} \right)^2,$$

where $(V_{jk}(\mathbf{a}_0)) = \mathrm{cov}\,(\nu_{\mathbf{a}_0})$, $\|\phi\|^p_{L^p(\mathbb{T}_N)} := \frac{1}{N} \sum_{x \in \mathbb{T}_N} |\phi(x)|^p$.

Proof of BG principle (Theorem 2)

- Itô-Tanaka trick to reduce dynamic problem to static one (bound by H^{-1}-norm, cf. Lecture No 3):

$$E^{\nu_{\mathbf{a}_0}}\Big[\sup_{0\le t\le T}\Big(\int_0^t F(\alpha_s)ds\Big)^2\Big] \underset{\text{roughly}}{\le} C\langle F,(-L^{\text{sym}})^{-1}F\rangle_{\nu_{\mathbf{a}_0}}.$$

- To estimate H^{-1}-norm by L^2-norm, we apply the spectral gap of the operator $-L^{\text{sym}}$, but this works on bounded region and depends on the size of region.
- $L^{\text{sym}}_{\mathbf{k},\ell}$: Symmetrized generator on $\Lambda_\ell=\{x;|x|\le\ell\}$ with $\sharp$particles$=\mathbf{k}$ on Λ_ℓ,
  $W(\mathbf{k},\ell):=$ (spectral gap of $-L^{\text{sym}}_{\mathbf{k},\ell})^{-1}$.
  $\Longrightarrow E^{\nu_{\mathbf{a}}}[W(\mathbf{k},\ell)^2]\le C\ell^4$ holds.
  We need some assumption on $(g_i)_{i=1}^n$ to show this.
- So, we need to confine ourselves in a bounded region of size ℓ by conditioning ($\to$ canonical ensemble).

- Static estimates: Decay estimate for canonical average as $\ell \to \infty$ to grandcanonical average (equivalence of ensembles) and Taylor expansion.
- To give some feeling, for $y \in \mathbb{R}^n$,

$$E^{\nu_{\mathbf{a}_0}}[f(\alpha)|\alpha^{(\ell)} = y] = \frac{E^{\nu_{\mathbf{a}_0}}[f(\alpha) \cdot 1_{\{\alpha^{(\ell)}=y\}}]}{\nu_{\mathbf{a}_0}(\alpha^{(\ell)} = y)}$$

$$\underset{(*)}{\sim} E^{\nu_y}[f(\alpha)]$$

$$\underset{\text{Taylor expansion}}{\sim} \langle f \rangle(\mathbf{a}_0) + \nabla \langle f \rangle(\mathbf{a}_0) \cdot (y - \mathbf{a}_0) + \frac{1}{2}\big(y - \mathbf{a}_0, D^2\langle f \rangle(\mathbf{a}_0)(y - \mathbf{a}_0)\big) + \cdots$$

- (*) is usually called the equivalence of ensembles and shown by applying local CLT:

$$\nu_{\mathbf{a}_0}(\alpha^{(\ell)} = y) \sim C_\ell e^{-c_\ell(y-\mathbf{a}_0, V^{-1}(y-\mathbf{a}_0))}.$$

- We use (by taking $y = \alpha^{(\ell)}$)

$$E^{\nu_{\mathbf{a}_0}}[f(\alpha)] = E^{\nu_{\mathbf{a}_0}}\Big[E^{\nu_{\mathbf{a}_0}}[f(\alpha)|\alpha^{(\ell)}]\Big],$$

and this leads to (the static version of) Theorem 2.

(End of proof of Theorem 2) □

Tightness of $\{Y_t^N\}$ in uniform topology in $D([0,T],\mathcal{S}'(\mathbb{T})^n)$

- (Mitoma's theorem) It is enough to show the tightness of $\{Y_t^{N,i}(G)\}$ in $D([0,T],\mathbb{R})$ for each test function $G \in C^\infty(\mathbb{T})$.
- Martingale term $\{M_t^{N,i}(G)\}$ has quadratic variation bounded in $L^4(\Omega)$, so that it is tight.
- BG principle gives a bound for drift term in Dynkin's formula.

Martingale problem approach (Gubinelli-Perkowski, 2020)

- Coupled KPZ-Burgers equation (canonical form) for $Y^i = \partial_u h^i$

$$\partial_t Y^i = \tfrac{1}{2}\partial_u^2 Y^i + \tfrac{1}{2}\Gamma^i_{jk}\partial_u(Y^j Y^k) + \partial_u \dot{W}^i.$$

- Formal generator (Lectures No 3, 4): $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}$, where

$$\mathcal{L}_0\Phi(Y) = \tfrac{1}{2}\sum_i \left(\int_{\mathbb{T}} \partial_u^2 D^2_{Y^i(u)}\Phi \, du + \int_{\mathbb{T}} \partial_u^2 Y^i(u)\cdot D_{Y^i(u)}\Phi \, du\right)$$

$$\mathcal{A}\Phi(Y) = \tfrac{1}{2}\sum_{i,j,k}\Gamma^i_{jk}\int_{\mathbb{T}} \partial_u(Y^j(u)Y^k(u))D_{Y^i(u)}\Phi \, du$$

for $\Phi = \Phi(Y)$. D, D^2 are Fréchet derivatives.

- Precise definition of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$:
  Let ν be the probability distribution on $\mathcal{S}'(\mathbb{T})^n$ of white noise in space. Let

$$L^2(\nu) \cong \Gamma L^2 := \bigoplus_{m=0}^{\infty} L^2(\mathbb{T}^m)^n \quad \text{(Fock space)}$$

  be the Wiener-Itô chaos decomposition.
- $\mathcal{D}(\mathcal{L}) := \{\varphi; \varphi^\sharp \in (-\mathcal{L}_0)^{-1}\Gamma L^2 \cap (1+\mathcal{N})^{-9/2}(-\mathcal{L}_0)^{-1/2}\Gamma L^2\}$,
  where φ, called controlled function, is a solution of

$$\varphi - (-\mathcal{L}_0)^{-1}\mathcal{A}^{\succ}\varphi = \varphi^\sharp,$$

  in controlled sense (i.e., first define singular products based on Gaussian structure by hand, and then others are usual calculus), $\mathcal{A}^{\succ}$ is a certain cut-off of $\mathcal{A}$ and $\mathcal{N}$ is a number operator.
- If $\mathcal{A}$ instead of $\mathcal{A}^{\succ}$, this is resolvent equation with $\lambda = 0$: $(\mathcal{L}_0 + \mathcal{A})\varphi = \mathcal{L}_0\varphi^\sharp$.

- [GP] showed that, for $\varphi^\sharp$ of this class, the solution φ exists, $\mathcal{D}(\mathcal{L})$: dense in ΓL^2 and $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to (-\mathcal{L}_0)^{1/2}\Gamma L^2$ is well-defined.
- [GP] also showed Kolmogorov backward equation $\partial_t \varphi = \mathcal{L}\varphi$ is solvable in controlled sense in $\varphi = \varphi(t, Y) \in \mathcal{D}(\mathcal{L})$ for wide class of initial values $\varphi(0) = \varphi_0$ (by Galerkin method + a priori estimates).
- Exponential L^2-ergodicity is also shown,

- $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem is well-posed.

  $\because$) Uniqueness is shown as follows:

$$\Phi(t, Y_t) - \Phi(0, Y_0) - \int_0^t (\partial_s \Phi + \mathcal{L}\Phi)(s, Y_s) ds$$

  is a martingale for $\Phi(t, \cdot) \in \mathcal{D}(\mathcal{L})$. Take $\Phi(t, Y) = \varphi(T - t, Y)$ with the solution φ of Kolmogorov equation. Then, $\varphi(T - t, Y_t) - \varphi(T, Y_0)$ is martingale. Take $t = T$ and we have $E_{Y_0}[\varphi_0(Y_T)] = \varphi(T, Y_0)$. This shows the uniqueness.

- Stationary solution of cylinder function martingale problem i.e., martingale property holds for tame functions

$$\Phi(Y) = f(\langle Y, \psi_1 \rangle, \dots, \langle Y, \psi_n \rangle)$$

  instead of $\Phi \in \mathcal{D}(\mathcal{L})$ satisfying Itô-Tanaka trick (or Kipnis-Varadhan type estimate), is a solution of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem.

  $\because$) Indeed, for $\varphi \in \mathcal{D}(\mathcal{L})$, let φ_M be the projection of φ to $\bigoplus_{m=0}^{M} L^2(\mathbb{T}^m)^n$. Then, φ_M is a tame function. By several a priori bounds, one can take the limit $M \to \infty$.

- We again use Itô-Tanaka trick (bound by H^{-1}-norm):

$$E\Big[\sup_{t\in[0,T]}\Big|\int_0^t \varphi(Y_s)ds\Big|^p\Big] \le C_T \int_0^T \|c_p^{\mathcal{N}}(-\mathcal{L}_0)^{-1/2}\varphi\|^p,$$

  where $c_p = \sqrt{p-1}$.
- Interpretation of nonlinear term $\Gamma^i_{jk}\partial_u(Y^jY^k)$:
  For $Y \in C([0,T], \mathcal{S}'(\mathbb{T})^n)$ and test function H,

$$A_t^{i,\varepsilon}(H) := \sum_{j,k}\Gamma^i_{jk}\int_0^t ds \int_{\mathbb{T}} \partial_u H(u)\langle Y_s^j, G_\varepsilon(\cdot - u)\rangle\langle Y_s^k, G_\varepsilon(\cdot - u)\rangle du,$$

  where $G_\varepsilon \to \delta_0$. Then, by Itô-Tanaka trick,

$$A_t(H) =^{\exists} \lim_{\varepsilon\downarrow 0} A_t^{i,\varepsilon}(H) \quad \text{in } L^2(\Omega, C([0,T],\mathbb{R})).$$

- The proof of Theorem 1 is completed by combining all these arguments. □

Trilinear condition

- Our $\Gamma^i_{jk}(\mathbf{a}_0)$ satisfies the trilinear condition (T) after rewriting it in a canonical form by change of time and magnitude.
- The scaling limit under Product measure $\nu_{\mathbf{a}_0}$ is "white noise" (at Burgers' level), so that this is consistent.
- As we noted, we have a heuristic proof of
  (T) $\Leftrightarrow$ "invariant measure = spatial white noise".
  (This is true at least in a discrete setting.)

Multi-color case

- $g_i(\mathbf{k}) = g(|\mathbf{k}|)\frac{k_i}{|\mathbf{k}|}$, $\mathbf{k} = (k_1, \ldots, k_n)$
- Frame condition holds at ρ_0 satisfying $\varphi'(\rho_0) = \frac{\varphi(\rho_0)}{\rho_0}$, where $\varphi(\rho) := \langle g \rangle(\rho)$ (defined in color-blind ensembles).

- In multi-color case, one can decouple our coupled KPZ equation as follows.
- $H := \sum_{i=1}^n h^i$ (color-blind system) satisfies the scalar-KPZ equation:

$$\partial_t H_t = c_1 \partial_u^2 H_t + c_2 \left(\sum_{i=1}^n a_0^i \right) (\partial_u H_t)^2 + c_3 \dot{W}, \quad \dot{W} := \sum_{i=1}^n \sqrt{a_0^i} \dot{W}^i,$$

with some constants c_1, c_2, c_3.
- On the other hand, $H_t^{ij} := a_0^j h^i - a_0^i h^j$ are OU processes:

$$\partial_t H_t^{ij} = c_1 \partial_u^2 H_t^{ij} + c_3 \dot{W}^{ij}, \quad \dot{W}^{ij} := \sqrt{a_0^i} a_0^j \dot{W}^i - \sqrt{a_0^j} a_0^i \dot{W}^j$$

- One can show that $\dot{W}$ and $\{\dot{W}^{ij}\}$ are independent, since the covariances vanish.
- In this case, the uniqueness of stationary energy solution of the coupled KPZ equation follows from the uniqueness for scalar-valued KPZ equation and OU processes, and independence of these processes.

Summary of this lecture

We discussed the derivation of coupled KPZ equation from multi-species zero-range processes:

- n-species zero-range processes
- Hydrodynamic limit, Linear fluctuation
- Nonlinear fluctuation leading to coupled KPZ equation
- Boltzmann-Gibbs principle
- Martingale problem
- Trilinear condition

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Thank you very much for your attention!