KPZ limit for interacting particle systems —Invariant measures of KPZ equation—

Tadahisa Funaki

Waseda University

November 24th+26th, 2020

Yau Mathematical Sciences Center, Mini-Course, Nov 17-Dec 17, 2020
Lecture No 3

- F-Quastel, Stoch. PDE: Anal. Comp. **3**, 2015

Plan of the course (10 lectures)

1 Introduction

2 Supplementary materials

Brownian motion, Space-time Gaussian white noise, (Additive) linear SPDEs, (Finite-dimensional) SDEs, Martingale problem, Invariant/reversible measures for SDEs, Martingales

3 Invariant measures of KPZ equation (F-Quastel, 2015)

4 Coupled KPZ equation by paracontrolled calculus (F-Hoshino, 2017)

5 Coupled KPZ equation from interacting particle systems (Bernardin-F-Sethuraman, 2020+)

5.1 Independent particle systems

5.2 Single species zero-range process

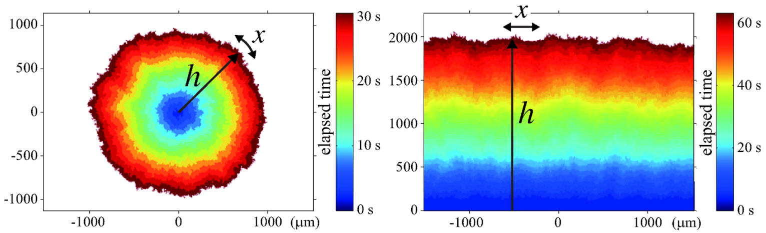
5.3 n-species zero-range process

5.4 Hydrodynamic limit, Linear fluctuation

5.5 KPZ limit=Nonlinear fluctuation

Plan of this lecture

Invariant measures of KPZ equation



(Takeuchi-Sano-Sasamoto-Spohn)

1. Renormalization, Cole-Hopf solution, Approximation-1

- In Lecture No 1, we introduced KPZ equation (1), the renormalized KPZ equation (2) and Cole-Hopf solution (3) of KPZ equation:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \tag{1}$$

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x), \tag{2}$$

$$h(t,x) := \log Z(t,x), \tag{3}$$

  where Z is the solution of multiplicative linear stochastic heat equation (SHE):

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x). \tag{4}$$

- We may consider on $\mathbb{R}$ or $\mathbb{T} = [0,1)$, but mostly on $\mathbb{R}$ in this lecture.
- The product of Z and $\dot{W}$ in (4) should be understood in Itô's sense (in mild form or in generalized functions' sense).

- As we saw, SHE (4) is well-posed and a heuristic application of Itô's formula to $h(t,x)$ in (3) leads to the renormalized KPZ equation (2).
- Our first goal is to give mathematically rigorous foundation to this procedure.
- The ill-posedness of KPZ equation (1) comes from the mismatch between the nonlinear term and the noise.
- We can not deal with the KPZ eq directly. We consider its approximation by replacing the noise by smooth one.
- However, the solution of the equation with the noise simply replaced by smooth one does not converge in the limit.
- We need to introduce some additional diverging factor to compensate in removing smoothness of the noise. This is called the renormalization.

1.1. Approximation-1: Simple

- Symmetric convolution kernel: Let $\eta \in C_0^\infty(\mathbb{R})$ s.t. $\eta(x) \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}} \eta(x)dx = 1$ be given, and set $\eta^\varepsilon(x) := \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$ for $\varepsilon > 0$.
- Smeared noise:
  $\dot{W}^\varepsilon(t,x) = \dot{W}(t) * \eta^\varepsilon(x) \equiv \langle \dot{W}(t), \eta^\varepsilon(x - \cdot)\rangle$
- Approximating equation-1: Let $h = h^\varepsilon$ be a solution of

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left((\partial_x h)^2 - c^\varepsilon\right) + \dot{W}^\varepsilon(t,x), \tag{5}$$

  where

$$c^\varepsilon = \int_{\mathbb{R}} \eta^\varepsilon(y)^2 dy \left(= \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})}\right).$$

- $c^\varepsilon \nearrow \infty$ as $\varepsilon \downarrow 0$. c^ε is called a renormalization. Without c^ε, the solution h^ε does not converge.
- Note that $\dot{W}^\varepsilon$ is smooth in x, but it remains stochastic in t. The solution h^ε of (5) is smooth in x.

1.2. Cole-Hopf solution

- As in Lecture No 1, consider the Cole-Hopf transform of $h = h^\varepsilon$ defined by $Z = Z^\varepsilon := e^h$, then Z satisfies

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\dot{W}^\varepsilon(t,x).$$

  (The product $Z\dot{W}^\varepsilon$ is defined in Itô's sense.)
- Indeed, apply Itô's formula for $z = e^h$ to see

$$\begin{aligned}\partial_t Z &= Z\partial_t h + \tfrac{1}{2}Z(\partial_t h)^2 \\ &= \tfrac{1}{2}Z\{\partial_x^2 h + (\partial_x h)^2 - c^\varepsilon\} + Z\dot{W}^\varepsilon + \tfrac{1}{2}Zc^\varepsilon \\ &= \tfrac{1}{2}\partial_x^2 Z + Z\dot{W}^\varepsilon,\end{aligned}$$

  since $Z\{\partial_x^2 h + (\partial_x h)^2\} = \partial_x^2 Z$.
- See next page for $(\partial_t h)^2 = c^\varepsilon$.
- In Lecture No 1, we computed $\partial_t h$ starting from Z. Here, conversely, we start from h and compute $\partial_t Z$.

- $(\partial_t h)^2 = c^\varepsilon$ or $(dh)^2 = c^\varepsilon dt$ is seen from

$$\begin{aligned}
(dh(t,x))^2 &= (dW^\varepsilon(t,x))^2 \\
&= \int \eta^\varepsilon(x-y)dW(t,y)dy \cdot \int \eta^\varepsilon(x-z)dW(t,z)dz \\
&= \iint \eta^\varepsilon(x-y)\eta^\varepsilon(x-z)\delta(y-z)dydz \cdot dt \\
&= \int \eta^\varepsilon(x-y)^2 dy \cdot dt = c^\varepsilon dt
\end{aligned}$$

- Recall $dW(t,y)dW(t,z) = \delta(y-z)dt$ from the relation of the covariance.
- The renormalization c^ε in (5) was chosen such that it cancels with this diverging Itô correction term.

- As we have shown, $Z = Z^\varepsilon$ is the solution of

$$\partial_t Z^\varepsilon = \tfrac{1}{2}\partial_x^2 Z^\varepsilon + Z^\varepsilon \dot{W}^\varepsilon(t,x).$$

- It is not difficult to show (Bertini-Giacomin 1997) that $Z^\varepsilon \to Z$ as $\varepsilon \downarrow 0$, the sol of the linear stochastic heat equation (defined in Itô's sense) (4):

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x),$$

  with a multiplicative noise. (4) is a well-posed equation.
- This implies $h^\varepsilon \to h_{CH}$ as $\varepsilon \downarrow 0$, i.e., the solution $h = h^\varepsilon$ of the approximating KPZ equation-1 converges to the Cole-Hopf solution of the KPZ equation defined by (3):

$$h_{CH}(t,x) := \log Z(t,x).$$

- Comparison theorem for (4): $Z(0) > 0 \Rightarrow Z(t) > 0$.

- The following is copied from Lecture No 1.
- The equation satisfied by h_{CH}:

$$\begin{aligned}\partial_t h_{CH} &= \frac{1}{Z}\partial_t Z - \frac{1}{2}\frac{1}{Z^2}(\partial_t Z)^2 \\ &= \frac{1}{Z}\left(\frac{1}{2}\partial_x^2 Z + Z\dot{W}\right) - \frac{1}{2}\delta_x(x) \\ &= \frac{1}{2}\left(\partial_x^2 h_{CH} + (\partial_x h_{CH})^2\right) + \dot{W} - \frac{1}{2}\delta_x(x)\end{aligned}$$

- Thus, for the Cole-Hopf solution h_{CH}, at least heuristically, we obtain the renormalized KPZ equation (2):

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x).$$

2. Approximation-2: Suitable to find invariant measures

- We introduce another KPZ approximating equation:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\big((\partial_x h)^2 - c^\varepsilon\big) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x), \qquad (6)$$

  where $\eta_2(x) = \eta * \eta(x)$, $\eta_2^\varepsilon(x) = \frac{1}{\varepsilon}\eta_2(\frac{x}{\varepsilon})$.
- Recall $c^\varepsilon = \eta_2^\varepsilon(0)$.
- General principle (Onsager relation, fluctuation-dissipation relation): Consider the SPDE

$$\partial_t h = F(h) + \dot{W},$$

  and let A be a certain operator. Then, the structure of the invariant measures essentially does not change for

$$\partial_t h = A^2 F(h) + A\dot{W}.$$

- Indeed, we will show in Proposition 1 below that the distribution of $B * \eta^\varepsilon(x)$, where B is the periodic Brownian motion (in case $\mathbb{T}$) or the two-sided Brownian motion (in case $\mathbb{R}$), is invariant for the sol. $h = h^\varepsilon$ of (6).

Explanation of fluctuation-dissipation relation

(reversible and finite-dimensional case, cf. Lecture No 2)

- Let $V \in C^1(\mathbb{R}^d)$ and consider SDE:

$$dX_t = -\frac{1}{2}\nabla V(X_t)dt + dB_t$$

- Then X_t is reversible under the measure $e^{-V}dx$.
- (Fluctuation-dissipation relation) For a matrix $A = (\alpha_{ij})_{1\leq i,j\leq d}$, consider SDE:

$$dY_t = -\frac{1}{2}A^*A\nabla V(Y_t)dt + AdB_t,$$

- Y_t is also reversible under $e^{-V}dx$.

- KPZ equation has an asymmetric part (growing part) so that the situation is not exactly the same ($\to$ Yaglom reversibility).
- However, as we expect, the 2nd Approximating SPDE (6) has a good property in its invariant (stationary) measures.
- Let ν^ε be the distribution of $\partial_x(B * \eta^\varepsilon(x))$, where B is the two-sided Brownian motion. ν^ε is independent of choice of $B(0)$.

Proposition 1

ν^ε is stationary for the tilt process $\partial_x h$ of the SPDE (6).

- At the KPZ level, the invariant measure is not a finite measure ($\to$ Thm 3 below).
- To avoid this, in Prop 1, we consider its slope (tilt), i.e. at the Burgers' level.

Two-sided Brownian motion: $\{B(x)\}_{x\geq 0}$ and $\{B(x)\}_{x\leq 0}$ are independent Brownian motions (conditioned on $B(0)$) regarding x as time parameter (for latter, take $-x$ as time parameter) and continuously connected at 0.

Sketch of the proof:

- Step 1: Consider on a discrete torus $\mathbb{T}_N = \{1, 2, \dots, N\}$. The discretization of $(\partial_x h)^2$ should be carefully chosen as

$$\frac{1}{3}\Big\{(h_{i+1} - h_i)^2 + (h_i - h_{i-1})^2 + (h_{i+1} - h_i)(h_i - h_{i-1})\Big\}, \ i \in \mathbb{T}_N$$

  Discrete version of ν^ε defined on $\mathbb{T}_N$ is invariant. $\rightarrow$ see next page how we apply the result in Lecture No 2.

- Step 2: Continuum limit as $N \to \infty$ leads to the result on $\mathbb{T}$. This can be easily extended to a large torus $M \cdot \mathbb{T} \simeq [-\frac{M}{2}, \frac{M}{2})$ of size M.
- Step 3: To show on $\mathbb{R}$, take an infinite-volume limit as $M \to \infty$ by usual tightness and martingale problem approach.

More on Step 1:

- Take $\alpha : \mathbb{Z} \to [0, \infty)$ such that $\alpha(i) = \alpha(-i)$ and $\alpha(i) = 0$ for $i : |i| \geq^{\exists} K$, instead of $\eta(x)$ in (6).
- For $h = (h_i \equiv h(i))_{i \in \mathbb{T}_N} \in \mathbb{R}^{\mathbb{T}_N}$, we define

$$\begin{aligned} \Delta h(i) &= h(i+1) + h(i-1) - 2h(i), \\ G_1(i, h) &= (h_{i+1} - h_i)^2 + (h_i - h_{i-1})^2, \\ G_2(i, h) &= (h_{i+1} - h_i)(h_i - h_{i-1}), \quad i \in \mathbb{T}_N, \end{aligned}$$

- Convolution of two functions β, γ on $\mathbb{T}_N$ is defined by $(\beta * \gamma)(i) = \sum_{k \in \mathbb{T}_N} \beta(i-k)\gamma(k)$, with $i - k$ understood in modulo N.
- We consider SDE for $h_t = (h_t(i))_{i \in \mathbb{T}_N} \in \mathbb{R}^{\mathbb{T}_N}$:

$$dh_t(i) = \frac{\lambda_1}{2} \Delta h_t(i) dt + \lambda_2 \{\alpha_2 * G_1(i, h_t) + \alpha_2 * G_2(i, h_t)\} dt + \lambda_3 dw_t^{\alpha}(i), \tag{7}$$

  where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are arbitrary constants, $\alpha_2 = \alpha * \alpha$, $w_t^{\alpha} = \alpha * w_t$ and $w_t = (w_t(i))_{i \in \mathbb{T}_N}$ are independent BMs.
- (7) is a discretization of (6) disregarding constant drift c^{ε}.

- We consider three operators on $\mathbb{R}^{\mathbb{T}_N}$: for $f \in C^2(\mathbb{R}^{\mathbb{T}_N})$,

$$\mathcal{L}_0^\alpha f(h) = \frac{\lambda_1}{2} \sum_{i\in\mathbb{T}_N} \Delta h(i) \frac{\partial f}{\partial h_i} + \frac{\lambda_3^2}{2} \sum_{i,j\in\mathbb{T}_N} \alpha_2(i-j) \frac{\partial^2 f}{\partial h_i \partial h_j},$$

$$\mathcal{A}_1^\alpha f(h) = \sum_{i\in\mathbb{T}_N} (\alpha_2 * G_1)(i,h) \frac{\partial f}{\partial h_i},$$

$$\mathcal{A}_2^\alpha f(h) = \sum_{i\in\mathbb{T}_N} (\alpha_2 * G_2)(i,h) \frac{\partial f}{\partial h_i},$$

- Then, $\mathcal{L}^\alpha := \mathcal{L}_0^\alpha + \lambda_2 \mathcal{A}_1^\alpha + \lambda_2 \mathcal{A}_2^\alpha$ is the generator of the SDE (7).
- Let $\mu_N(dh) = e^{-I_N^\alpha(h)} dh$ be the measure on $\mathbb{R}^{\mathbb{T}_N}$, where

$$I_N^\alpha(h) = \frac{\lambda_1}{2\lambda_3^2} \sum_{j\in\mathbb{T}_N} \{\alpha^{-1} * h(j+1) - \alpha^{-1} * h(j)\}^2$$

$$dh = \prod_{i\in\mathbb{T}_N} dh(i),$$

$$\alpha^{-1} = \text{inverse matrix of } \alpha = \{\alpha(i-j)\}_{i,j\in\mathbb{T}_N}.$$

Lemma 2

For every $f, g \in C_b^2(\mathbb{R}^{\mathbb{T}_N})$, we have the symmetry of $\mathcal{L}_0^\alpha$:

$$\int g(h)\mathcal{L}_0^\alpha f(h) d\mu_N = \int f(h)\mathcal{L}_0^\alpha g(h) d\mu_N.$$

In particular, $\int \mathcal{L}_0^\alpha f(h) d\mu_N = 0$. Moreover, we have

$$\int \mathcal{A}_1^\alpha f(h) d\mu_N = -\int \mathcal{A}_2^\alpha f(h) d\mu_N \quad i.e. \quad \int (\mathcal{A}_1^\alpha + \mathcal{A}_2^\alpha) f(h) d\mu_N = 0.$$

Accordingly, we have that $\int_{\mathbb{R}^{\mathbb{T}_N}} \mathcal{L}^\alpha f(h) d\mu_N = 0$.

- This lemma shows the infinitesimal invariance of μ_N for $\mathcal{L}^\alpha$ (→Recall Lecture No 2).
- In the finite-dimensional setting, infinitesimal invariance implies the invariance. We apply Echeveria's result (1982) by noting the well-posedness of the martingale problem corresponding to the SDE (7). End of Step 1 □

Remark:

- Infinitesimal invariance can be directly shown for the SPDE (6) based on Wiener-Itô chaos expansion of tame functions Φ of the form $\Phi(h) = f(\langle h, \varphi_1\rangle, \dots, \langle h, \varphi_n\rangle)$:

$$\int \mathcal{L}^\varepsilon \Phi(h) \nu^\varepsilon(dh) = 0,$$

where $\mathcal{L}^\varepsilon := \mathcal{L}_0^\varepsilon + \mathcal{A}^\varepsilon$ is (pre) generator of the SPDE (6) and

$$\mathcal{L}_0^\varepsilon \Phi(h) = \frac{1}{2} \int_{\mathbb{R}^2} D^2\Phi(x_1, x_2; h) \eta_2^\varepsilon(x_1 - x_2) dx_1 dx_2 + \frac{1}{2} \int_{\mathbb{R}} \partial_x^2 h(x) D\Phi(x; h) dx,$$

$$\mathcal{A}^\varepsilon \Phi(h) = \frac{1}{2} \int_{\mathbb{R}} \left((\partial_x h)^2 - c^\varepsilon\right) * \eta_2^\varepsilon(x) D\Phi(x; h) dx.$$

- Indeed, $\mathcal{L}_0^\varepsilon$ is symmetric, while $\mathcal{A}^\varepsilon$ is asymmetric:

$$\int \Psi \mathcal{L}_0^\varepsilon \Phi d\nu^\varepsilon = \int \Phi \mathcal{L}_0^\varepsilon \Psi d\nu^\varepsilon, \quad \int \Psi \mathcal{A}^\varepsilon \Phi d\nu^\varepsilon = -\int \Phi \mathcal{A}^\varepsilon \Psi d\nu^\varepsilon.$$

• F, in Séminaire de Probab., LNM **2137**, special issue for M. Yor, 2015 (for coupled KPZ equation)

- Combined with the well-posedness of $\mathcal{L}^\varepsilon$-martingale problem, which can be shown at least on $\mathbb{T}$, it is expected that the infinitesimal invariance implies Proposition 1. But this is not clear in infinite-dimensional setting (extension of Echeverria's result is unknown).
- Note that we have

$$\begin{aligned}\nu^\varepsilon : \text{ invariant } &\Leftrightarrow \int_{\mathcal{C}} e^{t\mathcal{L}^\varepsilon}\Phi(h)\nu^\varepsilon(dh) = \int_{\mathcal{C}} \Phi(h)\nu^\varepsilon(dh) \\ &\Rightarrow \int_{\mathcal{C}} \mathcal{L}^\varepsilon \Phi(h)\nu^\varepsilon(dh) = 0 \quad \text{(inf. invariance)}\end{aligned}$$

  for a wide class of Φ (and all $t \geq 0$).
- We can prove the last identity (integration by parts formula) due to the method of stochastic analysis. But, $\Leftarrow$ is unclear. □

3. Invariant measures of Cole-Hopf solution and SHE

- It's important to know the asymptotic behavior of the solutions of the KPZ equation as $t \to \infty$.
- The goal is to give a class of invariant (=stationary) measures of the stochastic heat equation (4):

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x)$$

  and for the Cole-Hopf solution of the KPZ equation (3):

$$h(t,x) := \log Z(t,x).$$

- We apply Proposition 1 and let $\varepsilon \downarrow 0$.
- We state the result only on $\mathbb{R}$, but it holds also on $\mathbb{T} = [0,1)$.

- [For Z] Let $\mu^c, c \in \mathbb{R}$ be the distribution of $e^{B(x)+cx}, x \in \mathbb{R}$, called geometric Brownian motion when $c = 0$, on $\mathcal{C}_+ = C(\mathbb{R}, (0, \infty))$, where $B(x), x \in \mathbb{R}$, is the two-sided Brownian motion such that
$$\mu^c(B(0) \in dy) = dy.$$
- [For h] Let ν^c be the distribution of $B(x) + cx$, BM with drift c, on $\mathcal{C} = C(\mathbb{R}, \mathbb{R})$.
- Note that these are not probability measures but infinite measures.

Theorem 3

$\{\mu^c\}_{c\in\mathbb{R}}$ *are invariant (stationary) under SHE* (4), *i.e.,*
$Z(0) \stackrel{law}{=} \mu^c \Rightarrow Z(t) \stackrel{law}{=} \mu^c$ *for all* $t \geq 0$ *and* $c \in \mathbb{R}$.
(or $E^{\mu^c}[f(Z(t))] =$ *const in* t *for a certain class of* f *on* $\mathcal{C}_+$*.)*

Corollary 4

$\{\nu^c\}_{c\in\mathbb{R}}$ *are invariant under the Cole-Hopf solution of the KPZ equation.*

- c means the average tilt (=slope) of the interface.
- We have different invariant measures for different average tilts.
- Reversibility does not hold, but a kind of Yaglom reversibility holds, cf. Remark above.

- (Scale invariance) If $Z(t,x)$ is a solution of SHE (4), then

$$Z^c(t,x) := e^{cx+\frac{1}{2}c^2 t} Z(t, x+ct)$$

is also a solution (with a new white noise). Therefore, once the invariance of μ^0 is shown, μ^c is also invariant for every $c \in \mathbb{R}$.

- Thus, we assume $c = 0$ and write $\mu = \mu^0$.
- Or, equivalently for $h(t,x)$, for every $c \in \mathbb{R}$,

$$h^c(t,x) := h_{CH}(t, x+ct) + cx + \tfrac{1}{2}c^2 t.$$

is a Cole-Hopf solution (with a new white noise).

- One expects μ^c, $c \in \mathbb{R}$ to be all extremal invariant measures (except constant multipliers), but this remains open; cf. F-Spohn for $\nabla\varphi$-interface model.

4. Proof of Theorem 3 and Corollary 4

4.1. Cole-Hopf transform for SPDE (6) (=Approximation-2)

- ν^ε in Proposition 1 converges to ν (=ν^0, i.e., $c = 0$, i.e. Wiener measure s.t. $\nu(B(0) \in dy) = dy$) as $\varepsilon \downarrow 0$.
- Therefore, our goal is to pass to the limit $\varepsilon \downarrow 0$ in the KPZ approximating equation (6):

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\left((\partial_x h)^2 - c^\varepsilon\right) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x).$$

- We consider its Cole-Hopf transform: $Z\,(\equiv Z^\varepsilon) := e^h$. Then, by Itô's formula, Z satisfies the SPDE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + A^\varepsilon(x,Z) + Z\dot{W}^\varepsilon(t,x), \tag{8}$$

where

$$A^\varepsilon(x,Z) = \frac{1}{2}Z(x)\left\{\left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon(x) - \left(\frac{\partial_x Z}{Z}\right)^2(x)\right\}.$$

- The term $A^\varepsilon(x,Z)$ looks vanishing as $\varepsilon \downarrow 0$.

- But this is not true. Indeed, under the average in time t, $A^\varepsilon(x, Z)$ can be replaced by a linear function $\frac{1}{24}Z$ (→ see Thm 5 below).
- The limit as $\varepsilon \downarrow 0$ (under stationarity of tilt),

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{24} Z + Z\dot{W}(t, x).$$

- Or, heuristically at KPZ level,

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \frac{1}{24} + \dot{W}(t, x).$$

- This shows for the solution $h^\varepsilon(t, x)$ of the KPZ approximating eq-2 (6):

$$h^\varepsilon \to h_{CH} + \frac{1}{24}t,$$

where $h_{CH} = h_{CH}(t, x)$ is the Cole-Hopf solution.
- "$+\frac{1}{24}t$" doesn't affect the invariant measure (→ see below)

4.2. Limit of $A^\varepsilon(x, Z)$ (Boltzmann-Gibbs principle)

- Asymptotic replacement of $A^\varepsilon(x, Z^\varepsilon(s))$ by $\frac{1}{24} Z^\varepsilon(s, x)$.
- To avoid the infiniteness of invariant measures, we view $h^\varepsilon(t, \rho) = \int h^\varepsilon(t, x)\rho(x)dx$ (height averaged by $\rho \in C_0^\infty(\mathbb{R}), \geq 0$, $\int \rho(x)dx = 1$) in modulo 1 (called wrapped process).

Theorem 5 (Boltzmann-Gibbs principle)

For every $\varphi \in C_0(\mathbb{R})$ satisfying supp $\varphi \cap$ supp $\rho = \emptyset$, we have that

$$\lim_{\varepsilon \downarrow 0} E^{\pi \otimes \nu^\varepsilon} \left[\left\{ \int_0^t ds \int_{\mathbb{R}} \Big(A^\varepsilon(x, Z^\varepsilon(s)) - \tfrac{1}{24} Z^\varepsilon(s, x) \Big) \varphi(x) dx \right\}^2 \right] = 0,$$

where π is the uniform measure for $h^\varepsilon(0, \rho) \in [0, 1)$ and ν^ε is the distribution of $B * \eta^\varepsilon$.

- Set

$$A^\varepsilon(\varphi, Z) := \int_{\mathbb{R}} \Big(A^\varepsilon(x, Z) - \tfrac{1}{24} Z(x) \Big) \varphi(x) dx$$

4.3. Proof of Theorem 5

(1) Reduction of equilibrium dynamic problem to static one:

- The expectation in Thm 5 is bounded by $(H^{-1}\text{-norm})^2$, which can be represented by a variational formula with $(H^1\text{-norm})^2$ = Dirichlet form:

$$E^{\pi\otimes\nu^\varepsilon}\Big[\Big\{\int_0^t ds\, A^\varepsilon(\varphi, Z_s^\varepsilon)\Big\}^2\Big]$$
$$\le Ct\|A^\varepsilon(\varphi, Z)\|_{-1,\varepsilon}^2 \quad (H^{-1}\text{-norm})$$
$$:= Ct \sup_{\Phi\in L^2(\pi\otimes\nu^\varepsilon)} \Big\{2E^{\pi\otimes\nu^\varepsilon}\left[A^\varepsilon(\varphi, Z)\Phi\right] - \langle \Phi, (-\mathcal{L}_0^\varepsilon)\Phi\rangle_{\pi\otimes\nu^\varepsilon}\Big\},$$

  where $\mathcal{L}_0^\varepsilon$ is the symmetric part of $\mathcal{L}^\varepsilon$. This is a generic bound in a stationary situation.
- In fact, roughly, writing $\mu = \pi\otimes\nu^\varepsilon$, $F = A^\varepsilon$,

$$E^\mu\Big[\Big\{\int_0^t ds\, F(Z_s)\Big\}^2\Big] = \int_0^t ds_1 \int_0^t ds_2 E^\mu[F(Z_{s_1})F(Z_{s_2})]$$
$$= 2\int_0^t ds_1 \int_0^{s_1} ds_2 E^\mu[F e^{(s_1-s_2)\mathcal{L}^\varepsilon} F]$$
$$\le 2t \int_0^\infty ds\, E^\mu[F e^{s\mathcal{L}_0^\varepsilon} F] = 2t\langle(-\mathcal{L}_0^\varepsilon)^{-1}F, F\rangle_\mu$$

Remark The above estimate can be extended to that on $E^{\mu}\left[\sup_{0\le t\le T}\left\{\int_0^t ds\, F(Z_s)\right\}^2\right]$ by the same H^{-1}-norm (with C changed), see Komorowski-Landim-Olla, "Fluctuations in Markov Processes", Springer, 2012, Lemma 2.4 (p.48). In the proof, backward martingale and Dynkin's formula are used to give a cancellation. This is sometimes called Itô-Tanaka trick. □

- Now we need to estimate the H^{-1}-norm, in which

$$2E^{\pi\otimes\nu^{\varepsilon}}\left[A^{\varepsilon}(\varphi,Z)\Phi\right]=E^{\pi}\left[Z_{\rho}E^{\nu^{\varepsilon}}\left[B^{\varepsilon}(\varphi,Z)\Phi(h(\rho),\nabla h)\right]\right],$$

where $Z_\rho=\exp\{\int_{\mathbb{R}}\log Z(x)\rho(x)dx\}$, $B^{\varepsilon}(x,Z)=2\frac{A^{\varepsilon}(x,Z)-\frac{1}{24}Z}{Z_\rho}$ and $B^{\varepsilon}(\varphi,Z)=\int_{\mathbb{R}}B^{\varepsilon}(x,Z)\varphi(x)dx$.

(2) The key is the following static bound:

- $\tilde{\mathcal{C}} := \mathcal{C}/\sim$, the quotient space of $\mathcal{C} = C(\mathbb{R}, \mathbb{R})$ under the equivalence relation $h \sim h + c$ for constants c.

Proposition 6

For $\Phi = \Phi(\nabla h) \in L^2(\tilde{\mathcal{C}}, \nu)$ s.t. $\|\Phi\|_{1,\varepsilon}^2 = \langle \Phi, (-\mathcal{L}_0^\varepsilon)\Phi\rangle_{\pi\otimes\nu^\varepsilon} < \infty$, and φ satisfying the condition of Theorem 5, we have that

$$\left|E^{\nu^\varepsilon}\left[B^\varepsilon(\varphi, Z)\Phi\right]\right| \le C(\varphi)\sqrt{\varepsilon}\|\Phi\|_{1,\varepsilon}, \tag{9}$$

with some positive constant $C(\varphi)$, which depends only on φ, for all ε: $0 < \varepsilon \le \frac{\delta}{4} \wedge 1$, where $\delta := \mathrm{dist}(\mathrm{supp}\,\varphi, \mathrm{supp}\,\rho)$.

- Once this proposition is shown, the proof of Theorem 5 is concluded, since the sup in the definition of H^{-1}-norm is bounded by

  $$\le Ct \sup_{\Phi}\{2eC(\varphi)\sqrt{\varepsilon}\|\Phi\|_{1,\varepsilon} - \|\Phi\|_{1,\varepsilon}^2\} = \mathrm{const}(\sqrt{\varepsilon})^2 \to 0.$$

- Recall $Z_\rho = e^{h(\rho)} \in [1, e]$ with $h(\rho) \in [0, 1]$.

Point of the proof of Proposition 6

(1) We first summarize Wiener-Itô chaos expansion ([FQ, p. 189∼])

- Recall that ν is the (two-sided) Wiener measure on $\widetilde{\mathcal{C}} := \mathcal{C}/\sim \;\cong \{B \in \mathcal{C}; B(0) = 0\}$, where $\mathcal{C} = C(\mathbb{R}, \mathbb{R})$.
  (Recall $h(\cdot) \sim h(\cdot) + c$.)
- Then, we have the orthogonal decomposition of $L^2(\widetilde{\mathcal{C}}, \nu)$:

  $L^2(\widetilde{\mathcal{C}}, \nu) = \oplus_{n=0}^{\infty} \mathcal{H}_n \cong \oplus_{n=0}^{\infty} \hat{L}^2(\mathbb{R}^n)$ (symmetric Fock space)
- Here, for $\varphi_n \in \hat{L}^2(\mathbb{R}^n)$, i.e. $\varphi_n \in L^2(\mathbb{R}^n)$ and symmetric in n-variables, define $I(\varphi_n)$ as the multiple Wiener integral:

$$I(\varphi_n) := \frac{1}{n!} \int_{\mathbb{R}^n} \varphi_n(x_1, \dots, x_n) dB(x_1) \cdots dB(x_n)$$

  and $\mathcal{H}_n$ is defined as

$$\mathcal{H}_n := \{I(\varphi_n) \in L^2(\widetilde{\mathcal{C}}, \nu); \varphi_n \in \hat{L}^2(\mathbb{R}^n)\}$$

  for $n \geq 1$ and $\mathcal{H}_0 := \{\text{const}\}$.
- $I(\varphi_n)$ is called *n*th order Wiener functional (chaos).

- Thus, for any $\Phi \in L^2(\widetilde{\mathcal{C}}, \nu)$, there exist $\varphi_n \in \hat{L}^2(\mathbb{R}^n)$, $n \geq 0$ such that

$$\Phi = \sum_{n=0}^{\infty} I(\varphi_n),$$

  and

  - $\|\Phi\|^2_{L^2(\nu)} = \sum_{n=0}^{\infty} \|I(\varphi_n)\|^2_{L^2(\nu)} = \sum_{n=0}^{\infty} \frac{1}{n!} \|\varphi_n\|^2_{L^2(\mathbb{R}^n)}$,
  - $(I(\varphi_n), I(\varphi_m))_{L^2(\nu)} \equiv E^\nu[I(\varphi_n) I(\varphi_m)] = 0 \quad \text{if} \quad n \neq m.$

  hold.
- Diagram formula ([FQ, Lemma 3.9]) gives the chaos expansion of the product $I(\varphi_{n_1}) \cdots I(\varphi_{n_m})$ ($\rightarrow$ see below).
- In particular, one can compute the expectation of Φ as $E[\Phi] = I(\varphi_0) = \varphi_0$.

(2) Now we come to the proof of Proposition 6

- Recalling that $\partial_x h = \partial_x (B * \eta^\varepsilon)$ under ν^ε, by Itô's formula

$$(\partial_x h)^2 = \left\{ \int_{\mathbb{R}} \eta^\varepsilon (x-y) dB(y) \right\}^2 = \Psi^\varepsilon(x) + c^\varepsilon, \quad (10)$$

  where $\Psi^\varepsilon(x) = \int_{\mathbb{R}^2} \eta^\varepsilon(x-x_1)\eta^\varepsilon(x-x_2)dB(x_1)dB(x_2)$, which is a 2nd order Wiener functional (chaos).

- In particular, the renormalization constant c^ε can be expressed as

$$c^\varepsilon = E\left[\left(\int_{\mathbb{R}} \eta^\varepsilon(x-y) dB(y) \right)^2 \right] \left(= \|\eta^\varepsilon\|^2_{L^2(\mathbb{R})} \right).$$

  and it is sometimes denoted by $c_\epsilon^{\vee}$.

- Therefore, transforming $\left(\frac{\partial_x Z}{Z}\right)^2 - c^\varepsilon$ back to $(\partial_x h)^2 - c^\varepsilon$,

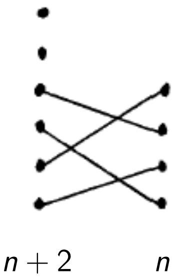
$$E^{\nu^\varepsilon}[B^\varepsilon(x,Z)\Phi] = E\left[2\frac{A^\varepsilon(x,Z) - \frac{1}{24}Z(x)}{Z_\rho}\Phi \right]$$

$$= E^{\nu^\varepsilon}\left[\frac{Z(x)}{Z_\rho} \left(\{\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)\} - \frac{1}{12} \right) \Phi \right],$$

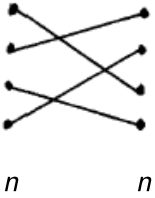
- To compute this expectation,
  - $\{\Psi^{\varepsilon} * \eta_2^{\varepsilon}(x) - \Psi^{\varepsilon}(x)\}$: 2nd order Wiener functional,
  - $\frac{1}{12}$: 0th order

  thus we need to pick up the 2nd order and 0th order terms of the products of two Wiener functionals $\frac{Z(x)}{Z_\rho} \times \Phi$.
- We apply the diagram formula to compute the Wiener chaos expansion of products of two functions.
- Diagrams γ to compute 2nd Wiener chaos and 0th order term in $\frac{Z(x)}{Z_\rho} \times \Phi$.

For 2nd Wiener chaos:



For 0th order term:



- [Chaos expansion of $\frac{Z(x)}{Z_\rho}$] Under ν,

$$\frac{Z(x)}{Z_\rho} = e^{B(x) - \int_{\mathbb{R}} B(y)\rho(y)dy}$$
$$= e^{a(x)} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \phi_x^{\otimes n}(u_1, \dots, u_n) dB(u_1) \cdots dB(u_n) \right\},$$

where,

$$\phi_x(u) = 1_{(-\infty, x]}(u) - \int_u^{\infty} \rho(y)dy,$$
$$a(x) = \tfrac{1}{2} \int_{\mathbb{R}} \phi_x(u)^2 du.$$

- Note that the kernel ϕ_x has jump.
- $\|\Phi\|_{1,\varepsilon}^2$ can be expressed by (∞-dimensional) Dirichlet form ($\to$ [FQ, Lemma 3.8]).
- Further details are left to [FQ].

(3) We only give a remark on the constant $\frac{1}{24}$

- The same factor $\frac{1}{24}$ $(24 = 4!)$ appears in several KPZ related papers such as [Bertini-Giacomin 1997], [Borodin-Corwin-Ferrari 2012], ...
- For general convolution kernel η, this constant is given by $J/2$, where

$$J = P(R_1 + R_3 > 0, R_2 + R_3 > 0) - P(R_1 > 0, R_2 > 0),$$

  and $\{R_i\}_{i=1}^3$ are i.i.d. r.v.s distributed under $\eta_2(x)dx$
- If η is symmetric,

$$\begin{aligned} P(R_1 + R_3 > 0, R_2 + R_3 > 0) &= P(R_1 - R_3 > 0, R_2 - R_3 > 0) \\ &= P(R_3 = \min R_i) = \tfrac{1}{3}, \end{aligned}$$

  so that $J = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.
- If the support of $\eta \subset [0, \infty)$ (or $\subset (-\infty, 0]$), then $J = 0$.
- See the next page for the reason that the above quantity J appears.

- Recall (10) to see that $\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)$ is 2nd order Wiener chaos with kernel:

$$2\Big\{ \int \eta^\varepsilon(y-x_1)\eta^\varepsilon(y-x_2)\eta_2^\varepsilon(x-y)dy - \eta^\varepsilon(x-x_1)\eta^\varepsilon(x-x_2)\Big\}.$$

- The product and sum (in n) of $(n+2)$th order chaos of $\frac{Z(x)}{Z_\rho}$, nth order chaos of Φ and the above quantity (2nd order) produces the quantity J. (Recall the kernel ϕ_x in $\frac{Z(x)}{Z_\rho}$ has jump.)
- This cancels with 0th order term $\frac{1}{12}E^{\nu^\varepsilon}\big[\frac{Z(x)}{Z_\rho}\Phi\big]$.
- The product and sum of $n \leftrightarrow \frac{Z(x)}{Z_\rho}$, $n+2 \leftrightarrow \Phi$ and above quantity $\leftrightarrow 2$ is bounded by the square root of Dirichlet form $\|\Phi\|_{1,\varepsilon}$. (End of the proof of Prop 6) □

4.4. Proof of Theorem 3 and Corollary 4

- Wrapping can be removed by showing uniform estimate:

$$\sup_{0<\varepsilon<1} E\left[\sup_{0\le t\le T} h^{\varepsilon}(t,\rho)^2\right] < \infty.$$

  Namely, height cannot move very fast. This is shown only on a torus (since we need Poincaré inequality).
- Under the stationary situation of the tilt processes, in the limit, we obtain the SHE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \tfrac{1}{24} Z + Z\dot{W}(t,x). \tag{11}$$

- This looks different from the original SHE (4), but the solution Z_t of (11) gives the solution $\tilde{Z}_t$ of (4) under the simple transformation $\tilde{Z}_t := e^{-\frac{t}{24}} Z_t$.
- This implies the invariance of the distribution of the geometric Brownian motion for the tilt process determined by the SHE (4), and therefore that of BM for Cole-Hopf solution.

- The above argument combined with Proposition 1 at approximating level shows the invariance of μ for tilt processes. ($\to$ Theorem 3)
- To rewrite this to the height processes h_t, we introduce the transformation $h^\varepsilon(x, Z) := \log(Z * \eta^\varepsilon(x))$. Then, the evolution of $h^\varepsilon(x, Z_t)$ is governed only by the tilt variables and the initial data $h^\varepsilon(x, Z_0)$. ($\to$ Corollary 4)

- Hoshino, SPA **128**, 2018 proved the convergence of the solutions of Approximating Eq-1 and Approximating Eq-2 as $\varepsilon \downarrow 0$ in non-stationary setting by applying paracontrolled calculus ($\to$ see also Lecture No 4 in coupled KPZ equation setting.)

5. Remarks from the viewpoint of interacting particle systems

- The stationary measure of the SHE (4) can be obtained by particle system approximation.
  - $\sigma_t = \{\sigma_t(i)\} \in \{\pm 1\}^{\mathbb{Z}}$: WASEP (with weak asymmetry $\varepsilon^{\frac{1}{2}}$)
  - ζ_t: height (or summed) process with height difference σ_t, sometimes called SOS-dynamics,
  - ξ_t^ε: (Discrete) Cole-Hopf transform of ζ_t scaled in space and time.
- [Bertini-Giacomin '97] showed that $\xi_t^\varepsilon(x) \Rightarrow Z_t(x)$, the solution of SHE (4) weakly as $\varepsilon \downarrow 0$ ($\rightarrow$ Lecture No 1).
- WASEP σ_t has Bernoulli product measure on $\mathcal{X} = \{\pm 1\}^{\mathbb{Z}}$ as its stationary measure

  $\Rightarrow$ $\lim_{\varepsilon \downarrow 0} \zeta_t^\varepsilon$ (fluctuation scaling limit, i.e., CLT) should have Wiener measure as its stationary measure (as ζ_t is the sum of σ_t).

  $\Rightarrow$ Z_t should have the distribution of geometric BM as its stationary measure.

Summary of this lecture.

1. KPZ equation:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{R}.$$

2. KPZ approximating equation-2 with $W^\varepsilon(t,x) = \langle W(t), \eta^\varepsilon(x-\cdot)\rangle$:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\left((\partial_x h)^2 - c^\varepsilon\right) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x)$$

   has invariant measure ν^ε (=distribution of $B * \eta^\varepsilon$).

3. Cole-Hopf transform $Z := e^h$ leads to the SPDE:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \tfrac{1}{2} Z \left\{ \left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon - \left(\frac{\partial_x Z}{Z}\right)^2 \right\} + Z\dot{W}^\varepsilon(t,x)$$

4. As $\varepsilon \downarrow 0$, one can replace the middle term by $\frac{1}{24}Z$ under time average and get the SPDE in the limit:

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + \tfrac{1}{24} Z + Z\dot{W}(t,x), \quad x \in \mathbb{R}.$$