KPZ limit for interacting particle systems —Independent and single species case—

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Lecture No 5-A

Plan of the course (10 lectures)

1. Introduction
2. Supplementary materials

   Brownian motion, Space-time Gaussian white noise, (Additive) linear SPDEs, (Finite-dimensional) SDEs, Martingale problem, Invariant/reversible measures for SDEs, Martingales
3. Invariant measures of KPZ equation (F-Quastel, 2015)
4. Coupled KPZ equation by paracontrolled calculus (F-Hoshino, 2017)
5. Coupled KPZ equation from interacting particle systems (Bernardin-F-Sethuraman, 2020+)
   - 5.1 Independent particle systems
   - 5.2 Single species zero-range process
   - 5.3 n-species zero-range process
   - 5.4 Hydrodynamic limit, Linear fluctuation
   - 5.5 KPZ limit=Nonlinear fluctuation

Plan of this lecture

5. Coupled KPZ equation from interacting particle systems

5.1 Independent particle systems on $\mathbb{T}_N$

1. Weakly asymmetric independent random walks on $\mathbb{T}_N$
2. Hydrodynamic limit
3. Invariant measures
4. Linear fluctuation
5. KPZ fluctuation

5.2 Single species zero-range process on $\mathbb{T}_N$

1. Model
2. Invariant measures
3. Hydrodynamic limit
4. Linear fluctuation, KPZ fluctuation

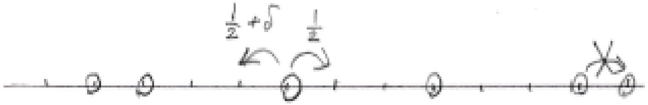
5.1 Independent particle systems on $\mathbb{T}_N$

- As a warm-up, we start with the system of particles moving independently with each other.
- Basic reference is: Kipnis-Landim, Scaling Limits of Interacting Particle Systems, Springer, 1999.
- In this book, Hydrodynamic limit (HDL=LLN) and (Equilibrium) Linear fluctuation (CLT) are discussed for interacting particle systems, mostly that called zero-range process on a d-dimensional square lattice $\mathbb{T}_N^d := (\mathbb{Z}/N\mathbb{Z})^d$ of large (microscopic) size N.
- In this lecture, we consider only the case $d = 1$, since this is relevant to discuss the KPZ equation.
- It is very interesting to study the systems with boundary effects. However, this requires additional efforts and we avoid it in this lecture.

Before starting, let's recall some of previous discussions.

Particle systems (Lecture No 1)

- We considered WASEP on $\mathbb{Z}$.
- configuration space $\mathcal{X} = \{\pm 1\}^{\mathbb{Z}} \ni \sigma = (\sigma(x))_{x\in\mathbb{Z}}$
- transition $\sigma \mapsto \sigma^{z,z+1}$ (exchange of configurations at z and $z+1$, i.e. jump of a particle at z to $z+1$ and vice versa).
- transition rate $c_{z,z+1}(\sigma)$
  ($\rightarrow$ determines how fast the transition occurs)
- $\rightarrow$ generator $Lf(\sigma) = \sum_{z\in\mathbb{Z}} c_{z,z+1}(\sigma)\{f(\sigma^{z,z+1}) - f(\sigma)\}$.
- $\rightarrow$ construction of particle systems σ_t: Markov proc. on $\mathcal{X}$
  - [distributional] based on semigroup e^{tL} on $C(\mathcal{X})$
  - [pathwise] based on "bell $\overset{\text{law}}{=}$ exp(λ)" of each particle and jump probability $\{p\}$.



From Lecture No 2

- Under some condition, "μ: invariant measure" $\longleftrightarrow$ "infinitesimal invariance $\int Lf\, d\mu = 0$ for a wide class of functions f".
- Dynkin's formula: If X_t is a Markov process with generator L, $M_t(f) := f(X_t) - \int_0^t Lf(X_s)ds$ is a martingale.
- Cross-variation of $M_t(f)$:

$$\langle M(f), M(g)\rangle_t = \int_0^t \{L(fg) - f\, Lg - g\, Lf\}(X_s)ds$$

  i.e., $M_t(f)M_t(g) - \langle M(f), M(g)\rangle_t$ is a martingale.
- In particular, quadratic variation is given by

$$\langle M(f)\rangle_t = \int_0^t \{Lf^2 - 2f\, Lf\}(X_s)ds$$

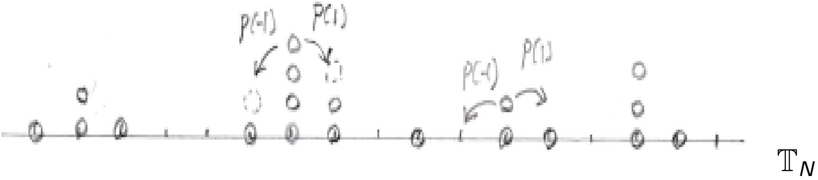
  i.e., $M_t(f)^2 - \langle M(f)\rangle_t$ is a martingale.

KPZ equation, Coupled KPZ equation (Lectures No 3 and 4)

- The solution we constructed was strong solution (in the sense of the SDEs). It is a function of driving terms $\mathbb{H}$, which are directly constructed from the noise $\dot{W}(t,x)$.
- In the theory of SDEs, another notion of the solution is a weak solution, that is, a solution in distribution's sense (or law's sense). The solution is characterized by martingale problem ($\rightarrow$ recall Lecture No 2).
- To show the limit theorem, it is more convenient to use the setting of martingale problem.
- Corresponding notion in KPZ equation is the energy solution.

1. Weakly asymmetric independent random walks on $\mathbb{T}_N$

- Let $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z} = \{1, 2, \ldots, N\}$ be the 1-dimensional integer lattice of size N with periodic boundary condition.
- We consider independent random walks on $\mathbb{T}_N$ with rate $p(1)$ of jumps to the right and $p(-1)$ to the left.
- No exclusion rule (different from WASEP).
- The configuration space is $\mathcal{X}_N = \{0, 1, 2, \ldots\}^{\mathbb{T}_N} \equiv \mathbb{Z}_+^{\mathbb{T}_N}$.
- Its element is denoted by $\eta = \{\eta_x\}_{x \in \mathbb{T}_N}$, where η_x represents the number of particles at site x.



- Transition: $\eta^{x,x+e} \in \mathcal{X}_N$ for $x \in \mathbb{T}_N$, $e = \pm 1$, is defined from η such that $\eta_x \geq 1$ by

$$(\eta^{x,x+e})_z = \begin{cases} \eta_x - 1 & \text{when } z = x \\ \eta_{x+e} + 1 & \text{when } z = x+e \\ \eta_z & \text{otherwise,} \end{cases}$$

  for $z \in \mathbb{T}_N$. $\eta^{x,x\pm 1}$ describes the configuration after one particle at x jumps to $x+1$ or $x-1$.
- Generator of weakly asymmetric independent RWs:

$$Lf(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{e=\pm 1} p(e) g(\eta_x) \{f(\eta^{x,x+e}) - f(\eta)\}$$

  for functions f on $\mathcal{X}_N$.
- Indep RW case: $g(\eta_x) = \eta_x$; one-particle jump rate is proportional to the particle number, i.e., each particle move independently with the same jump rate 1.
- Jump probability, weak asymmetry: $p(\pm 1) \equiv p_N(\pm 1) = \frac{1}{2} \pm \frac{c}{N}$ (HDL, linear fluctuation), or $= \frac{1}{2} \pm \frac{c}{\sqrt{N}}$ (KPZ) gives the probability of where to jump.

Microscopic time evolution

- L generates the Markov process $\eta(t) = \{\eta_x(t)\}_{x \in \mathbb{T}_N}$ on $\mathcal{X}_N$. This describes the microscopic time evolution of weakly asymmetric independent random walks on $\mathbb{T}_N$.
- As we recall, each particle (in $\eta(t)$) has a bell, which rings according to the exponential holding time exp(1) with (inverse) mean $\lambda = 1$. Once a bell of a certain particle rings, this particle makes a jump to the right or left according to the probability $p_N(\pm 1)$. Then, the system refreshes and repeats the same procedure.

Scaling from micro to macro

- [Scaling: time N^2] Consider $\eta^N(t) = \{\eta_x^N(t)\}_{x\in\mathbb{T}_N}$, the Markov process on $\mathcal{X}_N$ generated by $L_N = N^2 L$.
- N^2 means the time change from micro to macro. Microscopically, RWs spend long time (as $N \to \infty$)
- [Scaling: mass $\frac{1}{N}$, space $\frac{1}{N}$] The macroscopically scaled empirical measure on macroscopic space $\mathbb{T}(=[0,1)$ with periodic boundary) associated with $\eta \in \mathcal{X}_N$ is defined by

$$\alpha^N(dv;\eta) = \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_x \delta_{\frac{x}{N}}(dv), \quad v \in \mathbb{T}$$

- x/v denote the microscopic/macroscopic spatial variables.
- We denote (recall scaling in time N^2 so that space-time diffusive scaling)

$$\alpha^N(t, dv) = \alpha^N(dv;\eta^N(t)), \quad t \geq 0.$$

- Define $\langle \alpha, \phi\rangle = \int_{\mathbb{T}} \phi d\alpha$ for test functions ϕ and measures α on $\mathbb{T}$.

2. Hydrodynamic limit

- Here, we take $p_N(\pm 1) = \frac{1}{2} \pm \frac{c}{N}$, $c \in \mathbb{R}$ for jump probability.
- In particular, the order of weak asymmetry is $O(\frac{1}{N})$.

Theorem 1 (Hydrodynamic limit, LLN)

The macroscopic empirical measure of $\eta^N(t)$

$$\alpha^N(t, dv) \underset{N\to\infty}{\longrightarrow} \rho(t, v)dv \quad \textit{in probability}$$

(by multiplying any smooth test function $\phi \in C^\infty(\mathbb{T})$*), if this holds at* $t = 0$*. The limit density* $\rho(t, v)$ *is a unique weak solution of the linear heat equation with drift* $-2c$*:*

$$\partial_t \rho = \frac{1}{2}\Delta\rho - 2c\nabla\rho, \quad v \in \mathbb{T},$$

with an initial value $\rho_0(x) = \rho(0, x)$*, where* $\Delta = \partial_v^2, \nabla = \partial_v$*.*

[Proof] Though $g(\eta_x) = \eta_x$, we keep g in the computation. By Dynkin's formula,

$$\langle \alpha^N(t), \phi \rangle = \langle \alpha^N(0), \phi \rangle + \int_0^t N^2 L\{\langle \alpha, \phi \rangle\}(\eta^N(s))ds + M_t^N(\phi),$$

where $M_t^N(\phi)$ is a martingale with quadratic variation given by

$$\langle M^N(\phi) \rangle_t = \int_0^t \Big(N^2 L\{\langle \alpha, \phi \rangle^2\} - 2N^2 \langle \alpha, \phi \rangle L\{\langle \alpha, \phi \rangle\} \Big)(\eta^N(s))ds$$

and, noting $\langle \alpha, \phi \rangle = \frac{1}{N} \sum_x \eta_x \phi(\frac{x}{N})$,

$$N^2 L\langle \alpha, \phi \rangle = N \sum_x L\eta_x \, \phi(\tfrac{x}{N}),$$

$$N^2 L\langle \alpha, \phi \rangle^2 = \sum_{x,y} L(\eta_x \eta_y) \, \phi(\tfrac{x}{N})\phi(\tfrac{y}{N}).$$

Let's first compute $L\eta_x$ and $L(\eta_x \eta_y)$.

We have the following formulas:

(1) $L\eta_x = p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_{x-1}) - (p_N(-1) + p_N(1))g(\eta_x)$

(2) $L\eta_x^2 = p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_{x-1}) + (p_N(-1) + p_N(1))g(\eta_x)$
$+2\eta_x L\eta_x$

(3) $L(\eta_x\eta_{x+1}) = -\big(p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_x)\big)$
$+\eta_{x+1}L\eta_x + \eta_x L\eta_{x+1}$

(4) $L(\eta_x\eta_y) = \eta_y L\eta_x + \eta_x L\eta_y \quad \text{if } |x-y| \geq 2$

[Proof] The proof is elementary, but it is useful to see it for the first time. For (1), recall $L\eta_x = \sum_{z,e=\pm1} p_N(e)g(\eta_z)\{(\eta_x)^{z,z+e} - \eta_x\}$ and observe

$$(\eta_x)^{z,z+e} - \eta_x = \begin{cases} -1 & \text{if } z = x, \\ 1 & \text{if } z+e = x. \end{cases}$$

For (2), observe

$$(\eta_x^2)^{z,z+e} - \eta_x^2 = \begin{cases} (\eta_x - 1)^2 - \eta_x^2 = -2\eta_x + 1 & \text{if } z = x, \\ (\eta_x + 1)^2 - \eta_x^2 = 2\eta_x + 1 & \text{if } z+e = x, \end{cases}$$

$\pm 2\eta_x$ is absorbed in $2\eta_x L\eta_x$ and the effect of "+1" survives as an extra term.

For (3), observe

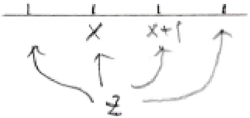
$$(\eta_x\eta_{x+1})^{z,z+1} - \eta_x\eta_{x+1} \qquad \text{(the case } e = 1)$$

$$= \begin{cases} (\eta_x + 1)\eta_{x+1} - \eta_x\eta_{x+1} = \eta_{x+1} & \text{if } z = x - 1, \\ (\eta_x - 1)(\eta_{x+1} + 1) - \eta_x\eta_{x+1} = \eta_x - \eta_{x+1} - 1 & \text{if } z = x, \\ \eta_x(\eta_{x+1} - 1) - \eta_x\eta_{x+1} = -\eta_x & \text{if } z = x + 1, \end{cases}$$

$$(\eta_x\eta_{x+1})^{z,z-1} - \eta_x\eta_{x+1} \qquad \text{(the case } e = -1)$$

$$= \begin{cases} (\eta_x - 1)\eta_{x+1} - \eta_x\eta_{x+1} = -\eta_{x+1} & \text{if } z = x, \\ (\eta_x + 1)(\eta_{x+1} - 1) - \eta_x\eta_{x+1} = -\eta_x + \eta_{x+1} - 1 & \text{if } z = x + 1, \\ \eta_x(\eta_{x+1} + 1) - \eta_x\eta_{x+1} = \eta_x & \text{if } z = x + 2, \end{cases}$$

We pick up only the effect of "-1" as an extra term and others are absorbed in $\eta_{x+1}L\eta_x + \eta_x L\eta_{x+1}$. The proof of (4) is immediate. □



We now come back to Dynkin's formula. By (1) and then shifting variables $x \mapsto x \pm 1$ and recalling $p_N(\pm 1) = \frac{1}{2} \pm \frac{c}{N}$,

$$\begin{aligned} N^2 L\langle \alpha, \phi \rangle &= N \sum_x \Big(p_N(-1) g(\eta_{x+1}) + p_N(1) g(\eta_{x-1}) \\ &\qquad - (p_N(-1) + p_N(1)) g(\eta_x) \Big) \phi(\tfrac{x}{N}) \\ &= N \sum_x g(\eta_x) \Big(p_N(-1)\phi(\tfrac{x-1}{N}) + p_N(1)\phi(\tfrac{x+1}{N}) - \phi(\tfrac{x}{N}) \Big) \\ &= N \sum_x g(\eta_x) \Big(\tfrac{1}{2} \big\{ \phi(\tfrac{x-1}{N}) + \phi(\tfrac{x+1}{N}) - 2\phi(\tfrac{x}{N}) \big\} \\ &\qquad + \tfrac{c}{N} \big\{ \phi(\tfrac{x+1}{N}) - \phi(\tfrac{x-1}{N}) \big\} \Big). \end{aligned}$$

Later, we will show $E[M_t(\phi)^2] \to 0$ as $N \to \infty$.

In particular, in case that $g(\eta_x) = \eta_x$, this is exactly written in α again (i.e. asymptotically closed in α) as

$$
\begin{aligned}
N^2 L\langle \alpha, \phi \rangle = \frac{1}{N} \sum_x \eta_x \Big(\frac{N^2}{2} \big\{ \phi(\tfrac{x-1}{N}) + \phi(\tfrac{x+1}{N}) - 2\phi(\tfrac{x}{N}) \big\} \\
+ cN \big\{ \phi(\tfrac{x+1}{N}) - \phi(\tfrac{x-1}{N}) \big\} \Big) \\
= \langle \alpha, \tfrac{1}{2}\Delta\phi + 2c\nabla\phi \rangle + O(\tfrac{1}{N}),
\end{aligned}
$$

by Taylor expansion for ϕ. At least, we need $\phi \in C^3(\mathbb{T})$. In the limit, we get weak form of the linear heat equation with drift $-2c$.

Finally let's show that $M_t^N(\phi)$ vanishes in the limit.

To show this, by (2), (3), (4), we see

$$
\begin{aligned}
&N^2 L\langle\alpha,\phi\rangle^2 - 2N^2\langle\alpha,\phi\rangle L\langle\alpha,\phi\rangle \\
&= \sum_x \Big(p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_{x-1}) + (p_N(-1)+p_N(1))g(\eta_x)\Big)\phi(\tfrac{x}{N})^2 \\
&\quad - \sum_x \Big(p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_x)\Big)\phi(\tfrac{x}{N})\phi(\tfrac{x+1}{N}) \\
&\quad - \sum_x \Big(p_N(-1)g(\eta_x) + p_N(1)g(\eta_{x-1})\Big)\phi(\tfrac{x-1}{N})\phi(\tfrac{x}{N}) \\
&\underset{(*)}{=} \sum_x \Big(p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_x)\Big)\Big(\phi(\tfrac{x}{N})^2 + \phi(\tfrac{x+1}{N})^2\Big) \\
&\quad - 2\sum_x \Big(p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_x)\Big)\phi(\tfrac{x}{N})\phi(\tfrac{x+1}{N}) \\
&= \sum_x \Big(p_N(-1)g(\eta_{x+1}) + p_N(1)g(\eta_x)\Big)\Big(\phi(\tfrac{x}{N}) - \phi(\tfrac{x+1}{N})\Big)^2 = O(N\cdot\tfrac{1}{N^2}) \to 0,
\end{aligned}
$$

at least if $g(\eta_x)$ behaves bounded under expectation. (*) follows by noting (1) the 3rd sum = the 2nd sum (by shifting $x \mapsto x+1$) and (2) for the 1st sum, we shift the 2nd and 3rd terms as $x \mapsto x+1$. This shows that the quadratic variation of $M_t^N(\phi)$ vanishes as $N \to \infty$. □

- Note that, in independent case, to show HDL, we don't use any information on the invariant measures of the system.
- As we saw in the proof, we have obtained asymptotically closed equation in $\alpha^N(t)$.

3. Invariant measures

- Poisson distribution on $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ with parameter $\alpha \geq 0$ is a probability measure
$$p \equiv p_\alpha = \{p_k = p_{\alpha,k}; k \in \mathbb{Z}_+\}$$
on $\mathbb{Z}_+$ given by
$$p_k = e^{-\alpha} \frac{\alpha^k}{k!}, \quad k \in \mathbb{Z}_+.$$
- Note the average $E^{p_\alpha}[k] := \sum_k k p_{\alpha,k} = \alpha$.
- We define $\nu_\alpha^N := p_\alpha^{\otimes \mathbb{T}_N}$ as a product of Poisson measure p_α on the configuration space $\mathcal{X}_N$.

Proposition 2 (Kipnis-Landim, Proposition 1.1, p.9)

For every $\alpha \geq 0$, ν_α^N is invariant under the time evolution of the weakly asymmetric independent random walks, that is,
$$\eta(0) \overset{law}{=} \nu_\alpha^N \Longrightarrow \eta(t) \overset{law}{=} \nu_\alpha^N \text{ for every } t \geq 0.$$

- This can be checked by showing the infinitesimal invariance $\int_{\mathcal{X}_N} Lf \, d\nu_\alpha^N = 0$ for every (bounded) function f on $\mathcal{X}_N$.
- In particular, invariant measure is not unique, but one parameter family of measures parametrized by average density α of particles.
- In symmetric case (i.e. $p(\pm 1) = \frac{1}{2}$), ν_N^α are reversible: $\int_{\mathcal{X}_N} f \, Lg \, d\nu_\alpha^N = \int_{\mathcal{X}_N} g \, Lf \, d\nu_\alpha^N$ holds.

4. Linear fluctuation

- Assume $\eta^N(0) \overset{law}{=} \nu_\alpha^N$ for some $\alpha > 0$, i.e., the system is in equilibrium.
- We consider the equilibrium fluctuation of independent random walks around its mean α taking $p_N(\pm 1) = \frac{1}{2} \pm \frac{c}{N}$ (same scaling as HDL):

$$Y^N(t, dv) := \frac{1}{\sqrt{N}} \sum_x (\eta_x^N(t) - \alpha) \delta_{\frac{x}{N}}(dv)$$
$$\fallingdotseq \sqrt{N}\big(\alpha^N(t, dv) - \alpha\, dv\big)$$

(regarding $\frac{1}{N} \sum_x \delta_{\frac{x}{N}}(dv) \fallingdotseq dv$).
- Note $\alpha^N(t, dv) \to \alpha dv$ in HDL (Theorem 1) when $\eta^N(0) \overset{law}{=} \nu_\alpha^N$.

In fact, we can easily analyze non-equilibrium fluctuation for independent RWs.

Theorem 3 (Equilibrium linear fluctuation, CLT)

$Y^N(t) \underset{N\to\infty}{\longrightarrow} Y(t)$ and the limit $Y(t) = Y(t,v)$ is a solution of the linear SPDE:

$$\partial_t Y = \frac{1}{2}\Delta Y - 2c\nabla Y + \sqrt{\alpha}\nabla \dot{W}(t,v), \quad v \in \mathbb{T},$$

where $\dot{W}(t,v)$ is the space-time Gaussian white noise with mean 0 and covariance structure

$$E[\dot{W}(t,v)\dot{W}(s,u)] = \delta(t-s)\delta(v-u).$$

[Proof] Note that $\langle Y^N(t), \phi\rangle = \sqrt{N}\big(\langle \alpha^N(t), \phi\rangle - \alpha\langle 1, \phi\rangle\big)$. Thus, recalling that Dynkin's formula showed

$$\langle \alpha^N(t), \phi\rangle = \langle \alpha^N(0), \phi\rangle + \int_0^t N^2 L\{\langle \alpha, \phi\rangle\}(\eta^N(s))ds + M_t^N(\phi),$$

we have

$$\langle Y^N(t), \phi\rangle = \langle Y^N(0), \phi\rangle + \int_0^t \sqrt{N} N^2 L\{\langle \alpha, \phi\rangle\}(\eta^N(s))ds + \sqrt{N} M_t^N(\phi).$$

However, by the computation we made above,

$$\begin{aligned}\sqrt{N} N^2 L\langle \alpha, \phi\rangle &= \sqrt{N}\langle \alpha, \tfrac{1}{2}\Delta\phi + 2c\nabla\phi\rangle + O(\tfrac{\sqrt{N}}{N}) \\ &= \langle Y^N, \tfrac{1}{2}\Delta\phi + 2c\nabla\phi\rangle + O(\tfrac{1}{\sqrt{N}}),\end{aligned}$$

since $\langle 1, \frac{1}{2}\Delta\phi + 2c\nabla\phi\rangle = 0$. This leads to the drift term of the limit SPDE.

To obtain $\sqrt{\alpha}\dot{W}(t,v)$ in the limit, we compute the limit of the quadratic variation of $\sqrt{N}M_t^N(\phi)$. Indeed, by the above computation and recalling $p(\pm 1) = \frac{1}{2} \pm \frac{c}{N}$, it is given by

$$(\sqrt{N})^2 \int_0^t \sum_x \Big(p_N(-1) g(\eta_{x+1}^N(s)) + p_N(1) g(\eta_x^N(s)) \Big) \Big(\phi(\tfrac{x}{N}) - \phi(\tfrac{x+1}{N}) \Big)^2 ds$$

$$= \int_0^t \frac{1}{2N} \sum_x \big(g(\eta_x^N(s)) + g(\eta_{x+1}^N(s)) \big) \Big\{ N\big(\phi(\tfrac{x+1}{N}) - \phi(\tfrac{x}{N}) \big) \Big\}^2 ds + O(\tfrac{1}{N}).$$

However, $g(\eta_x) = \eta_x$ and recall $\eta^N(s) \overset{law}{=} \nu_\alpha^N$ under equilibrium. Since ν_α^N is a product measure, LLN (for i.i.d. sequence) holds under this measure and we see, for every $x_0 \in \mathbb{T}_N$,

$$\frac{1}{2} \cdot \frac{1}{2\ell+1} \sum_{x:|x-x_0|\le \ell} \big(\eta_x^N(s) + \eta_{x+1}^N(s) \big) \underset{\ell\to\infty}{\longrightarrow} \alpha \quad \text{a.s.}$$

This is local ergodicity around x_0.

In non-equilibrium setting, this behaves as
$\cong \langle \alpha^N(s,\cdot), 1_{|v-\frac{x_0}{N}|\le \ell} \rangle \cdot \frac{N}{2\ell+1} \cong \rho(s, \frac{x_0}{N})$, which is the limit of the HDL.

In the above sum for $\langle \sqrt{N} M^N(\phi) \rangle_t$, $\left\{ N\big(\phi(\frac{x+1}{N}) - \phi(\frac{x}{N})\big) \right\}^2$ $\simeq \left\{ \nabla \phi(\frac{x}{N}) \right\}^2$, but it is not a constant but changes in x. However, ϕ is smooth so that it changes slowly in x. In other words, by taking $1 \ll \ell \ll N$ such that $\frac{N}{2\ell+1} \in \mathbb{N}$, one can rearrange the sum in x as

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N} = \frac{2\ell+1}{N} \sum_{x_0 \in \mathbb{T}_{\frac{N}{2\ell+1}}} \times \frac{1}{2\ell+1} \sum_{x : |x - x_0| \le \ell}$$

Then, for $x : |x - x_0| \le \ell$, $\nabla \phi(\frac{x}{N})^2$ can be replaced by $\nabla \phi(\frac{x_0}{N})^2$ with an error of $O(\frac{\ell}{N})$, which tends to 0. Thus, the local ergodicity shows that

$$\langle \sqrt{N} M^N(\phi) \rangle_t \to \alpha t \int_{\mathbb{T}} (\nabla \phi)^2 dv.$$

Accordingly, we have $\sqrt{N} M_t^N \to \sqrt{\alpha} \nabla W(t,v)$, where W is a time integral of $\dot{W}$. Indeed, by the covariance structure of W,

$$
\begin{aligned}
E[\langle \sqrt{\alpha} \nabla W(t,\cdot), \phi \rangle^2] &= \alpha E[\langle W(t,\cdot), \nabla \phi \rangle^2] \\
&= \alpha t \int_{\mathbb{T}^2} \nabla \phi(x) \nabla \phi(y) \delta(x-y) dxdy \\
&= \alpha t \|\nabla \phi\|^2_{L^2(\mathbb{T})}. \qquad \square
\end{aligned}
$$

- Note that the coefficient $\sqrt{\alpha}$ of the noise appears by the averaging effect (ergodic property=LLN) under the equilibrium measure ν_α.
- Invariant measure of the limit SPDE: Since the variance of Poisson distribution p_α is also α, by CLT, $\langle Y^N(0), \phi \rangle$ converges to Gaussian distribution $N(0, \alpha \|\phi\|^2_{L^2(\mathbb{T})})$ in law. In other words, $Y^N(0)$ converges in law to Y, which is $\sqrt{\alpha} \cdot$(spatial white noise on $\mathbb{T}$).

5. KPZ fluctuation

- In the independent particles case, KPZ limit yields only linear fluctuation.
- Similarly to the WASEP, let us consider larger weak asymmetry than HDL/Linear fluctuation:
  $p_N(\pm 1) = \frac{1}{2} \pm \frac{c}{\sqrt{N}}$.
- Then, in the SPDE obtained in Theorem 3, we have diverging drift $-2c\sqrt{N}\nabla Y$ (with error $\frac{1}{2}\frac{2c\sqrt{N}}{N}\langle Y^N, \phi''\rangle$ by Taylor expansion of $\phi(\frac{x+1}{N}) - \phi(\frac{x-1}{N})$).
- To cancel this, we need to introduce moving frame (with macroscopic speed $2c\sqrt{N}$) and define

$$Y^N(t, dv) := \frac{1}{\sqrt{N}} \sum_x (\eta_x^N(t) - \alpha)\delta_{\frac{x}{N} - 2c\sqrt{N}t}(dv)$$

- We assume $\eta^N(0) \overset{law}{=} \nu_\alpha^N$ also here.

- Then, the moving frame gives the effect $2c\sqrt{N}\nabla Y$ and this exactly cancels out the diverging drift. (One can see this through multiplying the test function ϕ.)
- There is no nonlinear effect and we can show that $Y^N(t) \underset{N\to\infty}{\longrightarrow} Y(t)$ and the limit $Y(t) = Y(t,v)$ is a solution of the linear SPDE:

$$\partial_t Y = \frac{1}{2}\Delta Y + \sqrt{\alpha}\nabla \dot{W}(t,v).$$

5.2 Single species zero-range process on $\mathbb{T}_N$

1. Model

- We consider particle system on $\mathbb{T}_N$. Particles interact with each other among those staying at the same site.
- Therefore, it is called zero-range process.
- In Lecture No 5-B, we will discuss several types of particles to derive coupled KPZ equation. Here, we consider one type of particles.
- The configuration space $\mathcal{X}_N = \{0, 1, 2, \dots\}^{\mathbb{T}_N}$ is the same as before.
- Generator of weakly asymmetric zero-range process has the same form as that of independent random walks:

$$Lf(\eta) = \sum_{x \in \mathbb{T}_N} \sum_{e=\pm 1} p(e) g(\eta_x) \{ f(\eta^{x,x+e}) - f(\eta) \}.$$

- $\eta^{x,x+e}$ (= configuration after one particle at x jumps to $x + e$), $p(e)$ ($=\frac{1}{2} \pm \frac{c}{N}$ or $\frac{1}{2} \pm \frac{c}{\sqrt{N}}$) are the same as before.

- Jump rate $g(\eta_x)$ was linear function (i.e. $g(k) = k$) for independent RWs, but here we consider nonlinear function. This gives the interaction among particles at the same site.
- In general, g satisfies $g(k) > 0$ for $k \geq 1$ and $g(0) = 0$.
- The one-particle jump rate among k particles is $\frac{g(k)}{k}$. If $g(k) = k$, it is 1 so that particles move independently.
- Under some assumptions on g, one can construct the processes $\eta(t)$ and $\eta^N(t)$ generated by L and $N^2 L$, respectively.
- The proof of HDL was straightforward (i.e. we got automatically a closed PDE in the limit) for independent case. But, in the interacting case, we need to clarify the structure of all invariant (or reversible) measures.

2. Invariant measures

- Let $p_\varphi = \{p_\varphi(k)\}_{k\in\mathbb{Z}_+}, \varphi \geq 0$ be the generalized Poisson distribution on $\mathbb{Z}_+$ defined by

$$p_\varphi(k) := \frac{1}{Z_\varphi}\frac{\varphi^k}{g(k)!}, \quad Z_\varphi = \sum_{k=0}^{\infty}\frac{\varphi^k}{g(k)!}.$$

  Here, $g(k)! = g(1)\cdots g(k)$ for $k \geq 1$ and $g(0)! = 1$.
- Recall that, for $g(k) = k$, p_φ is the Poisson distribution with mean φ.
- (cf. Kipnis-Landim, Prop 3.2, p.29) Under some assumptions on g, product measures $\bar\nu_\varphi \equiv \bar\nu_\varphi^N := p_\varphi^{\otimes\mathbb{T}_N}$ on $\mathcal{X}_N$ for $\varphi \in [0,\varphi^*)$ are invariant measures of zero-range process $\eta(t)$, where $\varphi^* := \liminf_{k\to\infty} g(k)$.
- For this, show the infinitesimal invariance $\int_{\mathcal{X}_N} Lf\, d\bar\nu_\varphi = 0$ for functions f on $\mathcal{X}_N$.
- Conversely, any invariant measure being translation-invariant is characterized as a convex combination of $\{\bar\nu_\varphi\}$, cf. [KL, p.40] on $\mathcal{X} = \mathbb{Z}_+^{\mathbb{Z}}$ (i.e. on infinite region $\mathbb{Z}$).

Change of the parameter $\varphi \mapsto \rho$ (density)

- We denote, for $\rho \geq 0$, that
$$\nu_\rho \equiv \nu_\rho^N := \bar{\nu}_{\varphi(\rho)} \left(\equiv \bar{\nu}^N_{\varphi(\rho)}\right)$$
by changing the parameter so that the mean of the marginal p_φ is ρ.
- In fact, $\varphi = \varphi(\rho)$ is determined by the relation
$$\rho = \varphi(\log Z_\varphi)' \left(= \frac{1}{Z_\varphi} \sum_{k=0}^{\infty} k \frac{\varphi^k}{g(k)!} =: \langle k \rangle_{p_\varphi}\right).$$
- Also, note that
$$\varphi = \langle g(k) \rangle_{p_\varphi} \left(:= \frac{1}{Z_\varphi} \sum_{k=1}^{\infty} \frac{\varphi^k}{g(k-1)!}\right).$$
- Moreover, differentiating $\rho = \varphi(\log Z_\varphi)'$ in φ, we see that
$$\varphi'(\rho) = \frac{\varphi(\rho)}{E_{\nu_\rho}[(\eta_0 - \rho)^2]} > 0.$$
- In particular, $\varphi = \varphi(\rho)$ is a strictly increasing function.
- Recall $\varphi(\rho) = \rho$ in the independent case.

3. Hydrodynamic limit

- As before, we consider the macroscopically scaled empirical measure on $\mathbb{T} = [0, 1)$ associated with $\eta \in \mathcal{X}_N$ defined by

$$\alpha^N(dv; \eta) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_x \delta_{\frac{x}{N}}(dv), \quad v \in \mathbb{T}.$$

- We also denote

$$\alpha^N(t, dv) = \alpha^N(dv; \eta^N(t)), \quad t \geq 0.$$

- For the microscopic system, we have introduced the scalings N^2 in time, $\frac{1}{N}$ in space and $\frac{1}{N}$ in mass.

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cf. F, Hydrodynamic limit for exclusion processes, Comm. Math. Stat., 2018. Based on a course at 北京交通大学

- We take $p_N(\pm 1) = \frac{1}{2} \pm \frac{c}{N}$.

Theorem 4 (Hydrodynamic limit)

The macroscopic empirical measure of $\eta^N(t)$

$$\alpha^N(t, dv) \underset{N\to\infty}{\longrightarrow} \rho(t, v)dv \quad \textit{in probability},$$

if this holds at $t = 0$. The limit density $\rho(t, v)$ is a unique weak solution of the nonlinear heat (diffusion) equation:

$$\partial_t \rho = \frac{1}{2}\Delta\varphi(\rho) - 2c\nabla\varphi(\rho), \quad v \in \mathbb{T},$$

with an initial value $\rho_0(x) = \rho(0, x)$.

- Here, from the above observation, the function $\varphi = \varphi(\rho)$ is defined by the ensemble average of g:

$$\varphi(\rho) \equiv \langle g\rangle(\rho) = E^{\nu_\rho}[g(\eta_0)],$$

  where E^{ν_ρ} is the expectation with respect to ν_ρ.
- Recall $\varphi'(\rho) > 0$ and it is known that $\varphi \in C^\infty(\mathbb{R}_+)$.
- Nonlinearity arises from the interaction.

For the proof of Theorem 4:

- Dynkin's formula and other computations hold as in the independent case: For each test function $\phi \in C^\infty(\mathbb{T})$,

$$\langle \alpha^N(t), \phi \rangle = \langle \alpha^N(0), \phi \rangle + \int_0^t N^2 L\{\langle \alpha, \phi \rangle\}(\eta^N(s))ds + M_t^N(\phi),$$

$$N^2 L\langle \alpha, \phi \rangle = \frac{1}{N} \sum_x g(\eta_x) \Big(\frac{N^2}{2} \big\{ \phi(\tfrac{x-1}{N}) + \phi(\tfrac{x+1}{N}) - 2\phi(\tfrac{x}{N}) \big\} + cN \big\{ \phi(\tfrac{x+1}{N}) - \phi(\tfrac{x-1}{N}) \big\} \Big),$$

- $\lim_{N\to\infty} E[M_t^N(\phi)^2] = 0$ holds.
- In independent case, $g(\eta_x) = \eta_x$ so that this was able to be written in $\alpha^N(s)$ again.
- However, in interacting case, this is not possible.

Heuristic derivation of the limit nonlinear PDE

- Local Equilibrium Ansatz: For each macroscopic time t and position $v \in \mathbb{T}$, the microscopic system $\eta^N(t)$ around (t, v) reaches one of the equilibrium states $\nu_{\rho(t,v)}$ with some $\rho(t, v) \geq 0$, i.e.,

  $\{\eta_x^N(t); |x - Nv| \leq \ell\} \overset{\text{law}}{\cong} \nu_{\rho(t,v)}(\{\eta_x; |x| \leq \ell\})$ holds.
- This looks plausible, since microscopic system $\eta^N(t)$ may reach equilibrium after spending long time. But, the equilibrium states are not unique so that it may depend on (t, v).
- Actually, it is more convenient to consider ν_ρ as a measure on $\mathcal{X} = \mathbb{Z}_+^{\mathbb{Z}}$ (i.e. on infinite region), since we let $N \to \infty$. (Indeed, this doesn't matter in our setting, since ν_ρ are product measures.)

- From local equilibrium Ansatz and by ergodicity under the long time average (Dynkin's formula involves time-integral, and it is actually large time-integral for microscopic system $\eta(s)$) or under the large spatial average, $g(\eta_x^N(s))$ would be replaced by its ensemble average $\langle g\rangle(\rho(s,v))$ for $\frac{x}{N}\sim v$ (from $|x-Nv|\le \ell$).
- Also $\eta_x^N(t)$ in $\langle \alpha^N(t),\phi\rangle$ would be replaced by $\langle \eta_0\rangle(\rho(t,v)) = \rho(t,v)$ for $\frac{x}{N}\sim v$.
- Thus, from Dynkin's formula, we would obtain

$$\langle \rho(t,\cdot),\phi\rangle = \langle \rho(0,\cdot),\phi\rangle + \int_0^t \langle \varphi(\rho(s,\cdot)), \tfrac{1}{2}\Delta\phi + 2c\nabla\phi\rangle ds$$

  in the limit.
- This is the limit equation in Theorem 4. □

Problem: How to make this rigorous?

- Method: Entropy method, relative entropy method

Entropy and Entropy production
(Originally due to Guo-Papanicolaou-Varadhan)

- Fix $\alpha > 0$ and take $\nu_\alpha \equiv \nu_\alpha^N$ as a reference measure.
- $d\mu^N(t) \equiv f_t^N(\eta) d\nu_\alpha^N(\eta) :=$ distribution of $\eta^N(t)$ on $\mathcal{X}_N$.
- Then, the density $f_t^N = f_t^N(\eta)$ satisfies Kolmogorov's forward equation:
$$\partial_t f_t^N = N^2 L^* f_t^N,$$
where L^* is the dual of the generator L of the process $\eta(t)$ with respect to ν_α^N.
- Indeed, L^* has a similar form to L, but with $p(e)$ replaced by $p(-e)$.
- Relative entropy: For two probability measures μ, ν on $\mathcal{X}_N$ such that $\mu \prec \nu$, *set*
$$H(\mu|\nu) := \int_{\mathcal{X}_N} \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} d\nu$$

Kipnis-Landim p.79∼, see also F. Comm. Math. Stat., 2018.

- Dirichlet form associated with the symmetric part of L defined by $L^{\text{sym}} := \frac{1}{2}(L + L^*)$ (i.e. $p(e) = \frac{1}{2}$):

$$\mathcal{D}_N(f,h) := -\int f\, L^{\text{sym}} h\, d\nu_\alpha$$
$$= \frac{1}{4}\sum_{x,e}\int g(\eta_x)\big(f(\eta^{x,x+e}) - f(\eta)\big)\big(h(\eta^{x,x+e}) - h(\eta)\big)\, d\nu_\alpha$$

- Entropy production (Information):

$$I_N(f) := \mathcal{D}_N(\sqrt{f}, \sqrt{f})$$

Proposition 5

Let us denote $H(\mu) := H(\mu|\nu_\alpha^N)$. Then, we have

$$\partial_t H(\mu^N(t)) \le -2N^2 I_N(f_t^N).$$

In particular, for space-time average of μ_t^N, we have

$$H\Big(\tfrac{1}{TN}\int_0^T \textstyle\sum_{x\in\mathbb{T}_N} \mu^N(t)\circ\tau_x^{-1} dt\Big) \le H(\mu^N(0)) \le CN,$$
$$I_N\Big(\tfrac{1}{TN}\int_0^T \textstyle\sum_{x\in\mathbb{T}_N} \mu^N(t)\circ\tau_x^{-1} dt\Big) \le \tfrac{1}{2N^2} H(\mu^N(0)) \le \tfrac{C}{N},$$

where $\tau_x, x \in \mathbb{T}_N$ denotes the spatial shift: $(\tau_x\eta)_z := \eta_{z+x}$.

[Proof] Last two inequalities follow by convexity of H and I_N, and noting $H(\mu^N(t)) \geq 0$. For the first estimate, recalling Kolmogorov's forward equation,

$$\begin{aligned}\partial_t H(\mu^N(t)) &= \partial_t \int f_t^N \log f_t^N d\nu_\alpha^N \\ &= \int \big(N^2 L^* f_t^N \cdot \log f_t^N + \partial_t f_t^N\big) d\nu_\alpha^N \\ &= N^2 \int f_t^N \cdot L \log f_t^N d\nu_\alpha^N.\end{aligned}$$

Since $a \log \frac{b}{a} \leq 2\sqrt{a}\,(\sqrt{b} - \sqrt{a})$ for $a, b > 0$, we have

$$\begin{aligned}\int f_t^N \cdot L \log f_t^N d\nu_\alpha^N &= \sum_{x,e} p(e) \int g(\eta_x) f_t^N(\eta) \log \tfrac{f_t^N(\eta^{x,x+e})}{f_t^N(\eta)} d\nu_\alpha^N \\ &\leq 2 \sum_{x,e} p(e) \int g(\eta_x) \sqrt{f_t^N(\eta)} \Big(\sqrt{f_t^N(\eta^{x,x+e})} - \sqrt{f_t^N(\eta)}\Big) d\nu_\alpha^N \\ &= 2 \int \sqrt{f_t^N(\eta)} L \sqrt{f_t^N(\eta)} d\nu_\alpha^N = 2 \int \sqrt{f_t^N(\eta)} L^{\mathrm{sym}} \sqrt{f_t^N(\eta)} d\nu_\alpha^N \\ &= -2 I_N(f_t^N).\end{aligned}$$

This shows the first estimate. □

- For a bounded local function $F = F(\eta)$ on $\mathcal{X}_N$ (depending only on finitely many $\{\eta_x\}$ independently of N) and $\ell \in \mathbb{N}$, define the sample average of F in the region of size 2ℓ and center x by

$$F_x^\ell(\eta) := \tfrac{1}{2\ell+1} \sum_{y:|y-x|\le\ell} \tau_y F(\eta)$$

where $\tau_y F(\eta) := F(\tau_y \eta)$ and $\tau_y \eta \in \mathcal{X}_N$ is defined by $(\tau_y \eta)_x := \eta_{x+y}$.
- Recall $\langle F\rangle(\alpha) := E^{\nu_\alpha}[F]$: ensemble average of F.
- Density of the space-time average of $\mu^N(t)$ with respect to ν_α:

$$\tilde{f}^N := \frac{1}{NT} \sum_{x\in\mathbb{T}_N} \int_0^T \tau_x f_t \, dt.$$

- By Proposition 5, we have

$$H(\tilde{f}^N) \equiv H(\tilde{f}^N d\nu_\alpha) \le CN, \quad I_N(\tilde{f}^N) \le \tfrac{C}{N}.$$

- The heuristic argument to replace $g(\eta_x)$ and η_x by their ensemble averages with mean $\rho(t, v)$ can be made rigorous by the next theorem.
- Instead of $\rho(t, v)$, we take sample average (density of particles) $\eta_0^{\varepsilon N}$ in a region of macroscopic size $\varepsilon > 0$.

Theorem 6 (Local ergodicity =Replacement lemma)

For every $\delta > 0$, we have

$$\overline{\lim_{\varepsilon \downarrow 0}} \, \overline{\lim_{N \to \infty}} \, P^{\tilde{f}^N d\nu_\alpha}\Big(|F_0^{\varepsilon N} - \langle F \rangle (\eta_0^{\varepsilon N})| > \delta \Big) = 0.$$

- We take $F(\eta) = g(\eta_0)$ or η_0 which are unbounded. To introduce cut-off, we apply the entropy bound "$H_N(\tilde{f}^N) \leq CN$" and the entropy inequality:
$$E^\mu[X] \leq \log E^\nu[e^X] + H(\mu|\nu).$$
- The proof of Theorem 6 is divided into two parts: one block estimate and two blocks estimate.

Theorem 7 (One block estimate)

$$\varlimsup_{\ell\to\infty} \varlimsup_{N\to\infty} \sup_f{}^{*} E^{f d\nu_\alpha}\left[|F_0^\ell - \langle F\rangle(\eta_0^\ell)|\right] = 0,$$

where $\sup_f^*$ *is taken over translation-invariant* f *(i.e.* $\tau_x f = f$*) such that* $I_N(f) \le \frac{C}{N}, H(f) \le CN$*.*

Theorem 8 (Two blocks estimate)

$$\varlimsup_{\ell\to\infty} \varlimsup_{\varepsilon\downarrow 0} \varlimsup_{N\to\infty} \sup_f{}^{*} \sup_{|y|\le\varepsilon N} E^{f d\nu_\alpha}\left[|\eta_y^\ell - \eta_0^{N\varepsilon}|\right] = 0,$$

where $\sup_f^*$ *is the same as above.*

- Theorems 7 and 8 imply Theorem 6.
- Indeed, $F_0^{\varepsilon N}$ in Thm 6 can be replaced by F_0^ℓ by rearranging the sum in $F_0^{\varepsilon N}$.
- $\eta_0^{\varepsilon N}$ in Thm 6 can be replaced by η_0^ℓ by Thm 8.
- After these replacement, we can apply Thm 7.

Rough idea of the proof of Theorem 7 (One block estimate)

- Let $\ell \in \mathbb{N}$ be fixed and consider the entropy production $I_\ell(f)$ on the domain of size ℓ. (We take pieces of Dirichlet form in this domain $\to$ see below.)
- Then, since it is proportional to the volume, we see $I_\ell(f) \le \frac{\ell}{N} I_N(f)$.
- Showing the tightness of ℓ-marginal distribution of $\{\tilde{f}^N d\nu_\alpha\}$ and recalling $I_N(\tilde{f}^N) \le \frac{C}{N}$, we see that every limit f, restricted on the configuration space $\mathcal{X}_\ell$ of this domain, satisfies $I_\ell(f) = 0$.
- This implies

$$\frac{1}{4} \sum_{2 \le x \le \ell-1, e=\pm 1} \int g(\eta_x) \{\sqrt{f(\eta^{x,x+e})} - \sqrt{f(\eta)}\}^2 d\nu_\alpha = 0.$$

- Therefore, we have $f(\eta^{x,x+e}) = f(\eta)$, which means that f is constant on the configurations in $\mathcal{X}_\ell$ with a fixed total number of particles.

- Thus the proof is reduced to show

$$\overline{\lim_{\ell\to\infty}} \sup_{\frac{j}{\ell}\le C} E^{\nu_j^\ell}[|F_0^\ell - \langle F\rangle(\eta_0^\ell)|] = 0,$$

where $\nu_j^\ell := \nu_\alpha\big|_{\mathcal{X}_\ell, \sum_{1\le x\le \ell}\eta_x = j}$ is the conditional distribution of ν_α on the space with j particles in the domain of size ℓ. ν_j^ℓ are called canonical ensembles. We can introduce cut-off C in density.
- By the equivalence of ensembles (shown by local CLT), the limits of ν_j^ℓ as $\ell \to \infty$ are superpositions of (grand canonical) ensembles $\{\nu_\alpha\}$.
- From this and also noting $\eta_0^\ell \to \alpha$ under ν_α, the proof is further reduced to show

$$\overline{\lim_{\ell\to\infty}} \sup_{\alpha\le C} E^{\nu_\alpha}[|F_0^\ell - \langle F\rangle(\alpha)|] = 0.$$

- But, this is nothing but the LLN for *i.i.d.* sequence (since F has finite-support).
- This completes the proof of one block estimate. □

- For the proof of Theorem 8 (Two blocks estimate), we move particles from one block to the other. The cost can be estimated by the entropy production $I_N(f)$.

Additional tasks to complete the proof of Theorem 4:

- We show the tightness of $\{\alpha^N(t,dv)\}_N$.
- We need the uniqueness of the weak solution of the limit nonlinear diffusion equation, roughly shown as follows.
- Set $u(t,v) := \nabla^{-1}\rho(t,v)$ and then u satisfies
$$\partial_t u = \tfrac{1}{2}\nabla\big(\varphi(\nabla u)\big) - 2c\varphi(\nabla u).$$
- Assume there are two solutions u_1, u_2 with the same initial value. Set $\bar{u} = u_1 - u_2$ and compute
$$\begin{aligned}
\partial_t \|\bar{u}\|_{L^2}^2 &= 2(\bar{u}, \partial_t \bar{u})_{L^2} \\
&= (\bar{u}, \nabla\varphi(\nabla u_1) - \nabla\varphi(\nabla u_2))_{L^2} - 4c(\bar{u}, \varphi(\nabla u_1) - \varphi(\nabla u_2))_{L^2} \\
&= -(\nabla\bar{u}, \varphi(\nabla u_1) - \varphi(\nabla u_2))_{L^2} - 4c(\bar{u}, \varphi(\nabla u_1) - \varphi(\nabla u_2))_{L^2} \\
&\le -c\|\nabla\bar{u}\|_{L^2}^2 + \tfrac{C}{\varepsilon}\|\bar{u}\|_{L^2}^2 + \varepsilon\|\nabla\bar{u}\|_{L^2}^2,
\end{aligned}$$
by noting $0 < c \le \varphi' \le C < \infty$. Taking $0 < \varepsilon < c$, this shows the uniqueness by Gronwall's lemma.
- This is called H^{-1}-method, since $\|u\|_{L^2} = \|\rho\|_{H^{-1}}$. □

4. Linear fluctuation, KPZ fluctuation

- Assume $\eta^N(0) \stackrel{law}{=} \nu_\alpha^N$ for some $\alpha > 0$, i.e., the system is in equilibrium.
- We consider the equilibrium fluctuation of zero-range process around its mean α taking $p_N(\pm 1) = \frac{1}{2} \pm \frac{c}{N}$ (same scaling as HDL):

$$Y^N(t, dv) := \frac{1}{\sqrt{N}} \sum_x (\eta_x^N(t) - \alpha) \delta_{\frac{x}{N}}(dv).$$

Theorem 9 (Equilibrium linear fluctuation, CLT)

$Y^N(t) \underset{N \to \infty}{\longrightarrow} Y(t)$ and the limit $Y(t) = Y(t, v)$ is a solution of the linear SPDE:

$$\partial_t Y = \frac{1}{2}\varphi'(\alpha)\Delta Y - 2c\varphi'(\alpha)\nabla Y + \sqrt{\varphi(\alpha)}\nabla \dot{W}(t, v),$$

where $\dot{W}(t, v)$ is the space-time Gaussian white noise.

[Proof] As in the proof of HDL, we apply Dynkin's formula and the limit of the martingale term can be handled similarly to the independent RWs except that we get $\sqrt{\varphi(\alpha)}\nabla W(t,v)$ as the limit of $g(\eta_x)$ instead of η_x:

$$\frac{1}{2}\cdot\frac{1}{2\ell+1}\sum_{|x|\le\ell}\left(g(\eta_x^N(s))+g(\eta_{x+1}^N(s))\right)\underset{\ell\to\infty}{\longrightarrow}\varphi(\alpha)\quad \text{a.s.}$$

by ergodicity under ν_α^N. To study the limit of the drift term, we need the following theorem. □

(1st order) Boltzmann-Gibbs principle in equilibrium
(=combination of local average and Taylor expansion)

Theorem 10

For a local function $F(\eta)$ and $G\in C(\mathbb{T})$, we have

$$\lim_{N\to\infty}E^{\nu_\alpha}\Big[\Big(\int_0^t ds\frac{1}{\sqrt{N}}\sum_{x\in\mathbb{T}_N}G(\tfrac{x}{N})\tau_x V_F(\eta(s))\Big)^2\Big]=0$$

where $V_F(\eta)=F(\eta)-\langle F\rangle(\alpha)-\langle F\rangle'(\alpha)(\eta_0-\alpha)$.

- Noting that $\langle g\rangle(\alpha)=\varphi(\alpha)$, by Theorem 10 (BG principle), we can show Theorem 9.
- Nonequilibrium fluctuation: Chang-H.T. Yau CMP **145** 1992, Jara-Menezes arXiv:1810.09526

KPZ fluctuation

- We introduce a different (larger) scaling $p_N(\pm 1)=\frac{1}{2}\pm\frac{c}{\sqrt{N}}$ for jump probability.
- We introduce moving frame to cancel $\langle F\rangle'(\alpha)$ in this scaling.
- Then, under the scaling, the next term in Taylor expansion becomes $O(1)$ and we roughly have

$$F(\eta)\sim\frac{1}{2}\langle F\rangle''(\alpha)(\eta_0^\ell-\alpha)^2,$$

  where recall $\eta_0^\ell=\frac{1}{2\ell+1}\sum_{|x|\le\ell}\eta_x$.
- This is called (2nd order) Boltzmann-Gibbs principle and will be discussed in Lecture No 5-B.

Summary of this lecture.

1. Independent random walks: We made explicit computations based on Dynkin's formula and the formula of quadratic variation. This is the linear theory.
2. Single species zero-range process on $\mathbb{T}_N$: Mostly we discussed the hydrodynamic scaling limit due to the method of entropy and entropy production and derived nonlinear diffusion equation in the limit.
3. We stated 1st and 2nd order Boltzmann-Gibbs principle in equilibrium to study fluctuation limits.