KPZ limit for interacting particle systems —Coupled KPZ equation by paracontrolled calculus—

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Lecture No 4

- F-Hoshino, JFA, **273**, 2017
- F, in "Stochastic Dynamics Out of Equilibrium", Springer 2019

Plan of the course (10 lectures)

1 Introduction
2 Supplementary materials

    Brownian motion, Space-time Gaussian white noise, (Additive) linear SPDEs, (Finite-dimensional) SDEs, Martingale problem, Invariant/reversible measures for SDEs, Martingales

3 Invariant measures of KPZ equation (F-Quastel, 2015)
4 Coupled KPZ equation by paracontrolled calculus (F-Hoshino, 2017)
5 Coupled KPZ equation from interacting particle systems (Bernardin-F-Sethuraman, 2020+)
    5.1 Independent particle systems
    5.2 Single species zero-range process
    5.3 n-species zero-range process
    5.4 Hydrodynamic limit, Linear fluctuation
    5.5 KPZ limit=Nonlinear fluctuation

Plan of this lecture

Coupled KPZ equation by paracontrolled calculus

1. Multi-component coupled KPZ equation
   - Motivation: nonlinear fluctuating hydrodynamics
   - Trilinear condition (T)
2. Two approximating equations, local well-posedness, invariant measure
   - Convergence results due to paracontrolled calculus
   - Difference of two limits
   - Main theorems (Theorems 1 and 2)
3. Global existence for a.s.-initial values under invariant (stationary) measure
4. Ertaș-Kardar's example
   - not satisfying (T) but having invariant measure
5. Role of trilinear condition (T)
   - Invariant measure, renormalizations (for 4th order terms)
6. Extensions of Ertaș-Kardar's example
7. Proof of main theorems (Theorems 1 and 2)
8. Remarks for the case with diffusion constant σ

1. Multi-component coupled KPZ equation

- In Lectures No 1 and No 3, we studied scalar-valued KPZ equation (1) and the renormalized KPZ equation (2):

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \tag{1}$$

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x). \tag{2}$$

- In this lecture, we consider on $\mathbb{T} = [0,1)$.
- We used the Cole-Hopf transformation and Cole-Hopf solution $h(t,x) := \log Z(t,x)$, where Z is the solution of multiplicative linear stochastic heat equation.
- In this lecture, we consider a system of KPZ equations.
- For such equation, one cannot apply Cole-Hopf transformation in general.
- The method we use in the present part works also for scalar-valued equations (1) and (2).

- Our equation in this lecture has the following form.
- $\mathbb{R}^d$-valued KPZ eq for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ on $\mathbb{T}$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \dot{W}^\beta \qquad (\sigma,\Gamma)_{KPZ}$$

- We use Einstein's convention. i.e., the sums $\sum_{\beta,\gamma}, \sum_\beta$ are omitted.
- $\dot{W}(t,x) = (\dot{W}^\alpha(t,x))_{\alpha=1}^d \big(\equiv \dot{W}(t,x) \big)$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise with covariance structure:

$$E[\dot{W}^\alpha(t,x)\dot{W}^\beta(s,y)] = \delta^{\alpha\beta}\delta(x-y)\delta(t-s).$$

- $\delta^{\alpha,\beta}$ is Kronecker's δ. This means that $(\dot{W}^\alpha(t,x))_{\alpha=1}^d$ are independent $\mathbb{R}$-valued space-time Gaussian white noises.

- Coupled KPZ equation is ill-posed, since noise is irregular and conflicts with nonlinear term. ($h^\alpha \in C_{t,x}^{\frac{1}{4}-,\frac{1}{2}-}$ a.s. when $\Gamma = 0$)
- We need to introduce approximations with smooth noises and renormalization for $(\sigma, \Gamma)_{KPZ}$. Indeed, one can introduce two types of approximations: one is simple, the other is suitable to find invariant measures (Lecture No 3: $d = 1$, F-Quastel 2015).

- The constants $\Gamma^{\alpha}_{\beta\gamma}$ satisfy bilinear condition

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} \;\text{ for all }\; \alpha, \beta, \gamma, \qquad \textbf{(B)}$$

  and (we sometimes assume) trilinear condition

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} = \Gamma^{\gamma}_{\beta\alpha} \;\text{ for all }\; \alpha, \beta, \gamma. \qquad \textbf{(T)}$$

  (cf. Ferrari-Sasamoto-Spohn 2013, Kupiainen-Marcozz 2017)
- $\sigma = (\sigma^{\alpha}_{\beta})$ is an invertible matrix.
- Similar SPDE appears to discuss motion of loops on a manifold, cf. Funaki 1992, Bruned-Gabriel-Hairer--Zambotti 2019; Dirichlet form approach, Röckner--Wu-Zhu-Zhu 2020, Chen-Wu-Zhu-Zhu 2020+.

$$\partial_t h^{\alpha} = \tfrac{1}{2}\partial_x^2 h^{\alpha} + \tfrac{1}{2}\Gamma^{\alpha}_{\beta\gamma}\partial_x h^{\beta}\partial_x h^{\gamma} + \sigma^{\alpha}_{\beta}\dot{W}^{\beta} \qquad (\sigma, \Gamma)_{KPZ}$$

- Since σ is invertible, $\hat{h} = \sigma^{-1} h$ transforms $(\sigma, \Gamma)_{KPZ}$ to $(I, \hat{\Gamma} = \sigma \circ \Gamma)_{KPZ}$, where

$$(\sigma \circ \Gamma)^{\alpha}_{\beta\gamma} := (\sigma^{-1})^{\alpha}_{\alpha'} \Gamma^{\alpha'}_{\beta'\gamma'} \sigma^{\beta'}_{\beta} \sigma^{\gamma'}_{\gamma}.$$

  Thus, the KPZ equation with $\sigma = I$ is considered as a canonical form.
- The operation (coordinate change) $\Gamma \mapsto \sigma \circ \Gamma$ keeps the bilinearity, but not the trilinearity.
- We should say (σ, Γ) satisfies trilinear condition, iff $\hat{\Gamma} := \sigma \circ \Gamma$ satisfies (T).
- Thus, in the following, we assume $\sigma = I$. In Section 8, we remark how the results are modified for general σ.

Motivation to study the coupled KPZ equation

- Coupled KPZ equation appears in the study of nonlinear fluctuating hydrodynamics for a system with d-conserved quantities by taking 2nd order terms into account. The problem goes back to Landau.
  cf. Spohn-Ferrari-Sasamoto-Stoltz JSP 2013, '14, '15.
- If some of $\Gamma^{\alpha}_{\beta\gamma}$ are degenerate, then the solution involves different (anomalous) scalings such as Diffusive=OU, KPZ, $\frac{5}{3}$, $\frac{3}{2}$-Lévy scalings (they look different behavior in time-correlation functions).

Coupled KPZ equation with additional drifts

- Consider the equation with additional drift $c^\alpha \in \mathbb{R}$ for each component assuming $\sigma = I$:

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + c^\alpha \partial_x h^\alpha + \dot{W}^\alpha.$$

- This equation can be easily reduced to the case $c^\alpha = 0$.
- Indeed, if (h^α) is a solution of this equation, $\tilde{h}^\alpha(t,x) := h^\alpha(t, x - c^\alpha t)$ satisfies the same equation with $c^\alpha = 0$ and a new noise $\widetilde{\dot{W}^\alpha}(t,x) := \dot{W}^\alpha(t, x - c^\alpha t)$, which is also an $\mathbb{R}^d$-valued space-time Gaussian white noise.

Why trilinear condition (T) plays a role: one reason

- For simplicity, consider $(\sigma, \Gamma)_{KPZ}$ without noise and at Burgers level for $u^\alpha := \partial_x h^\alpha$:

$$\partial_t u^\alpha = \tfrac{1}{2}\partial_x^2 u^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x(u^\beta u^\gamma).$$

- If (T) is satisfied, the usual method of energy estimate works:

$$\begin{aligned}
\partial_t \|u(t)\|^2_{L^2(\mathbb{T})} &= \partial_t \sum_\alpha \int_{\mathbb{T}} (u^\alpha)^2 dx \\
&= 2\sum_\alpha (u^\alpha, \partial_t u^\alpha)_{L^2} \\
&= \sum_\alpha (u^\alpha, \partial_x^2 u^\alpha)_{L^2} + \sum_{\alpha,\beta,\gamma} \Gamma^\alpha_{\beta\gamma}\big(u^\alpha, \partial_x(u^\beta u^\gamma)\big)_{L^2} \\
&= -\|\partial_x u\|^2_{L^2(\mathbb{T})} \le 0,
\end{aligned}$$

  by integration by parts.
- The term with Γ vanishes by interchanging the role of α, β, γ if Γ satisfies (T) ($\rightarrow$ see next page).

- Indeed,

$$\sum_{\alpha,\beta,\gamma} \Gamma^{\alpha}_{\beta\gamma} \big(u^{\alpha}, \partial_x (u^{\beta} u^{\gamma})\big)_{L^2} = \sum_{\alpha,\beta,\gamma} \Gamma^{\alpha}_{\beta\gamma} \int_{\mathbb{T}} u^{\alpha} \cdot \partial_x (u^{\beta} u^{\gamma}) dx$$
$$= - \sum_{\alpha,\beta,\gamma} \Gamma^{\alpha}_{\beta\gamma} \int_{\mathbb{T}} \partial_x u^{\alpha} \cdot u^{\beta} u^{\gamma} dx$$
$$\underset{(\mathrm{T})}{=} 0,$$

  since (LHS) $= 2 \times (-\text{RHS})$.
- This is similar to Navier-Stokes equation (or Euler equation).

2. Two approximating equations, local well-posedness, invariant measure

- We will extend the results for scalar-valued equation in Lecture No 3 (i.e. $d = 1$) to coupled equation.
- We replace the noise by smeared one.
  As in Lecture No 3, take a symmetric convolution kernel:
  $$\eta^\varepsilon(x) := \frac{1}{\varepsilon}\eta(\tfrac{x}{\varepsilon}) \underset{\varepsilon\downarrow 0}{\longrightarrow} \delta_0.$$
- Approximating equation-1 (simple): For $h^\alpha = h^{\varepsilon,\alpha}$,

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon \delta^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \dot{W}^\alpha * \eta^\varepsilon, \tag{3}$$

  where $c^\varepsilon = \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})} - 1 \,(= O(\frac{1}{\varepsilon}))$ and $B^{\varepsilon,\beta\gamma}$ $(= O(\log\frac{1}{\varepsilon})$ in general) is another renormalization factor.
- The renormalization $B^{\varepsilon,\beta\gamma}$ was unnecessary in the scalar-valued case, and also in coupled case under (T).

- Approx. equation-2 (suitable to find invariant measure):
  For $\tilde{h}^{\alpha} = \tilde{h}^{\varepsilon,\alpha}$

$$\partial_t \tilde{h}^{\alpha} = \frac{1}{2}\partial_x^2 \tilde{h}^{\alpha} + \frac{1}{2}\Gamma^{\alpha}_{\beta\gamma}(\partial_x \tilde{h}^{\beta}\partial_x \tilde{h}^{\gamma} - c^{\varepsilon}\delta^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^{\varepsilon} + \dot{W}^{\alpha} * \eta^{\varepsilon}, \tag{4}$$

  with a renormalization factor $\tilde{B}^{\varepsilon,\beta\gamma}$, where $\eta_2^{\varepsilon} = \eta^{\varepsilon} * \eta^{\varepsilon}$.
- The idea behind (4) is the fluctuation-dissipation relation.
- Renormalization factor $c^{\varepsilon} \equiv$  $= O(\frac{1}{\varepsilon})$ is from 2nd order terms in the expansion, while Renormalization factors $B^{\varepsilon,\beta\gamma}$ and $\tilde{B}^{\varepsilon,\beta\gamma} = O(log\frac{1}{\varepsilon})$ are from 4th order terms involving $C^{\varepsilon} =$ , $D^{\varepsilon} =$  (see → Section 7).
- For the solution of (4) (with $\tilde{B} = 0$), F (Yor volume, 2015) showed (on $\mathbb{R}$), under the trilinear condition (T), the infinitesimal invariance of the distribution of $B * \eta^{\varepsilon}(x)$, where B is the $\mathbb{R}^d$-valued two-sided Brownian motion (with $x \in \mathbb{R}$) (see → Thm 2-(2)).

- Our goal is to study the limits of the solutions of Approx-Eq-1 (3) and Approx-Eq-2 (4) as $\varepsilon \downarrow 0$.
- As we saw, when $d = 1$ and $\Gamma = \sigma = 1$, the solution of (3) with $B^\varepsilon = 0$ converges as $\varepsilon \downarrow 0$ to the Cole-Hopf solution h_{CH} of the KPZ equation, while the solution of (4) with $\tilde{B}^\varepsilon = 0$ converges to $h_{CH} + \frac{1}{24}t$.
- Note that log-renormalization factors do not appear, when $d = 1$.
- The method of F-Quastel is based on the Cole-Hopf transform, which is not available for the coupled equation with multi-components in general.
- Instead, we use the paracontrolled calculus due to Gubinelli-Imkeller-Perkowski 2015.
- In particular, we study the difference between these two limits.

Summary of results of F-Hoshino 2017

- Convergence of h^ε and $\tilde{h}^\varepsilon$ and Local well-posedness of coupled KPZ eq $(\sigma, \Gamma)_{KPZ}$ by applying paracontrolled calculus due to Gubinelli-Imkeller-Perkowski 2015. (Cole-Hopf doesn't work for coupled eq. in general. In 1D, we used it and showed Boltzmann-Gibbs principle, FQ 2015.)
- Approx-Eq-2 fits to identify invariant measure under (T).
- Global solvability for a.s.-initial data under an invariant measure under (T) (similar to Da Prato-Debussche).
- Combine this with strong Feller property (i.e. continuity of probability in initial value, Hairer-Mattingly 2016).
- Global well-posedness (existence, uniqueness) under (T) Ergodicity and uniqueness of invariant measure.
- A priori estimates for Approx-Eq-1 (3) under (T).

Convergence of h^ε and $\tilde h^\varepsilon$ and Local well-posedness of coupled KPZ eq $(\sigma,\Gamma)_{KPZ}$ (we take $\sigma = I$): $\mathcal{C}^\kappa = (\mathcal{B}^\kappa_{\infty,\infty}(\mathbb{T}))^d$, $\kappa \in \mathbb{R}$ denotes $\mathbb{R}^d$-valued (Hölder-)Besov space on $\mathbb{T}$ (see → Sect 7).

Theorem 1

(1) Assume $h_0 \in \mathcal{C}^{0+} := \cup_{\delta>0}\mathcal{C}^\delta$, then a unique solution h^ε of (3) exists up to some $T^\varepsilon \in (0,\infty]$ and $\bar T = \liminf_{\varepsilon\downarrow 0} T^\varepsilon > 0$ holds. With a proper choice of $B^{\varepsilon,\beta\gamma}$, h^ε converges in prob. to some h in $C([0,T],\mathcal{C}^{\frac12-\delta})$ for every $\delta > 0$ and $0 < T \le \bar T$.

(2) Similar result holds for the solution $\tilde h^\varepsilon$ of (4) with some limit $\tilde h$. Under proper choices of $B^{\varepsilon,\beta\gamma}$ and $\tilde B^{\varepsilon,\beta\gamma}$, we can actually make $h = \tilde h$.

$$\partial_t h^\alpha = \tfrac12 \partial_x^2 h^\alpha + \tfrac12 \Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon \delta^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \dot W^\alpha * \eta^\varepsilon \qquad (3)$$

$$\partial_t \tilde h^\alpha = \tfrac12 \partial_x^2 \tilde h^\alpha + \tfrac12 \Gamma^\alpha_{\beta\gamma}(\partial_x \tilde h^\beta \partial_x \tilde h^\gamma - c^\varepsilon \delta^{\beta\gamma} - \tilde B^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \dot W^\alpha * \eta^\varepsilon \qquad (4)$$

$\mathcal{C}^\kappa$ is defined in Fourier analytic way. In particular, for $\kappa \in (0,\infty)\setminus\mathbb{N}$, $\mathcal{C}^\kappa = \{u \in C_b^k; \partial_x^k u \text{ is } (\kappa-k)\text{-Hölder continuous}\}$, where $k = [\kappa]$ is the integer part of κ. Note that for $\kappa \in \mathbb{N}$, $C_b^\kappa \subsetneq \mathcal{C}^\kappa$.

Results under (T): Unnecessity of Log-Renormalizations, Invariant measure = Wiener measure, difference of two limits

Theorem 2

Assume the trilinear condition (**T**).
(1) Then, $B^{\varepsilon,\beta\gamma}, \tilde{B}^{\varepsilon,\beta\gamma} = O(1)$ *so that the solutions of* (3) *with* $B = 0$ *and* (4) *with* $\tilde{B} = 0$ *converge. In the limit, we have*

$$\tilde{h}^\alpha(t,x) = h^\alpha(t,x) + c^\alpha t, \quad 1 \le \alpha \le d,$$

where

$$c^\alpha = \tfrac{1}{24} \sum_{\gamma_1,\gamma_2} \Gamma^\alpha_{\beta\gamma} \Gamma^\beta_{\gamma_1\gamma_2} \Gamma^\gamma_{\gamma_1\gamma_2}.$$

(2) Moreover, the distribution of $(\partial_x B)_{x\in\mathbb{T}}$ *(*$B =$ *periodic BM) is invariant under the tilt process* $u = \partial_x h$ *(or periodic Wiener measure on the quotient space* $\mathcal{C}^{\frac{1}{2}-\delta}/\sim$ *where* $h \sim h + c$*).*

Proofs of Theorems 1 and 2 → Section 7

3. Global existence for a.s.-initial values under stationary meas

- We assume (T) and initial value $h(0)$ is given by $h(0,0) = 0$ and $u(0) := \partial_x h(0) \underset{\text{law}}{=} (\partial_x B)_{x \in \mathbb{T}}$ (i.e., stationary). Then, similarly to Da Prato-Debussche (2002, for 2D stochastic Navier-Stokes equation; Galerkin approximation), $u = \partial_x h$ satisfies

Theorem 3

For every $T > 0, p \geq 1, \delta > 0$, we have

$$E\left[\sup_{t \in [0,T]} \|u(t; u_0)\|^p_{-\frac{1}{2}-\delta}\right] < \infty$$

In particular, $T_{survival}(u(0)) = \infty$ for a.a.-$u(0)$.

- Global existence for all given $u(0)$: In the scalar-valued case, this is immediate, since the limit is Cole-Hopf solution. Hairer-Mattingly 2016 proved this for coupled equation by showing the strong Feller property on $\mathcal{C}^{\alpha-1}, \alpha \in (0, \frac{1}{2})$.

- For Approx-Eq-1 (3), under (T), we have

$$\sum_{\alpha,\beta,\gamma} \Gamma^{\alpha}_{\beta\gamma} \int_{\mathbb{T}} u^{\alpha} \partial_x (u^{\beta} u^{\gamma}) dx = 0.$$

  This shows a priori estimate and global well-posedness for (3) at least if $h(0) \in H^1(\mathbb{T})$.
- Therefore, Theorem 1-(1) holds globally in time if $h(0) \in H^1(\mathbb{T})$.
- We expect Theorem 1-(2) also holds globally in time (by showing strong Feller property for (4)).

4. Ertaș and Kardar's example

Unnecessity of Log-Renormalizations and $^\exists$Invariant measure without (T)

- Example (Ertaș and Kardar 1992: $d = 2$)

$$\begin{aligned} \partial_t h^1 &= \tfrac{1}{2}\partial_x^2 h^1 + \tfrac{1}{2}\{\lambda_1(\partial_x h^1)^2 + \lambda_2(\partial_x h^2)^2\} + \dot{W}^1, \\ \partial_t h^2 &= \tfrac{1}{2}\partial_x^2 h^2 + \lambda_1 \partial_x h^1 \partial_x h^2 + \dot{W}^2 \end{aligned} \qquad \text{(EK)}$$

  Γ satisfies (T) only when $\lambda_1 = \lambda_2$ ($\Gamma^1_{11} = \lambda_1, \Gamma^1_{22} = \lambda_2, \Gamma^2_{12} = \lambda_1$).

- However, under the transform $\hat{h} = sh$ with $s = \begin{pmatrix} \lambda_1 & (\lambda_1\lambda_2)^{1/2} \\ \lambda_1 & -(\lambda_1\lambda_2)^{1/2} \end{pmatrix}$, (EK) is transformed into

$$\partial_t \hat{h}^\alpha = \tfrac{1}{2}\partial_x^2 \hat{h}^\alpha + \tfrac{1}{2}(\partial_x \hat{h}^\alpha)^2 + s^\alpha_\beta \dot{W}^\beta. \qquad (\mathrm{EK}_T)$$

  i.e. nonlinear term is decoupled, but noise is coupled.

- Namely, $\hat{\Gamma} = s \circ \Gamma$ in (EK_T) is given by $\hat{\Gamma}^\alpha_{\alpha\alpha} = 1, = 0$ otherwise, so that $\hat{\Gamma}$ satisfies (T). But, (EK) is the canonical form (with $\sigma = I$) and not (EK_T).

- (EK) doesn't satisfy (T).
- However, since nonlinear term is decoupled in (EK$_T$), the Cole-Hopf transform $Z^\alpha = \exp \hat{h}^\alpha$ works for each component so that global well-posedness follows.
- In particular, log-renormalization factors are unnecessary.
- Invariant measure exists whose marginals are Wiener measures, but the joint distribution of such invariant measure is unclear (presumably non-Gaussian).
- Indeed, because of the tightness of marginals, Cesàro mean $\mu_T = \frac{1}{T}\int_0^T \mu(t)dt$ of the distributions $\mu(t)$ of $\partial_x \hat{h}(t)$ having an initial distribution $\otimes_\alpha \mu_\alpha$ is tight on the space $\mathcal{C}^{-\frac{1}{2}-}/\sim$, so that the limit (along subsequence) of μ_T as $T \to \infty$ is an invariant measure.
  (Recall $h \sim \tilde{h}$ if $h = \tilde{h} + c$) (cf. Liggett, 1985, p.11)

5. Role of trilinear condition (T)

Reason of unnecessity of log-renormalization factors

- Formulas of Renormalization factors $B^{\varepsilon,\beta\gamma}$, $\tilde{B}^{\varepsilon,\beta\gamma}$ (→see Section 7):

$$B^{\varepsilon,\beta\gamma} = F^{\beta\gamma} C^\varepsilon + 2G^{\beta\gamma} D^\varepsilon, \quad \tilde{B}^{\varepsilon,\beta\gamma} = F^{\beta\gamma} \tilde{C}^\varepsilon + 2G^{\beta\gamma} \tilde{D}^\varepsilon,$$

where

$$F^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma}_{\gamma_1\gamma_2}, \quad G^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma_1}_{\gamma\gamma_2},$$

$$C^\varepsilon + 2D^\varepsilon = -\frac{1}{12} + O(\varepsilon), \quad \tilde{C}^\varepsilon + 2\tilde{D}^\varepsilon = 0,$$

($C^\varepsilon = c_\epsilon$, $D^\varepsilon = c_\epsilon$ from Wiener expansion)

- Trilinear condition (T) $\Longleftrightarrow$ "$F = G$" $\Longleftrightarrow$ $B, \tilde{B} = O(1)$
- But, for unnecessity of log-renormalization factors, what we really need is: "$\Gamma B, \Gamma\tilde{B} = O(1)$". This holds if $\Gamma F = \Gamma G$.

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon \delta^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \dot{W}^\alpha * \eta^\varepsilon \quad (3)$$

$$\partial_t \tilde{h}^\alpha = \frac{1}{2}\partial_x^2 \tilde{h}^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}^\beta \partial_x \tilde{h}^\gamma - c^\varepsilon \delta^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \dot{W}^\alpha * \eta^\varepsilon \quad (4)$$

- "$\Gamma F = \Gamma G$" holds iff Γ satisfies the condition

$$\Gamma^{\alpha}_{\beta\gamma}\Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma}_{\gamma_1\gamma_2} = \Gamma^{\alpha}_{\beta\gamma}\Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma_1}_{\gamma\gamma_2}, \quad {}^{\forall}\alpha$$

- This holds under (T) and also for Ertaş-Kardar's example.
- We can summarize as

$$\begin{aligned} (T) &\Longleftrightarrow \text{“}F = G\text{”} \\ &\Longrightarrow \text{“}\Gamma F = \Gamma G\text{”} \\ &\Longleftrightarrow \text{Unnecessity of log-renormalization factors} \end{aligned}$$

Infinitesimal invariance (to explain the role of (T) heuristically)

- $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}$: (pre) generator of coupled KPZ eq ($\sigma = I$).
- $\mathcal{L}_0$ is the generator of OU (Ornstein-Uhlenbeck)-part, while $\mathcal{A}$ is that of nonlinear part (we ignore renormalization factors):

$$\mathcal{L}_0\Phi = \frac{1}{2}\sum_{\alpha}\left\{\int_{\mathbb{T}} D^2_{h^\alpha(x)}\Phi\, dx + \int_{\mathbb{T}} \ddot{h}^\alpha(x) D_{h^\alpha(x)}\Phi\, dx\right\}$$

$$\mathcal{A}\Phi = \frac{1}{2}\sum_{\alpha,\beta,\gamma} \Gamma^\alpha_{\beta\gamma}\int_{\mathbb{T}} \dot{h}^\beta(x)\dot{h}^\gamma(x) D_{h^\alpha(x)}\Phi\, dx,$$

  where D, D^2 denote 1st and 2nd Fréchet derivatives, and $\dot{h}^\beta(x) := \partial_x h^\beta(x)$, $\ddot{h}^\alpha(x) := \partial_x^2 h^\alpha(x)$.
- In Lecture No 2, we wrote down the generator of finite-dimensional SDE by applying Itô's formula.
- SPDE is an infinite-dimensional version of SDE with infinite-dimensional BM $W(t,x)$ (recall it was constructed by a formal Fourier series). This generates the infinite-dimensional Laplacian $\frac{1}{2}\sum_\alpha \int_{\mathbb{T}} D^2_{h^\alpha(x)} \cdot\, dx$.

- Since h is not differentiable, the argument is heuristic.
- The infinitesimal invariance $(ST)_{\mathcal{L}}$ for ν
  $\underset{\text{def}}{\Longleftrightarrow}$ "$\int \mathcal{L}\Phi d\nu = 0, {}^{\forall} \Phi$"
- If the invariant measure ν is Gaussian, $(ST)_{\mathcal{L}_0}$ is the condition for 2nd order Wiener chaos of Φ, while $(ST)_{\mathcal{A}}$ is that for 3rd order Wiener chaos of Φ. Therefore, the condition $(ST)_{\mathcal{L}}$ is separated into two conditions:

$$(ST)_{\mathcal{L}} \Longleftrightarrow (ST)_{\mathcal{L}_0} + (ST)_{\mathcal{A}}$$

- $\mathcal{L}_0$ is (well-known) OU-operator and $(ST)_{\mathcal{L}_0}$ determines $\nu =$ Wiener measure.

Trilinear condition (T) $\Longleftrightarrow$ Wiener meas ν satisfies $(ST)_{\mathcal{A}}$

- We have the integration-by-parts formula for $\nu =$ Wiener measure (actually we need to discuss at ε-level, since h is not differentiable at $\varepsilon = 0$):

$$\int \mathcal{A}\Phi d\nu = -\frac{1}{2}\Gamma^{\alpha}_{\beta\gamma} c^{\beta\gamma}_{\alpha},$$

where

$$c^{\beta\gamma}_{\alpha} \equiv c^{\beta\gamma}_{\alpha}(\Phi) := E^{\nu}\left[\Phi \int_{\mathbb{T}} \dot{h}^{\beta}(x)\dot{h}^{\gamma}(x)\ddot{h}^{\alpha}(x)dx\right].$$

- Indeed, heuristically,
  $\nu \propto e^{-\frac{1}{2}|\dot{h}|^2} dh$ and $D_{h^{\alpha}(x)} e^{-\frac{1}{2}|\dot{h}|^2} = \ddot{h}^{\alpha}(x) e^{-\frac{1}{2}|\dot{h}|^2}$.
- c has the following properties:
  (1) (bilinearity) $c^{\beta\gamma}_{\alpha} = c^{\gamma\beta}_{\alpha}$
  (2) (integration by parts on $\mathbb{T}$) $c^{\beta\gamma}_{\alpha} + c^{\gamma\alpha}_{\beta} + c^{\alpha\beta}_{\gamma} = 0$
- In particular, $c^{\alpha\alpha}_{\alpha} = 0, {}^{\forall}\alpha$. When $d = 1$, this implies $(ST)_{\mathcal{A}}$: $\int \mathcal{A}\Phi d\nu = 0$ for ${}^{\forall}\Phi$.

- If Γ satisfies (T), by (2) for $c_\alpha^{\beta\gamma}$

$$\Gamma^\alpha_{\beta\gamma} c_\alpha^{\beta\gamma} = \frac{1}{3}\Gamma^\alpha_{\beta\gamma}(c_\alpha^{\beta\gamma} + c_\beta^{\gamma\alpha} + c_\gamma^{\alpha\beta}) = 0$$

  Therefore, (T) implies $(ST)_\mathcal{A}$.
- Conversely, $(ST)_\mathcal{A}$ implies (T). In fact, by (2) for $c_\alpha^{\beta\gamma}$

$$\begin{aligned} 0 \underset{(ST)_\mathcal{A}}{=} -2\int \mathcal{A}\Phi d\nu &= \Gamma^\alpha_{\beta\gamma} c_\alpha^{\beta\gamma} \\ &= \sum_{\alpha\neq\beta}(\Gamma^\alpha_{\beta\beta} - \Gamma^\beta_{\alpha\beta})c_\alpha^{\beta\beta} + 2\sum_{\alpha>\beta>\gamma}(\Gamma^\alpha_{\beta\gamma} - \Gamma^\gamma_{\alpha\beta})c_\alpha^{\beta\gamma} \\ &\quad + 2\sum_{\beta>\alpha>\gamma}(\Gamma^\alpha_{\beta\gamma} - \Gamma^\gamma_{\alpha\beta})c_\alpha^{\beta\gamma} \end{aligned}$$

  and $c_\alpha^{\beta\beta}, c_\alpha^{\beta\gamma}(\alpha > \beta > \gamma, \beta > \alpha > \gamma)$ move freely.

- Ertaș-Kardar's example does not satisfy (T), but has an invariant measure. This should be "non-separating class" (i.e. $(ST)_{\mathcal{L}} \Longleftrightarrow (ST)_{\mathcal{L}_0} + (ST)_{\mathcal{A}}$ does not hold) and the invariant measure is presumably non-Gaussian (but has Gaussian marginal).

6. Extensions of Ertaș-Kardar's example

We give extensions to d-component system.

Extension-1: nonlinear term decoupling to scalar-KPZ eq's (but noise term is correlated)

- If Γ has the form

$$\Gamma^{\alpha}_{\beta\gamma} = \sum_{\alpha'} (s^{-1})^{\alpha}_{\alpha'} s^{\alpha'}_{\beta} s^{\alpha'}_{\gamma},$$

  with invertible matrix s, the nonlinear term of the coupled KPZ equation is decoupled for $\hat{h}^{\alpha} = s^{\alpha}_{\beta} h^{\beta}$

$$\partial_t \hat{h}^{\alpha} = \tfrac{1}{2}\partial_x^2 \hat{h}^{\alpha} + \tfrac{1}{2}(\partial_x \hat{h}^{\alpha})^2 + s^{\alpha}_{\beta}\sigma^{\beta}_{\gamma}\dot{W}^{\gamma}. \qquad (\mathrm{EK})_{\mathrm{ext}}$$

- The above Γ may not satisfy the trilinear condition.

- However, since nonlinear term is decoupled in $(\mathrm{EK})_{\mathrm{ext}}$, the Cole-Hopf transform $Z^\alpha = \exp \hat{h}^\alpha$ works for each component so that global well-posedness (global existence of h in time) follows.
- Moreover, Log-renormalization factors are unnecessary.
- Invariant measure exists whose marginals are Wiener measures (with diffusion coefficients), but the joint distribution of such invariant measure is unclear.

Extension-2: nonlinear term decoupling to coupled KPZ eq's satisfying (T) (but noise term is correlated)

- Consider KPZ ($\sigma = I, \Gamma$).
- This has an invariant measure if $^\exists s \in GL(d)$, $^\exists$decomposition $\Delta = \cup_{i=1}^k I_i$ (disjoint) of $\{1, \dots, d\}$ such that
  - $s \circ \Gamma$ is decoupled under Δ,
    i.e., $(s \circ \Gamma)^\alpha_{\beta\gamma} = 0$ if $\{\alpha, \beta, \gamma\} \not\subset I_i$ for $^\forall i$
  - $(\sigma_i, s \circ \Gamma|_{I_i})$ are trilinear i.e., $\sigma_i \in GL(|I_i|)$
    and $\sigma_i \circ (s \circ \Gamma|_{I_i})$ satisfy (T),

  where $\sigma_i = \sqrt{(\sum_{\gamma=1}^d s^\alpha_\gamma s^\beta_\gamma)_{\alpha,\beta \in I_i}}$ and $\Gamma|_{I_i} = (\Gamma^\alpha_{\beta\gamma})|_{\alpha,\beta,\gamma \in I_i}$.
- Γ does not satisfy (T) in general.

One can prove infinitesimal invariance for subclasses of Φ.
(e.g., reflection-inv or shift-inv for each component)

Conjecture: For every Γ, invariant measure exists.

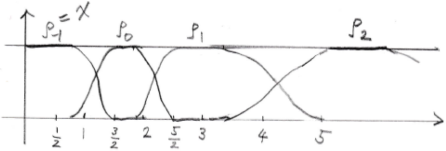
7. Proof of Theorems 1 and 2

Besov space and paraproducts

First we quickly introduce Besov space and paraproducts due to Fourier analysis. Basic reference is
Gubinelli-Imkeller-Perkowski, Forum Math., Pi, **3**, 2015.

Dyadic partition of unity

- χ, ρ: symmetric functions on $\mathbb{R}$ such that
  - $\operatorname{supp} \chi \subset [-\frac{4}{3}, \frac{4}{3}]$
  - $\operatorname{supp} \rho \subset [-\frac{8}{3}, -\frac{3}{4}] \cup [\frac{3}{4}, \frac{8}{3}]$
  - $\sum_{j=-1}^{\infty} \rho_j(z) = 1$,
    where $\rho_{-1}(z) := \chi(z)$, $\rho_j(z) = \rho(\frac{z}{2^j}), j \geq 0$
  - $\operatorname{supp} \rho_i \cap \operatorname{supp} \rho_j = \emptyset$ if $|i - j| \geq 2$



Littlewood-Paley blocks

- $\mathcal{F}$: Fourier transform for $u \in \mathcal{S}'(\mathbb{R})$
- $\Delta_j u := \mathcal{F}^{-1}(\rho_j \mathcal{F} u), \quad j \geq -1$
- Note $u = \sum_{j=-1}^{\infty} \Delta_j u$ for any $u \in \mathcal{S}'(\mathbb{R})$.

Besov space   scalar-valued case (i.e. $d = 1$), $\kappa \in \mathbb{R}$

- $\mathcal{B}^{\kappa}_{\infty,\infty}(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}); \|u\|_{\mathcal{B}^{\kappa}_{\infty,\infty}} < \infty\}$, where
  $$\|u\|_{\mathcal{B}^{\kappa}_{\infty,\infty}} := \sup_{j \geq -1} 2^{j\kappa} \|\Delta_j u\|_{L^\infty(\mathbb{R})}.$$
- $\mathcal{B}^{\kappa}_{\infty,\infty}(\mathbb{T})$ is a class of $u \in \mathcal{B}^{\kappa}_{\infty,\infty}(\mathbb{R})$ which are periodic with period 1 (or sometimes 2π)
- We denote $C^\kappa := \mathcal{B}^{\kappa}_{\infty,\infty}(\mathbb{T})$.
- In particular, for $\kappa \in (0, \infty) \setminus \mathbb{N}$,
  $$C^\kappa = \{u \in C^k_b; \partial^k_x u \text{ is } (\kappa - k)\text{-Hölder continuous}\},$$
  where $k = [\kappa]$ is the integer part of κ.
- Note that for $\kappa \in \mathbb{N}$, $C^\kappa_b \subsetneq C^\kappa$.
- Recall $\mathcal{C}^\kappa = (C^\kappa(\mathbb{T}))^d$ denotes $\mathbb{R}^d$-valued Besov space.

Bony's paraproducts scalar-valued case

- For two distributions $f, g \in \mathcal{S}'(\mathbb{T})$
  - $f \prec g := \sum_{i,j=-1: i \le j-2}^{\infty} \Delta_i f \Delta_j g$: paraproduct
  - $f \circ g := \sum_{i,j=-1: |i-j| \le 1}^{\infty} \Delta_i f \Delta_j g$: resonant term
- Littlewood-Paley decomposition of product fg:

$$fg = f \prec g + f \circ g + g \prec f.$$

- (Bony's estimates)
  - $a \lesssim b$ means $a \le^{\exists} C\, b$
  - For $\alpha > 0$ and $\beta \in \mathbb{R}$, $\|u \prec v\|_{C^\beta} \lesssim \|u\|_{L^\infty} \|v\|_{C^\beta}$.
  - For $\alpha \neq 0$ and $\beta \in \mathbb{R}$, $\|u \prec v\|_{C^{(\alpha \wedge 0)+\beta}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta}$.
  - For $\alpha + \beta > 0$, $\|u \circ v\|_{C^{\alpha+\beta}} \lesssim \|u\|_{C^\alpha} \|v\|_{C^\beta}$.
- Mollifier estimates (how mollifier improves regularity, convergence as $\varepsilon \downarrow 0$), Schauder estimates (how parabolic operator improves regularity), commutator estimates (commutator makes sense, even if each term has no meaning)

Driving terms $\mathbb{H}$, local-in-time solvability and continuity in $\mathbb{H}$

- We think of the noise as the leading term and the nonlinear term as its perturbation by putting (small parameter) $a > 0$ in front of the nonlinear term, though we eventually take $a = 1$.

$$\mathcal{L}h^\alpha = \frac{a}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \dot{W}^\alpha,$$

where $\mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2$.

- We expand the solution h of the coupled KPZ eq $(I, \Gamma)_{KPZ}$ in a: $h^\alpha = \sum_{k=0}^\infty a^k h_k^\alpha$. Then, we have

$$\sum_{k=0}^\infty a^k \mathcal{L}h_k^\alpha = \dot{W}^\alpha + \frac{a}{2}\sum_{k_1,k_2=0}^\infty a^{k_1+k_2}\Gamma^\alpha_{\beta\gamma}\partial_x h_{k_1}^\beta \partial_x h_{k_2}^\gamma.$$

- Comparing the terms of order $a^0, a^1, a^2, a^3, \dots$ in both sides and noting the bilinearity condition (B), we obtain the followings:

$$
\begin{aligned}
\mathcal{L}h_0^\alpha &= \dot{W}^\alpha, \\
\mathcal{L}h_1^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h_0^\beta \partial_x h_0^\gamma, \\
\mathcal{L}h_2^\alpha &= \Gamma^\alpha_{\beta\gamma}\partial_x h_0^\beta \partial_x h_1^\gamma, \\
\mathcal{L}h_3^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h_1^\beta \partial_x h_1^\gamma + \Gamma^\alpha_{\beta\gamma}\partial_x h_0^\beta \partial_x h_2^\gamma, \\
&\dots
\end{aligned}
$$

- $h_0^\alpha \in C^{\frac{1}{2}-}$ is linear OU (Ornstein-Uhlenbeck)-process and well-defined:

$$
\begin{aligned}
h_0^\alpha(t,x) = &\int_0^t \int_{\mathbb{T}} p(t-s,x,y)dW^\alpha(s,y)dy \\
&+ \int_{\mathbb{T}} h_0^\alpha(0,y)p(t,x,y)dy
\end{aligned}
$$

where p is the heat kernel on $\mathbb{T}$.

- To define h_1^α, we need to define the product $\partial_x h_0^\beta \partial_x h_0^\gamma$ (product of two generalized functions), but this is ill-defined.
- Indeed, h_0^α is a 1st order Wiener functional (chaos) of $\dot{W}$, so that $\partial_x h_0^\beta \partial_x h_0^\gamma$ is considered as a sum of (2nd+0th) order Wiener chaos of $\dot{W}$.
- To define h_1^α, similarly as we did in Lecture No 3, we take only 2nd order part and cut the diverging 0th order part.
- This procedure corresponds to the renormalization ($\rightarrow$ see below).
- Assume $h_1^\alpha \in C^{1-}$ (and $\in \mathcal{H}_2$) is defined in the above sense (note $-\frac{1}{2} - \frac{1}{2} + 2 = 1$, $+2$ is by Schauder effect).
- $h_2^\alpha \in C^{\frac{3}{2}-}$ (and $\in \mathcal{H}_3 \oplus \mathcal{H}_1$) (note $-\frac{1}{2} + 0 + 2 = \frac{3}{2}$).

- We denote $h_0^\alpha, h_1^\alpha, h_2^\alpha$ with stationary initial values by $H_0^\alpha, H_1^\alpha, H_2^\alpha$ and call driving terms.
- After defining $H_0^\alpha, H_1^\alpha, H_2^\alpha$ in the above way, the KPZ equation for $h^\alpha = H_0^\alpha + H_1^\alpha + H_2^\alpha + h_{\geq 3}^\alpha$ can be rewritten as
$$\mathcal{L} h_{\geq 3}^\alpha = \Phi^\alpha + \mathcal{L} h_3^\alpha, \tag{5}$$
where $\Phi^\alpha = \Phi^\alpha(H_0, H_1, H_2, h_{\geq 3})$ is given by
$$\begin{aligned}\Phi^\alpha =& \Gamma_{\beta\gamma}^\alpha \partial_x h_{\geq 3}^\beta \partial_x H_0^\gamma + \Gamma_{\beta\gamma}^\alpha (\partial_x H_2^\beta + \partial_x h_{\geq 3}^\beta) \partial_x H_1^\gamma \\ &+ \tfrac{1}{2} \Gamma_{\beta\gamma}^\alpha (\partial_x H_2^\beta + \partial_x h_{\geq 3}^\beta)(\partial_x H_2^\gamma + \partial_x h_{\geq 3}^\gamma).\end{aligned}$$
- To define $h_{\geq 3}^\alpha$, we need to introduce four more objects as driving terms:
$$H_{3,1}^{\beta\gamma} = \tfrac{1}{2} \partial_x H_1^\beta \partial_x H_1^\gamma, \qquad H_{3,2}^{\beta\gamma} = \partial_x H_0^\beta \circ \partial_x H_2^\gamma,$$
$$H_{3,3}^\alpha = \text{solution of "}\mathcal{L} H_{3,3}^\alpha = \partial_x H_0^\alpha\text{"}, \qquad H_{3,4}^{\beta\gamma} = \partial_x H_{3,3}^\beta \circ \partial_x H_0^\gamma.$$
- First two terms $H_{3,1}^{\beta\gamma}, H_{3,2}^{\beta\gamma}$ ($\in \mathcal{H}_4 \oplus \mathcal{H}_2$) appear in h_3^α.
- $H_{3,3}^\alpha, H_{3,4}^{\beta\gamma}$ appear to solve (5), to take care of $\partial_x H_0^\gamma$ in Φ^α.

- $\mathbb{H} := (H_0^{\alpha}, H_1^{\alpha}, H_2^{\alpha}, H_{3,1}^{\beta\gamma}, H_{3,2}^{\beta\gamma}, H_{3,3}^{\alpha}, H_{3,4}^{\beta\gamma})$ are called driving terms. The class of driving terms $\mathcal{H}_{KPZ}^{\kappa}$ is defined for $\kappa \in (\frac{1}{3}, \frac{1}{2})$ (i.e. $\kappa = \frac{1}{2}-$) as follows:

$$
\begin{aligned}
\mathcal{H}_{KPZ}^{\kappa} = & C([0,T],\mathcal{C}^{\kappa}) \times C([0,T],\mathcal{C}^{2\kappa}) \\
& \times \{C([0,T],\mathcal{C}^{\kappa+1}) \cap C^{\frac{1-\kappa}{2}}([0,T],\mathcal{C}^{2\kappa})\} \\
& \times C([0,T],\mathcal{C}^{2\kappa-1}) \times C([0,T],\mathcal{C}^{2\kappa-1}) \\
& \times C([0,T],\mathcal{C}^{\kappa+1}) \times C([0,T],\mathcal{C}^{2\kappa-1}).
\end{aligned}
$$

- Once $\mathbb{H}$ is given, the rest can be analyzed by deterministic argument.
- The following theorem (deterministic part) is due to the paracontrolled calculus and fixed point theorem.

Theorem 4

Let $\mathbb{H} \in \mathcal{H}_{KPZ}^{\kappa}$ be given. Then, the above equation (5) for $h_{\geq 3}$ is solvable (in a proper space controlled by driving terms) up to time $T = T(\|h_{\geq 3}(0)\|_{\mu+1}, \|\mathbb{H}\|), \mu \in (\frac{1}{3}, \kappa)$ and the solution map $h = S(h_{\geq 3}(0), \mathbb{H})$ is continuous in $(h_{\geq 3}(0), \mathbb{H})$.

(1) Coupled KPZ Approximating equation-1

- By replacing $\dot{W}^\alpha$ by $\dot{W}^\alpha * \eta^\varepsilon$ and introducing the renormalization factors $c^\varepsilon \delta^{\beta\gamma}, C^{\beta\gamma}, D^{\beta\gamma}$, we have the expansion for the coupled KPZ approx. eq-1 (simple) (3):

$$\begin{aligned}
\mathcal{L}h_0^\alpha &= \dot{W}^\alpha * \eta^\varepsilon, \\
\mathcal{L}h_1^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h_0^\beta \partial_x h_0^\gamma - c^\varepsilon \delta^{\beta\gamma}), \\
\mathcal{L}h_2^\alpha &= \Gamma^\alpha_{\beta\gamma}\partial_x h_0^\beta \partial_x h_1^\gamma, \\
\mathcal{L}h_3^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h_1^\beta \partial_x h_1^\gamma - C^{\varepsilon,\beta\gamma}) + \Gamma^\alpha_{\beta\gamma}(\partial_x h_0^\beta \partial_x h_2^\gamma - D^{\varepsilon,\beta\gamma}).
\end{aligned}$$

- See next pages for renormalization constants $c^\varepsilon, C^{\varepsilon,\beta\gamma}, D^{\varepsilon,\beta\gamma}$.
- From this, we see that $h^\varepsilon = (h^{\varepsilon,\alpha}) := S(h_{\geq 3}(0), \mathbb{H}^\varepsilon)$ solves

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon \delta^{\beta\gamma} - C^{\varepsilon,\beta\gamma} - 2D^{\varepsilon,\beta\gamma}) + \dot{W}^\alpha * \eta^\varepsilon,$$

i.e., (3) with

$$B^{\varepsilon,\beta\gamma} = C^{\varepsilon,\beta\gamma} + 2D^{\varepsilon,\beta\gamma}.$$

- Theorem 4 (especially, continuity in $\mathbb{H}$) combined with the convergence of driving terms $\mathbb{H}^\varepsilon$ (multiple Wiener integrals in $\dot{W}^\alpha$'s cut 0th order terms) to $\mathbb{H}$ shows Theorem 1-(1).

- We especially see how c^ε is determined.
- The stationary solution $h_0^\alpha \in C^{\frac{1}{2}-}$ of the first equation "$\mathcal{L}h_0^\alpha = \dot{W}^\alpha * \eta^\varepsilon$" is given by

$$h_0^\alpha(t,x) = \int_{-\infty}^t \int_{\mathbb{T}} p(t-s,x,y)d(W^\alpha * \eta^\varepsilon)(s,y)dy,$$

  where p is the heat kernel on $\mathbb{T}$. $\dot{W}(t,x)$ is extended for $t<0$ as an $\mathbb{R}^d$-valued space-time Gaussian white noise on $\mathbb{R}\times\mathbb{T}$.
- The renormalization constant c^ε is defined by

$$c^\varepsilon = E\big[(\partial_x h_0^\alpha(t,x))^2\big].$$

  Note $E\big[\partial_x h_0^\beta(t,x)\partial_x h_0^\gamma(t,x)\big] = 0$ if $\beta \neq \gamma$.
- Since

$$\partial_x h_0^\alpha(t,x) = \int_{-\infty}^t \int_{\mathbb{T}} dy \int_{\mathbb{T}} \partial_x p(t-s,x,y)\eta^\varepsilon(y-z)dW^\alpha(s,z)dz,$$

  we have by Itô isometry

$$c^\varepsilon = \int_{-\infty}^t ds \int_{\mathbb{T}} dz \Big(\int_{\mathbb{T}} \partial_x p(t-s,x,y)\eta^\varepsilon(y-z)dy\Big)^2$$

- Thus, using Chapman-Kolmogorov identity,

$$\begin{aligned} c^{\varepsilon} &= \int_0^{\infty} ds \int_{\mathbb{T}} dz \Big(\int_{\mathbb{T}} \partial_x p(s,x,y_1 - z) \eta^{\varepsilon}(y_1) dy_1 \Big) \\ &\qquad \times \Big(\int_{\mathbb{T}} \partial_x p(s,x,y_2 - z) \eta^{\varepsilon}(y_2) dy_2 \Big) \\ &= \int_0^{\infty} ds \int_{\mathbb{T}^2} -\partial_x^2 p(2s,y_1,y_2) \eta^{\varepsilon}(y_1) \eta^{\varepsilon}(y_2) dy_1 dy_2 \\ &= -\int_0^{\infty} ds \int_{\mathbb{T}^2} \partial_s p(2s,y_1,y_2) \eta^{\varepsilon}(y_1) \eta^{\varepsilon}(y_2) dy_1 dy_2 \\ &= \int_{\mathbb{T}^2} \eta^{\varepsilon}(y_1) \eta^{\varepsilon}(y_2) \big(\delta(y_1 - y_2) - 1 \big) dy_1 dy_2 \\ &= \|\eta^{\varepsilon}\|^2_{L^2(\mathbb{T})} - 1. \end{aligned}$$

- -1 appears on $\mathbb{T}$, but not on $\mathbb{R}$.

- In terms of Fourier transform, we also have the following formula:

$$c^\varepsilon = \sum_{k\neq 0} \varphi_\varepsilon^2(k)$$

where $\varphi(k) = \mathcal{F}\eta(k), \varphi_\varepsilon(k) = \varphi(\varepsilon k)$.
- In fact, by Plancherel's identity and noting $\varphi_\varepsilon(0) = 1$, this also shows $c^\varepsilon = \|\eta^\varepsilon\|^2_{L^2(\mathbb{T})} - 1$.
- Similarly, and using diagram formula similar to Lecture No 3 but now in the noise $\dot{W}(t,x)$, the fourth order renormalization factors can be computed as

$$C^{\varepsilon,\beta\gamma} = F^{\beta\gamma} C^\varepsilon \text{ with } C^\varepsilon = \frac{1}{4\pi^2} \sum_{k_1,k_2}^{*} \frac{\varphi_\varepsilon(k_1)^2 \varphi_\varepsilon(k_2)^2}{k_1^2 + k_1 k_2 + k_2^2},$$

$$D^{\varepsilon,\beta\gamma} = G^{\beta\gamma} D^\varepsilon \text{ with } D^\varepsilon = -\frac{1}{4\pi^2} \sum_{k_1,k_2}^{*} \frac{(k_1+k_2)\varphi_\varepsilon(k_1)^2 \varphi_\varepsilon(k_2)^2}{k_2(k_1^2 + k_1 k_2 + k_2^2)},$$

where $\varphi(k) = \mathcal{F}\eta(k), \varphi_\varepsilon(k) = \varphi(\varepsilon k)$, $\sum^{*}$ means the sum over k_1, k_2 s.t. $k_1 \neq 0$, $k_2 \neq 0$, $k_1 + k_2 \neq 0$ and

$$F^{\beta\gamma} = \Gamma^\beta_{\gamma_1\gamma_2} \Gamma^\gamma_{\gamma_1\gamma_2},$$

$$G^{\beta\gamma} = \Gamma^\beta_{\gamma_1\gamma_2} \Gamma^{\gamma_1}_{\gamma\gamma_2}.$$

- **Remark**: Our notation and those in [Hairer, Gubinelli, ...] studying the case $d = 1$ (i.e. scalar-valued case) correspond with each other as follows:

$H_0 = X_\epsilon$, $H_1 = X_\epsilon$, $H_2 = X_\epsilon$, $h_{\geq 3} = X_\epsilon$ $+ X_\epsilon$ $+ \cdots$,

$c^\varepsilon \delta^{\beta\gamma} = c_\epsilon$, $C^{\varepsilon,\beta\gamma} = c_\epsilon$, $D^{\varepsilon,\beta\gamma} = c_\epsilon$.

(2) Coupled KPZ Approximating equation-2

- We do similar for the coupled KPZ equation with $*\eta_2^\varepsilon$ for the nonlinear term. Then, by the expansion, we have

$$
\begin{aligned}
\mathcal{L}\tilde{h}_0^\alpha &= \dot{W}^\alpha, \\
\mathcal{L}\tilde{h}_1^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_0^\gamma) * \eta_2^\varepsilon, \\
\mathcal{L}\tilde{h}_2^\alpha &= \Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_1^\gamma) * \eta_2^\varepsilon, \\
\mathcal{L}\tilde{h}_3^\alpha &= \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}_1^\beta \partial_x \tilde{h}_1^\gamma) * \eta_2^\varepsilon + \Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_2^\gamma) * \eta_2^\varepsilon.
\end{aligned}
$$

Theorem 5

There exists a solution map $\tilde{h} = S_\varepsilon(h_{\geq 3}(0), \mathbb{H})$. Note that S_ε means that the equation has the factor $*\eta_2^\varepsilon$.

Furthermore, we have:

Theorem 6

If $h^\varepsilon_{\geq 3}(0) \to h_{\geq 3}(0)$ in $\mathcal{C}^{\mu+1}$ and $\mathbb{H}^\varepsilon \to \mathbb{H}$ in $\mathcal{H}^\kappa_{KPZ}$, then we have that $S_\varepsilon(\overline{h}^\varepsilon_{\geq 3}(0), \mathbb{H}^\varepsilon) \to S(h_{\geq 3}(0), \mathbb{H})$.

- By replacing $\dot{W}^\alpha$ by $\dot{W}^\alpha * \eta^\varepsilon$ and introducing the renormalization factors $-c^\varepsilon \delta^{\beta\gamma}, \tilde{C}^{\beta\gamma}, \tilde{D}^{\beta\gamma}$, we have the expansion related to the coupled KPZ approx. eq-2 (suitable for studying inv measures) (4):

$$\mathcal{L}\tilde{h}_0^\alpha = \dot{W}^\alpha * \eta^\varepsilon,$$
$$\mathcal{L}\tilde{h}_1^\alpha = \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_0^\gamma - c^\varepsilon \delta^{\beta\gamma}) * \eta_2^\varepsilon,$$
$$\mathcal{L}\tilde{h}_2^\alpha = \Gamma^\alpha_{\beta\gamma}\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_1^\gamma * \eta_2^\varepsilon,$$
$$\mathcal{L}\tilde{h}_3^\alpha = \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}_1^\beta \partial_x \tilde{h}_1^\gamma - \tilde{C}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}_0^\beta \partial_x \tilde{h}_2^\gamma - \tilde{D}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon.$$

- From this, we see that $\tilde{h}^\varepsilon = (\tilde{h}^{\varepsilon,\alpha}) := S_\varepsilon(h_{\geq 3}(0), \mathbb{H}^\varepsilon)$ solves

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon \delta^{\beta\gamma} - \tilde{C}^{\varepsilon,\beta\gamma} - 2\tilde{D}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \dot{W}^\alpha * \eta^\varepsilon,$$

  i.e., (4) with
$$\tilde{B}^{\varepsilon,\beta\gamma} = \tilde{C}^{\varepsilon,\beta\gamma} + 2\tilde{D}^{\varepsilon,\beta\gamma}.$$

- Theorems 5, 6 together with the convergence of driving terms show Theorem 1-(2).

Difference of solutions of two approximating eq-1 and -2

- We show Theorem 2-(1).
- From the above computation, the difference of solutions of two approximating equations with $B^{\varepsilon,\beta\gamma}, \tilde{B}^{\varepsilon,\beta\gamma} = 0$ are given by

$$\tilde{h}^{\varepsilon,\alpha}_{\tilde{B}=0} - h^{\varepsilon,\alpha}_{B=0} = \left(\tilde{h}^{\varepsilon,\alpha}_{\tilde{B}} - h^{\varepsilon,\alpha}_{B}\right) + \frac{t}{2}\Gamma^{\alpha}_{\beta\gamma}\left(\tilde{B}^{\varepsilon,\beta\gamma} - B^{\varepsilon,\beta\gamma}\right)$$

  and by Theorem 1-(2), $\left(\tilde{h}^{\varepsilon,\alpha}_{\tilde{B}} - h^{\varepsilon,\alpha}_{B}\right) \to 0$.
- In particular, we have

$$\lim_{\varepsilon\downarrow 0}\left(\tilde{h}^{\varepsilon,\alpha}_{\tilde{B}=0} - h^{\varepsilon,\alpha}_{B=0}\right) = \frac{t}{2}\Gamma^{\alpha}_{\beta\gamma}\lim_{\varepsilon\downarrow 0}\left(\tilde{B}^{\varepsilon,\beta\gamma} - B^{\varepsilon,\beta\gamma}\right).$$

- We can explicitly compute the renormalization factors:

$$C^{\varepsilon,\beta\gamma} = F^{\beta\gamma} C^{\varepsilon} \text{ with } C^{\varepsilon} = \frac{1}{4\pi^2} \sum_{k_1,k_2}^{*} \frac{\varphi_\varepsilon(k_1)^2 \varphi_\varepsilon(k_2)^2}{k_1^2 + k_1 k_2 + k_2^2},$$

$$D^{\varepsilon,\beta\gamma} = G^{\beta\gamma} D^{\varepsilon} \text{ with } D^{\varepsilon} = -\frac{1}{4\pi^2} \sum_{k_1,k_2}^{*} \frac{(k_1+k_2)\varphi_\varepsilon(k_1)^2 \varphi_\varepsilon(k_2)^2}{k_2(k_1^2 + k_1 k_2 + k_2^2)},$$

$$\tilde{C}^{\varepsilon,\beta\gamma} = F^{\beta\gamma} \tilde{C}^{\varepsilon} \text{ with } \tilde{C}^{\varepsilon} = \frac{1}{4\pi^2} \sum_{k_1,k_2}^{*} \frac{\varphi_\varepsilon(k_1)^2 \varphi_\varepsilon(k_2)^2 \varphi_\varepsilon(k_1+k_2)^4}{k_1^2 + k_1 k_2 + k_2^2},$$

$$\tilde{D}^{\varepsilon,\beta\gamma} = G^{\beta\gamma} \tilde{D}^{\varepsilon} \text{ with } \tilde{D}^{\varepsilon} = -\frac{1}{4\pi^2} \sum_{k_1,k_2}^{*} \frac{(k_1+k_2)\varphi_\varepsilon(k_1)^2 \varphi_\varepsilon(k_2)^2 \varphi_\varepsilon(k_1+k_2)^4}{k_2(k_1^2 + k_1 k_2 + k_2^2)},$$

where $\varphi(k) = \mathcal{F}\eta(k), \varphi_\varepsilon(k) = \varphi(\varepsilon k)$, $\sum^{*}$ means the sum over k_1, k_2 s.t. $k_1 \neq 0$, $k_2 \neq 0$, $k_1 + k_2 \neq 0$ and

$$F^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2} \Gamma^{\gamma}_{\gamma_1\gamma_2},$$
$$G^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2} \Gamma^{\gamma_1}_{\gamma\gamma_2}.$$

- Assume the trilinear condition (T). Then, as we already saw, we have

$$F^{\beta\gamma} = G^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma}_{\gamma_1\gamma_2}.$$

- Thus,

$$\begin{aligned}\tilde{B}^{\varepsilon,\beta\gamma} - B^{\varepsilon,\beta\gamma} &= (\tilde{C}^{\varepsilon,\beta\gamma} + 2\tilde{D}^{\varepsilon,\beta\gamma}) - (C^{\varepsilon,\beta\gamma} + 2D^{\varepsilon,\beta\gamma}) \\ &= F^{\beta\gamma}\big((\tilde{C}^{\varepsilon} + 2\tilde{D}^{\varepsilon}) - (C^{\varepsilon} + 2D^{\varepsilon})\big).\end{aligned}$$

- However, by the explicit computation (for scalar-valued case),

$$\tilde{C}^{\varepsilon} + 2\tilde{D}^{\varepsilon} = 0, \quad C^{\varepsilon} + 2D^{\varepsilon} = -\tfrac{1}{12} + O(\varepsilon).$$

- Therefore, in the limit, we have

$$\tilde{h}^{\alpha}_{\tilde{B}=0}(t,x) = h^{\alpha}_{B=0}(t,x) + c^{\alpha}t, \quad 1 \le \alpha \le d,$$

where

$$c^{\alpha} := \frac{1}{24}\Gamma^{\alpha}_{\beta\gamma}F^{\beta\gamma} = \frac{1}{24}\sum_{\gamma_1,\gamma_2}\Gamma^{\alpha}_{\beta\gamma}\Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma}_{\gamma_1\gamma_2}.$$

- This concludes the proof of Theorem 2-(1).

Invariant measure

- We finally give the outline of the proof of Theorem 2-(2).
- We actually consider the coupled KPZ-Burgers equation for $u^\alpha := \partial_x h^\alpha$ as in Lecture No 3.
- We move to the Fourier mode $\{u^{\alpha,k}\}_{k\in\mathbb{Z}}$ and introduce a cut-off, i.e. we use Galerkin approximation.
- We show the infinitesimal invariance of Gaussian measure with cut-off by applying Echeveria's criterion for the finite-dimensional SDE. Trilinear condition (T) is essential (as we saw at least heuristically above).
- Moreover, the energy estimate holds uniformly in cut-off by noting that the nonlinear term cancels under (T).
- We finally take the limit.

8. Remarks for the case with diffusion constant σ

- Coupled KPZ approx. eq-1: Simple

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon A^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \sigma^\alpha_\beta \dot{W}^\beta * \eta^\varepsilon, \quad (6)$$

where $A^{\beta\gamma} = \sum_{\delta=1}^d \sigma^\beta_\delta \sigma^\gamma_\delta$, $c^\varepsilon = \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})} - 1$ and $B^{\varepsilon,\beta\gamma}$ ($= O(-\log\varepsilon)$ in general) is another renormalization factor.

- Coupled KPZ approx. eq-2: suitable for studying inv measures

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon A^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon \quad + \sigma^\alpha_\beta \dot{W}^\beta * \eta^\varepsilon, \quad (7)$$

with a renormalization factor $\tilde{B}^{\varepsilon,\beta\gamma}$.

- For the solution of (7) (with $\tilde{B} = 0$), F ('15, Yor volume) showed (on $\mathbb{R}$), under the additional (trilinear) condition:

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = \hat{\Gamma}^{\gamma}_{\alpha\beta} = \hat{\Gamma}^{\gamma}_{\beta\alpha} \tag{8}$$

  for all $\alpha, \beta\gamma$ (second equality is by bilinearity), where

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} := \tau^{\alpha}_{\alpha'} \Gamma^{\alpha'}_{\beta'\gamma'} \sigma^{\beta'}_{\beta} \sigma^{\gamma'}_{\gamma}, \quad \tau = \sigma^{-1},$$

  the (infinitesimal) invariance of the distribution of $(\sigma B) * \eta^{\varepsilon}(x)$, where B is the $\mathbb{R}^d$-valued two-sided Brownian motion (with $x \in \mathbb{R}$).
- Our goal is to study the limits of the solutions of (6) and (7) as $\varepsilon \downarrow 0$.

Theorem 1 holds with σ^α_β.

Theorem 7 (cf. Theorem 2)

Assume trilinear condition (8). Then, $B^{\varepsilon,\beta\gamma}, \tilde{B}^{\varepsilon,\beta\gamma} = O(1)$ so that the solutions of (6) with $B = 0$ and (7) with $\tilde{B} = 0$ converge. In the limit, we have

$$\tilde{h}^\alpha(t,x) = h^\alpha(t,x) + c^\alpha t, \quad 1 \le \alpha \le d,$$

where

$$c^\alpha = \frac{1}{24} \sum_{\gamma,\gamma'} \sigma^\alpha_\beta \hat{\Gamma}^\beta_{\alpha'\alpha''} \hat{\Gamma}^{\alpha'}_{\gamma\gamma'} \hat{\Gamma}^{\alpha''}_{\gamma\gamma'}.$$

Moreover, the distribution of $\{\sigma B\}_{x\in\mathbb{T}}$ (note: infinite measure) is invariant under h. Or, the distribution of $\{\sigma \partial_x B\}_{x\in\mathbb{T}}$ (finite measure) is invariant under the tilt process $\partial_x h$.

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon A^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \sigma^\alpha_\beta \dot{W}^\beta * \eta^\varepsilon \qquad (6)$$

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon A^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \sigma^\alpha_\beta \dot{W}^\beta * \eta^\varepsilon \qquad (7)$$

- (cf. Theorem 3) Under the trilinear condition (8), global existence holds for a.s.-initial values under stationary measure, and then for all given $u(0)$ as before.

Summary of this lecture.

1. Coupled KPZ equation (mostly with $\sigma = I$):

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \dot{W}^\alpha, \quad x \in \mathbb{T}.$$

2. For $^\forall\Gamma$, convergence of two approximating solutions $h^\varepsilon, \tilde{h}^\varepsilon$ and local well-posedness of coupled KPZ equation (σ, Γ) by applying paracontrolled calculus.

3. For Γ satisfying (T), Wiener measure is invariant and global well-posedness of coupled KPZ equation holds, first for a.a.-initial values under stationary measure, then for all initial values.

4. $(T) \Longleftrightarrow$ "$F = G$" $\Longleftrightarrow (ST)_{\mathcal{A}}$ for Wiener meas. ν
   $\Longrightarrow$ "$\Gamma F = \Gamma G$" $\Longleftrightarrow$ Cancellation of log-renormalization factors

5. Extensions of Ertaş-Kardar's example