KPZ limit for interacting particle systems —Supplementary materials—

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Lecture No 2

Plan of the course (10 lectures)

1 Introduction

2 Supplementary materials

Brownian motion, Space-time Gaussian white noise, (Additive) linear SPDEs, (Finite-dimensional) SDEs, Martingale problem, Invariant/reversible measures for SDEs, Martingales

3 Invariant measures of KPZ equation (F-Quastel, 2015)

4 Coupled KPZ equation by paracontrolled calculus (F-Hoshino, 2017)

5 Coupled KPZ equation from interacting particle systems (Bernardin-F-Sethuraman, 2020+)

- 5.1 Independent particle systems
- 5.2 Single species zero-range process
- 5.3 *n*-species zero-range process
- 5.4 Hydrodynamic limit, Linear fluctuation
- 5.5 KPZ limit=Nonlinear fluctuation

Plan of this lecture

Supplementary materials

1. Brownian motion
2. Construction of space-time Gaussian white noise
3. (Additive) Linear SPDEs
4. (Finite-dimensional) SDEs, their invariant measures, reversible measures
5. Martingales

1. Brownian motion

- Brownian motion is a fundamental object in stochastic analysis. In our case, it will be used to construct space-time Gaussian white noise. It also appears as an invariant measure of KPZ equation.

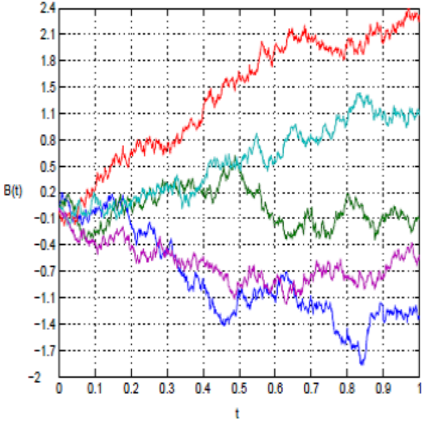
[Definition] (Brownian motion) An $\mathbb{R}$-valued process $B = (B_t)_{t \geq 0} = (B_t(\omega))_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called a Brownian motion if

(1) $B_0 = 0$ a.s.

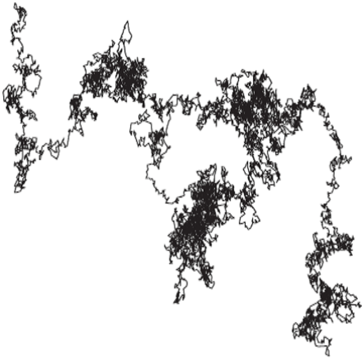
(2) $B_t(\omega)$ is continuous in t for ${}^\forall \omega \in \Omega$

(3) For every $0 = t_0 <^\forall t_1 < \cdots <^\forall t_n, {}^\forall n \in \mathbb{N}$, the increments $\{B_{t_i} - B_{t_{i-1}}\}_{1 \leq i \leq n}$ are independent and distributed under $N(0, t_i - t_{i-1})$ (i.e. Gaussian, mean 0, variance $t_i - t_{i-1}$).

A function $X : \Omega \to \mathbb{R}$ which is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable is called a random variable. A collection of $\mathbb{R}$-valued random variables $X = \{X(t)\}_{t \geq 0}$ defined on a probability space (so that $X(t) = X(t, \omega)$) is called a stochastic process or process.



5 trials of BMs



2D BM

- The condition (3) is equivalent to

$$P(B_{t_i} - B_{t_{i-1}} \in A_i, 1 \le i \le n) = \int_{A_1} dx_1 \int_{A_2} dx_2 \cdots \int_{A_n} dx_n \prod_{i=1}^{n} p(t_i - t_{i-1}, x_i)$$

for ${}^\forall A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$, where $p(t, x)$ is the heat kernel:

$$p(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \quad t > 0,\ x \in \mathbb{R}.$$

- Or under the transformation $x_i = y_i - y_{i-1}, 1 \le i \le n$ with $y_0 = 0$, this is further equivalent to

$$P(B(t_i) \in A_i, 1 \le i \le n) = \int_{A_1} dy_1 \int_{A_2} dy_2 \cdots \int_{A_n} dy_n \prod_{i=1}^{n} p(t_i - t_{i-1}, y_{i-1}, y_i)$$

for ${}^\forall A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$, where

$$p(t, x, y) := p(t, x - y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \quad t > 0,\ x, y \in \mathbb{R}.$$

- $p(t, x, y)$ is called the transition probability (density) of the BM.

- The distribution of Brownian motion on the path space $\mathcal{C} := C([0,\infty),\mathbb{R})$ is called the Wiener measure.
- In other words, the Wiener measure is the image measure of P (on Ω) under the map $\Omega \ni \omega \mapsto B(\omega) = (B_t(\omega))_{t\geq 0} \in \mathcal{C}$.

- The property

$$E[(B_t - B_s)^2] = |t-s|$$

or

$$E[(B_t - B_s)^{2n}] = C_n|t-s|^n, \quad n \in \mathbb{N}$$

roughly implies $\frac{1}{2}$-Hölder continuity of B_t in t.

- More precisely, the modulus of continuity of BM is given by

$$\limsup_{\substack{t_2 - t_1 = \varepsilon \downarrow 0 \\ 0 \leq t_1 < t_2 \leq 1}} \frac{|B_{t_2} - B_{t_1}|}{\sqrt{2\varepsilon \log 1/\varepsilon}} = 1 \quad \text{a.s.}$$

- Brownian motion has a (diffusive) scale invariance: $B^c := (cB_{t/c^2})_{t\geq 0}$ has the same distribution as B for all $c \neq 0$.

- B_t is a martingale, i.e., $E[B_t|\mathcal{F}_s^B] = B_s$ if $t \geq s \geq 0$ w.r.t. the natural filtration $(\mathcal{F}_t^B)_{t\geq 0}$ of BM i.e. $\mathcal{F}_t^B := \sigma\{B_s; 0 \leq s \leq t\}$ ($\rightarrow$ see below).
- Its quadratic variation is given by $\langle B \rangle_t = t$, i.e. $B_t^2 - t$ is a martingale ($\rightarrow$ see below).

- B_t is neither differentiable nor of bounded variation, so that the (Stieltjes-)integral $\int_0^t f(s,\omega)dB_s$ can not be defined in a usual sense.

Stochastic integral

- It is definable only in stochastic (Itô's) sense. Roughly,

$$\int_0^t f(s,\omega)dB_s := \lim_{|\Delta|\to 0} \sum_{i=1}^{n} f(s_{i-1},\omega)\big(B_{s_i}(\omega) - B_{s_{i-1}}(\omega)\big),$$

  in $L^2(\Omega)$, where $\Delta = \{0 = s_0 < s_1 < \cdots < s_n = t\}$ is a division of the interval $[0,t]$ and $|\Delta| = \max_i(s_i - s_{i-1})$.
- $M_t := \int_0^t f(s,\omega)dB_s$ is a martingale ($\to$ see below).
- Itô isometry:

$$E[M_t^2] = \int_0^t E[f^2(s)]ds$$

- Or, the quadratic variation of M_t is given by

$$\langle M\rangle_t = \int_0^t f^2(s)ds$$

  (i.e. $M_t^2 - \langle M\rangle_t$ is a martingale $\to E[M_t^2 - \langle M\rangle_t] = 0$ $\to$ Itô isometry).

- The formal derivative $\dot{B}_t$ of B_t (though it is not differentiable) called the white noise is δ-correlated:

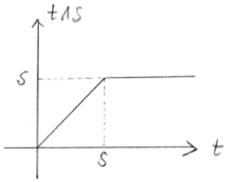
$$E[\dot{B}_t \dot{B}_s] = \delta(t-s) \quad (= \delta_0(t-s)).$$

- Heuristically, since $E[B_t B_s] = t \wedge s = G(t,s)$, taking the derivative in t, we would have

$$E[\dot{B}_t B_s] = 1_{(0,s]}(t) = 1_{[t,\infty)}(s).$$

Next, taking the derivative in s,

$$E[\dot{B}_t \dot{B}_s] = \frac{d}{ds} 1_{[t,\infty)}(s) = \delta_t(s) = \delta(t-s).$$



2. Construction of space-time Gaussian white noise

- Take $\{\psi_k\}_{k=1}^{\infty}$: CONS of $L^2(D, dx)$, $D \subset \mathbb{R}^d$ or $\mathbb{T}^d$, and $\{B_t^k\}_{k=1}^{\infty}$: independent 1D BMs, and consider a formal Fourier series:

$$W(t,x) = \sum_{k=1}^{\infty} B_t^k \psi_k(x). \tag{1}$$

  (This doesn't converge in $L^2(D)$.)

- Then, by independence of B^k and $E[B_t^k B_s^k] = t \wedge s$, one would expect to have that

$$E[W(t,x)W(s,y)] = \sum_{k=1}^{\infty} (t \wedge s)\psi_k(x)\psi_k(y) = (t \wedge s)\delta(x-y).$$

- Thus, as we saw $\frac{\partial}{\partial s}\frac{\partial}{\partial t}(t \wedge s) = \delta(t-s)$ to derive $E[\dot{B}_t \dot{B}_s] = \delta(t-s)$, the time derivative $\dot{W}(t,x) := \frac{\partial}{\partial t} W(t,x)$ would have the covariance structure:

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y). \tag{2}$$

- One can define $W(t,\cdot)$ as an H-valued process by properly taking a Hilbert space $H\,(\supset L^2(D))$.

3. (Additive) Linear SPDEs

3.1. Regularity of solutions of linear SPDE on $\mathbb{T}^d$ or $\mathbb{R}^d$

- Consider the linear SPDE, dropping nonlinear term in KPZ equation, on $\mathbb{T}^d$:

$$\partial_t h = \tfrac{1}{2}\Delta h + \dot{W}(t,x), \quad x \in \mathbb{T}^d.$$

- Then, $h(t,x) \in C^{\frac{2-d}{4}-,\frac{2-d}{2}-} \left(:= \bigcap_{\delta>0} C^{\frac{2-d}{4}-\delta,\frac{2-d}{2}-\delta} \right)$ a.s.
- In fact, regularity in x is seen as follows. Let $\{\psi_k\}_{k=1}^\infty, \{\lambda_k\}_{k=1}^\infty$ be normalized eigenfunctions (CONS of $L^2(\mathbb{T}^d)$) and corresponding eigenvalues of $-\Delta$.
- Then it is well-known (Weyl's law): $\lambda_k \sim k^{2/d}$ as $k \to \infty$.
- We define Sobolev norms for $s \in \mathbb{R}$:

$$\|h\|_{H^s}^2 := ((1-\Delta)^s h, h)_{L^2} = \sum_{k=1}^\infty (1+\lambda_k)^s (h,\psi_k)_{L^2}^2.$$

- $h_k(t) := (h(t), \psi_k)_{L^2}$ satisfy SDEs ($\to$ see below):

$$dh_k(t) = -\tfrac{1}{2}\lambda_k h_k(t)dt + dB_k(t)$$

with independent Brownian motions $\{B_k := (W(t), \psi_k)_{L^2}\}_k$, and this can be solved as (Duhamel's formula)

$$h_k(t) = e^{-\frac{1}{2}\lambda_k t} h_k(0) + \int_0^t e^{-\frac{1}{2}\lambda_k (t-s)} dB_k(s).$$

- Assuming $h(0) = 0$ for simplicity, by Itô isometry, we have

$$E\left[\|h(t)\|_{H^s}^2\right] = E\left[\sum_k (1+\lambda_k)^s \int_0^t e^{-\lambda_k(t-s)} ds\right]$$
$$\sim \sum_k \frac{(1+\lambda_k)^s}{\lambda_k} \sim \sum_k k^{\frac{2}{d}(s-1)}$$

Thus

$$E\left[\|h(t)\|_{H^s}^2\right] < \infty \;\Leftrightarrow\; \tfrac{2}{d}(s-1) < -1 \;\Leftrightarrow\; s < \tfrac{2-d}{2}.$$

- The linear SPDE is well-posed only when $d = 1$ and in this case, we have $h \in C^{\frac{1}{4}-,\frac{1}{2}-}([0,\infty)\times\mathbb{T})$ a.s. as we mentioned in Lecture No 1.

3.2 Higher order SPDEs (generalization of linear SPDEs)

- Let us consider linear stochastic PDEs (OU processes) on $\mathbb{R}^d$ replacing $\frac{1}{2}\partial_x^2$ by A and dropping nonlinear term:

$$\partial_t h = Ah + \dot{W}(t,x), \quad x \in \mathbb{R}^d. \tag{3}$$

- $\dot{W}(t,x)$ is the space-time Gaussian white noise on $\mathbb{R}^d$.
- $A = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C_b^\infty(\mathbb{R}^d)$, $m \in \mathbb{N}$, $D^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^d}\right)^{\alpha_d}$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$.
- The coefficients satisfy the uniform ellipticity condition:

$$\inf_{x,\sigma\in\mathbb{R}^d, |\sigma|=1} (-1)^{m+1} \sum_{|\alpha|=2m} a_\alpha(x)\sigma^\alpha > 0,$$

  where $\sigma^\alpha = \sigma_1^{\alpha_1} \cdots \sigma_d^{\alpha_d}$ for $\sigma = (\sigma_1, \dots, \sigma_d) \in \mathbb{R}^d$.
- It is expected that "larger m" implies better regularity.

- The solution of (3) is defined in a generalized functions' sense (by multiplying test functions $\varphi \in C_0^\infty(\mathbb{R})$) or in a mild form (via Duhamel's principle):

$$h(t) = e^{tA}h(0) + \int_0^t e^{(t-s)A} dW(s).$$

  The last term is defined as a stochastic integral.
- We can show that, if $2m > d$,

$$h(t,x) \in C^{\alpha-,\beta-}((0,\infty)\times\mathbb{R}^d), \quad \text{a.s.},$$

  where $\alpha = \frac{2m-d}{4m}$ and $\beta = \frac{2m-d}{2}$.
- If $A = \Delta$, then $m = 1$ and $\alpha = \frac{2-d}{4}, \beta = \frac{2-d}{2}$. This recovers the result in §3.1.

- The necessity of the condition "$2m > d$" can be seen from

$$E\left[\left\{\int_0^t e^{(t-s)A}dW(s)\right\}^2\right] = \int_0^t ds \int_{\mathbb{R}^d} p^2(t-s,x,y)dy$$
$$= \int_0^t p(2s,x,x)ds \sim \int_0^t s^{-\frac{d}{2m}}ds < \infty \quad \text{iff} \quad d < 2m,$$

  where $p(t,x,y)$ is the kernel of the integral operator e^{tA} (cf. F, Osaka J. Math, 1991)
- For the first line, we applied the Itô isometry for the stochastic integrals w.r.t. $W(t)$:

$$E\left[\left\{\int_0^t \int_{\mathbb{R}^d} \varphi(s,y,\omega)dW(s,y)\right\}^2\right] = E\left[\int_0^t ds \int_{\mathbb{R}^d} \varphi^2(s,y,\omega)dy\right].$$

4. (Finite-dimensional) SDEs, its invariant measures, reversible measures

4.1 Stochastic differential equations (SDEs)

- Let the followings be given:

$$\alpha = (\alpha_{ij}(x))_{i,j=1}^d : \mathbb{R}^d \to \mathbb{R}^{d\times d} \ (d \times d \text{ matrices})$$
$$b = (b_i(x))_{i=1}^d : \mathbb{R}^d \to \mathbb{R}^d \ (\text{vector field on } \mathbb{R}^d)$$
$$B_t = (B_t^j)_{j=1}^d : \ d\text{-dimensional Brownian motion}$$

- Consider SDE for $X_t = (X_t^i)_{i=1}^d \in \mathbb{R}^d$:

$$dX_t = \alpha(X_t)dB_t + b(X_t)dt,$$

or componentwisely written as

$$dX_t^i = \sum_{j=1}^d \alpha_{ij}(X_t)dB_t^j + b_i(X_t)dt, \quad 1 \le i \le d$$

- More precisely, X_t is defined by means of the stochastic integral equation:

$$X_t^i = X_0^i + \sum_{j=1}^{d} \int_0^t \alpha_{ij}(X_s) dB_s^j + \int_0^t b_i(X_s) ds, \quad 1 \leq i \leq d.$$

- Similarly to ODEs, if the coefficients α, b are (globally) Lipschitz continuous, the SDE has a unique (strong = pathwise) solution, that is, $(\mathcal{F}_t^B)$-adapted (measurable) solution, where $\mathcal{F}_t^B := \sigma\{B_s; 0 \leq s \leq t\}$ is the natural filtration of BM.
- Define the generator associated with the SDE as

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i},$$

where $a_{ij}(x) := \sum_{k=1}^{d} \alpha_{ik}(x) \alpha_{jk}(x)$ or $a = \alpha\alpha^*$ as a matrix.

- For $f \in C^2(\mathbb{R}^d)$, by Itô's formula (especially with Itô correction term $\frac{1}{2}\cdots$) noting $dB_t^i dB_t^j = \delta^{ij} dt$, we have

$$\begin{aligned} df(X_t) &= \sum_i \partial_{x_i} f(X_t) dX_t^i + \tfrac{1}{2} \sum_{i,j} \partial_{x_i}\partial_{x_j} f(X_t) dX_t^i dX_t^j \\ &= \sum_i \partial_{x_i} f(X_t) dX_t^i + \tfrac{1}{2} \sum_{i,j} \partial_{x_i}\partial_{x_j} f(X_t) \sum_k \alpha_{ik}(X_t)\alpha_{jk}(X_t) dt \\ &= Lf(X_t)dt + \sum_{i,j} \partial_{x_i} f(X_t)\alpha_{ij}(X_t) dB_t^j. \end{aligned}$$

- This means (Dynkin's formula)

$$f(X_t) = f(X_0) + \int_0^t Lf(X_s)ds + M_t(f),$$

where

$$M_t(f) := \sum_{i,j} \int_0^t \partial_{x_i} f(X_s)\alpha_{ij}(X_s) dB_s^j$$

is given as a stochastic integral, so that it is a martingale ($\rightarrow$ see below).

4.2 Martingale problem

- In particular, under the law $\mathbb{P}$ of $X = (X_t)_{t\geq 0}$ on the path space $\mathcal{C} = C([0,\infty), \mathbb{R}^d)$,

$$f(w_t) - f(x_0) - \int_0^t Lf(w_s)ds$$

  is a martingale (w.r.t. the natural filtration) for every $f \in C^2(\mathbb{R}^d)$, where $w = (w_t)_{t\geq 0}$ denotes an element of $\mathcal{C}$.
- A probability measure $\mathbb{P}$ on $\mathcal{C}$, which has this property, is called the solution of L-martingale problem.
- [Stroock-Varadhan] If $a(x) = (a_{ij}(x))$ is (bounded and) continuous and uniformly positive definite, and b is (bounded and) measurable, then the L-martingale problem has a unique solution.

4.3 Invariant measures, reversible measures

- μ: invariant measure

$$\underset{\text{def}}{\iff} E^{\mu}[f(X_0)] = E^{\mu}[f(X_t)], \quad {}^{\forall} f \in C_b(\mathbb{R}^d)$$

i.e., law of X_t is invariant in t.
E^{μ} means the initial distribution of $X_t = \mu$.

- Invariant measure appears as a limit law of X_t as $t \to \infty$, so it is important to study.
- μ: reversible measure

$$\underset{\text{def}}{\iff} E^{\mu}[f(X_0)g(X_t)] = E^{\mu}[g(X_0)f(X_t)], \quad {}^{\forall} f, g$$

i.e., law of (X_0, X_t) = law of (X_t, X_0).

- This (combined with Markov property) implies reversibility: For every $T > 0$, laws on the path space $C([0, T], \mathbb{R}^d)$ of two processes $\{X_t\}_{t \in [0,T]}$ and $\{X_{T-t}\}_{t \in [0,T]}$ are the same.
- reversible $\Rightarrow$ invariant

- μ: infinitesimally invariant

$$\underset{\text{def}}{\iff} E^\mu[Lf(X_0)] = \int_{\mathbb{R}^d} Lf(x)\mu(dx) = 0, \ {}^\forall f \in \mathcal{D}(L)(\supset C_b^2(\mathbb{R}^d))$$

- μ: infinitesimally reversible

$$\underset{\text{def}}{\iff} \int_{\mathbb{R}^d} g(x)Lf(x)\mu(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\mu(dx), \ {}^\forall f, g \in \mathcal{D}(L)$$

- invariant $\Rightarrow$ infinitesimally invariant
- Indeed, by Dynkin's formula (or Itô's formula as we saw)

$$0 \underset{\text{martingale}}{=} E^\mu[M_t(f)] \underset{\text{by invariance}}{=} \int_0^t E^\mu[Lf(X_s)]ds$$

Take the derivative in t, then we have the inf. invariance:

$$0 = E^\mu[Lf(X_0)]$$

- Converse is also known. i.e.
  "invariance $\Leftrightarrow$ inf. invariance" under some condition,
    e.g., Echeveria's result (under the well-posedness of
      the martingale problem).

- reversible $\Rightarrow$ inf. reversible
- "reversible $\Leftrightarrow$ inf. reversible" under some condition,
    e.g., Fukushima-Stroock's result

- Example: $V \in C^1(\mathbb{R}^d)$ is given, and consider

$$dX_t = -\tfrac{1}{2}\nabla V(X_t)dt + dB_t$$

$$L = \tfrac{1}{2}\Delta - \tfrac{1}{2}\nabla V \cdot \nabla \quad \left(= \tfrac{1}{2}\sum_{i=1}^{d} \tfrac{\partial^2}{\partial x_i^2} - \tfrac{1}{2}\sum_{i=1}^{d} \tfrac{\partial V}{\partial x_i}\tfrac{\partial}{\partial x_i} \right)$$

$$L^*\Phi = \frac{1}{2}\sum_{i=1}^{d} \tfrac{\partial}{\partial x_i}\left(\tfrac{\partial \Phi}{\partial x_i} + \tfrac{\partial V}{\partial x_i}\Phi \right) = 0 \qquad \text{for } \Phi = e^{-V}$$

- Dirichlet form approach:

$$\begin{aligned} \mathcal{D}(f,g) :=& \frac{1}{2}\int \nabla f \cdot \nabla g \, e^{-V} dx \\ =& -\int f \, Lg \, e^{-V} dx \\ =& -\int g \, Lf \, e^{-V} dx, \end{aligned}$$

- In particular, reversibility of $\mu = e^{-V}dx$ for X_t follows.

- Taking a matrix $A = (\alpha_{ij})_{1\le i,j\le d}$, we modify the Dirichlet form as

$$\tilde{\mathcal{D}}(f,g) := \frac{1}{2}\int A\nabla f \cdot A\nabla g \, e^{-V} dx$$
$$= -\int f \, \tilde{L} g \, e^{-V} dx,$$

  where
$$\tilde{L} g = \frac{1}{2} A^* A \Delta g - \frac{1}{2} A^* A \nabla V \cdot \nabla g.$$

- (Fluctuation-dissipation relation) The corresponding SDE is changed as

$$dY_t = -\frac{1}{2} A^* A \nabla V(Y_t) dt + A dB_t.$$

- $\mu = e^{-V} dx$ is reversible also for Y_t.
- This will be applied in Lecture 3.
- In SPDEs, this idea is applied for
  TDGL (time-dependent Ginzburg-Landau) equation:
  non-conservative type $\longleftrightarrow$ conservative type
  Stoch Allen-Cahn equation $\longleftrightarrow$ Stoch Cahn-Hilliard eq

5. Martingales

5.1. Definition

- $(\Omega, \mathcal{F}, P)$: Probability space
- $(\mathcal{F}_t) \equiv (\mathcal{F}_t)_{t\geq 0}$: filtration (or reference family)

  $\underset{\text{def}}{\Longleftrightarrow}$ · Each $\mathcal{F}_t$ is a sub σ-filed of $\mathcal{F}$

  · increasing in t, i.e. $0 \leq s < t \Longrightarrow \mathcal{F}_s \subset \mathcal{F}_t$

  · right continuous, i.e., For ${}^{\forall}t \geq 0$, $\mathcal{F}_t = \mathcal{F}_{t+}$, where $\mathcal{F}_{t+} := \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$
- $X = (X_t)_{t\geq 0}$: a stochastic process such that, for ${}^{\forall}\omega \in \Omega$, $t \in [0, \infty) \mapsto X_t(\omega) \in \mathbb{R}$ is right continuous and has left limits at each t. Such process is called càdlàg (continue à droite limites à gauche).

[Definition] X is called $(\mathcal{F}_t)$-martingale, if it satisfies

(1) $(\mathcal{F}_t)$-adapted: For ${}^\forall t \geq 0$, X_t is $\mathcal{F}_t$-measurable (in ω).

(2) Integrable: For ${}^\forall t \geq 0$, $E[|X_t|] < \infty$.

(3) For $0 \leq {}^\forall s < {}^\forall t$, $E[X_t | \mathcal{F}_s] = X_s \quad a.s.$

In (3), if $E[X_t|\mathcal{F}_s] \geq X_s$ $a.s.$ holds, X is called sub-martingale. If $E[X_t|\mathcal{F}_s] \leq X_s$ $a.s.$, it is called super-martingale. □

- Note that (3) $\iff E[X_t, A] = E[X_s, A]$ for every $A \in \mathcal{F}_s$, where $E[X, A] := \int_A X dP$.
- For martingales/sub-martingales, Doob's (maximal) inequality, Burkholder's inequality, Doob's optional sampling theorem, Sub-martingale convergence theorem, Doob-Meyer decomposition of sub-martingales are known. ($\rightarrow$ Ikeda-Watanabe, Karatzas-Shreve, Revuz-Yor, Le Gall)

5.2. Useful properties

- (Dynkin's formula) Let L be the generator of (jump) Markov process η_t on $\mathcal{X}$. For a function f on $\mathcal{X}$,

$$M_t(f) := f(\eta_t) - \int_0^t Lf(\eta_s)ds \quad (\text{or} - f(\eta_0))$$

is a martingale (with respect to natural filtration).
- (cross variation) For functions f, g on $\mathcal{X}$, the cross-variation of $M_t(f)$ and $M_t(g)$ is given by

$$\langle M(f), M(g)\rangle_t = \int_0^t \{L(fg) - f\,Lg - g\,Lf\}(\eta_s)ds$$

i.e., $M_t(f)M_t(g) - \langle M(f), M(g)\rangle_t$ is a martingale.
- Taking $f = g$, this implies that the quadratic variation of $M_t(f)$ is given by

$$\langle M(f)\rangle_t = \int_0^t \{Lf^2 - 2f\,Lf\}(\eta_s)ds$$

i.e., $M_t(f)^2 - \langle M(f)\rangle_t$ is a martingale.

- Note that two different definitions of quadratic variation are known for jump process $M_t(f)$. The other one is

$$[M(f)]_t = \lim_{|\Delta|\to 0} \sum_{i=1}^{n} \{M_{t_i}(f) - M_{t_{i-1}}(f)\}^2,$$

  where $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ is a division of the interval $[0, t]$ and $|\Delta| = \max_{1 \le i \le n} |t_i - t_{i-1}|$.
- In general, $[M(f)]_t \neq \langle M(f) \rangle_t$ in case with jumps.