KPZ limit for interacting particle systems —Introduction—

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Lecture No 1

Plan of the course (10 lectures)

1 Introduction

2 Supplementary materials

   Brownian motion, Space-time Gaussian white noise, (Additive) linear SPDEs, (Finite-dimensional) SDEs, Martingale problem, Invariant/reversible measures for SDEs, Martingales

3 Invariant measures of KPZ equation (F-Quastel, 2015)

4 Coupled KPZ equation by paracontrolled calculus (F-Hoshino, 2017)

5 Coupled KPZ equation from interacting particle systems (Bernardin-F-Sethuraman, 2020+)

   5.1 Independent particle systems

   5.2 Single species zero-range process

   5.3 n-species zero-range process

   5.4 Hydrodynamic limit, Linear fluctuation

   5.5 KPZ limit=Nonlinear fluctuation

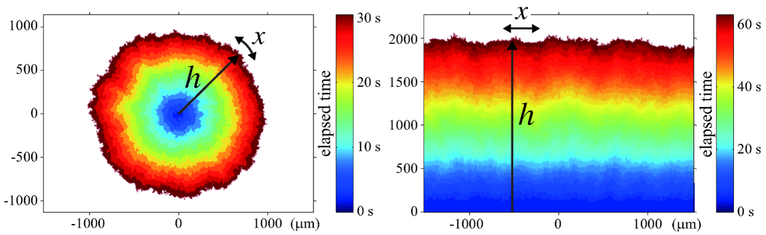
Plan of Lecture No 1

Introduction to the course

1. KPZ equation
2. Heuristic derivation of KPZ equation (following the original KPZ paper, 1986)
3. Reason for KPZ equation to attract a lot of attention
4. Ill-posedness, Renormalization
5. Cole-Hopf solution, Multiplicative linear stochastic heat equation, Itô's formula
6. KPZ equation from interacting particle systems (WASEP)
7. Quick overview of the course

1. KPZ equation

- The KPZ (Kardar-Parisi-Zhang, 1986) equation describes the motion of growing interface with random fluctuation.



(Takeuchi-Sano-Sasamoto-Spohn)

- (Right Fig) $h = h(t, x) \in \mathbb{R}$ denotes height of interface measured from the x-axis at time t and position x.
- Video of combustion experiment by Laser shot: srep00034-s2.mov, srep00034-s3.mov (Takeuchi-Sano)

- KPZ is an equation for height function $h(t,x)$:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{T} \ \text{(or } \ \mathbb{R}). \quad (1)$$

  where $\mathbb{T} \equiv \mathbb{R}/\mathbb{Z} = [0,1)$.
- We consider in 1D on a whole line $\mathbb{R}$ or on a finite interval $\mathbb{T}$ under periodic boundary condition.
- The coefficients $\frac{1}{2}$ are not important, since we can change them under some scaling.
- $\dot{W}(t,x)$ is a space-time Gaussian white noise with mean 0 and covariance structure:

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y). \qquad (2)$$

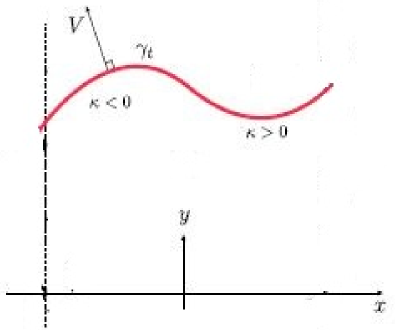
- This means that the noise is independent if (t,x) is different, since "Gaussian property+0-correlation" means independence.
- However, $\dot{W}(t,x)$ is realized only as a generalized function (distribution).

2. Heuristic derivation of KPZ equation

- We give a derivation of KPZ equation following the original KPZ paper 1986.
- Consider a motion of interface (curve) growing upward with normal velocity:

$$V = \kappa + A,$$

where κ is the (signed) curvature and $A > 0$ is a constant.

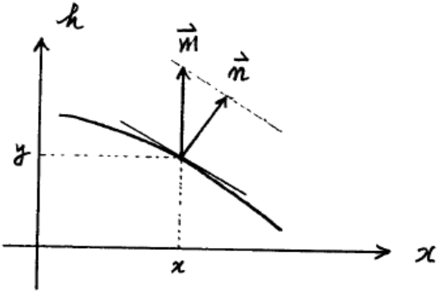


- The interface dynamics can be described by an equation for its height function $h(t,x)$ assuming that the interface in $\mathbb{R}^2$ is represented as a graph:

$$\gamma_t = \{(x,y); y = h(t,x), x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

- The dynamics "$V = \kappa + A$" can be rewritten into the following nonlinear PDE for $h(t,x)$

$$\partial_t h = \frac{\partial_x^2 h}{1+(\partial_x h)^2} + A(1+(\partial_x h)^2)^{1/2} \qquad (3)$$



- Indeed, (3) can be derived as follows.
- First, note that the normal vector $\vec{n}$ to the curve
  $\gamma_h = \{(x, y); y = h(x), x \in \mathbb{R}\} \subset \mathbb{R}^2$
  at the point (x, y) is given by

$$\vec{n} = \frac{1}{\left(1 + (\partial_x h(x))^2\right)^{1/2}} \begin{pmatrix} -\partial_x h(x) \\ 1 \end{pmatrix}$$

  pf) $\vec{n} \perp \begin{pmatrix} 1 \\ \partial_x h(x) \end{pmatrix}$ (= tangent vector to γ_h) and $|\vec{n}| = 1$. □

- The interface growth to the direction $\vec{n}$ is equivalent to the growth of the height function h to the vertical direction $\vec{m}$, where

$$\vec{m} = \begin{pmatrix} 0 \\ \left(1 + (\partial_x h(x))^2\right)^{1/2} \end{pmatrix}$$

  pf) We can check $(\vec{m} - \vec{n}) \perp \vec{n}$ □

- The curvature of the curve $\gamma_h = \{y = h(x)\}$ at (x, y) is given by
$$\kappa = \frac{\partial_x^2 h(x)}{\left(1 + (\partial_x h(x))^2\right)^{3/2}}.$$
- Summarizing these observations, the interface growing equation $V = \kappa + A$ can be written as
$$\partial_t h = \left\{ \frac{\partial_x^2 h}{(1 + (\partial_x h)^2)^{3/2}} + A \right\} (1 + (\partial_x h)^2)^{1/2},$$
i.e. we obtain (3):
$$\partial_t h = \frac{\partial_x^2 h}{1 + (\partial_x h)^2} + A(1 + (\partial_x h)^2)^{1/2},$$
for the height function $h = h(t, x)$.

- If we consider $\tilde{h} := h - At$ instead of h by subtracting the constant growth factor At and write h for $\tilde{h}$ again, we obtain that

$$\partial_t h = \frac{\partial_x^2 h}{1 + (\partial_x h)^2} + A\left\{(1 + (\partial_x h)^2)^{1/2} - 1\right\}$$
$$\simeq \partial_x^2 h + \frac{A}{2}(\partial_x h)^2,$$

i.e.

$$\partial_t h = \partial_x^2 h + \frac{A}{2}(\partial_x h)^2,$$

at least if $|\partial_x h|$ is small, i.e., if we take the leading effect of this equation.

- Note that $u := \partial_x h$ is a solution of (viscous) Burgers equation:

$$\partial_t u = \partial_x^2 u + \frac{A}{2}\partial_x u^2.$$

- Kardar-Parisi-Zhang equation (KPZ, 1986) is obtained by taking the fluctuation effect due to space-time independent noise $\dot{W}(t,x)$ into account:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t,x).$$

- Here $h = h(t,x,\omega)$ and $\dot{W}(t,x) = \dot{W}(t,x,\omega)$ is the space-time Gaussian white noise defined on a certain probability space $(\Omega, \mathcal{F}, P)$ with mean 0 and covariance structure

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(x-y)\delta(t-s).$$

- We took $A = 1$ and put $\frac{1}{2}$ in front of $\partial_x^2 h$.
- Only leading terms are taken in the equation.
- This simplification is essential in view of the scaling property or universality related to the KPZ equation.

Mathematically, everything is built on a probability space $(\Omega, \mathcal{F}, P)$, i.e. Ω is a set, $\mathcal{F}$ is a σ-field of Ω, P is a measure on $(\Omega, \mathcal{F})$ s.t. $P(\Omega) = 1$.

3. Reason for KPZ equation to attract a lot of attention

- $\frac{1}{3}$-power law (instead of $\frac{1}{2}$-law in usual CLT): Fluctuation of height function at a single point $x = 0$:

$$h(t,0) \asymp c_1 t + c_2 t^{\frac{1}{3}} \zeta_{TW},$$

  in particular, $\mathrm{Var}(h(t,0)) = O(t^{\frac{2}{3}})$, as $t \to \infty$, i.e. the fluctuations of $h(t,0)$ are of order $t^{\frac{1}{3}}$. Subdiffusive behavior different from CLT (=diffusive behavior).
- The limit distribution of $h(t,0)$ under scaling is given by the so-called Tracy-Widom distribution ζ_{TW} (different depending on initial distributions). (instead of Gaussian distribution in CLT)
- KPZ universality class, 1:2:3 scaling, KPZ fixed point
- Integrable Probability

- Singular ill-posed SPDEs:
  - Hairer: Regularity structures, KPZ equation, dynamic $P(\phi)_d$-model, Parabolic Anderson model
  - Gubinelli-Imkeller-Perkowski: Paracontrolled calculus (harmonic analytic method)
  - The solution map is continuous in "$\dot{W}^\varepsilon$ and their (finitely many) polynomials".
  - Renormalization is required (called subcritical case).
- Microscopic interacting particle systems
  - Bertini-Giacomin (1997) was the first to this direction.
  - This is one of main purposes of this course.

4. Ill-posedness, Renormalization

- Nonlinearity and roughness of noise conflict with each other.
- $\dot{W}(t,x) \in C^{-\frac{d+1}{2}-} := \bigcap_{\delta>0} C^{-\frac{d+1}{2}-\delta}$ a.s. if $x \in \mathbb{T}^d$ or $\mathbb{R}^d$.
  (Construction will be discussed later $\to$ Lecture No 2).
- C^α: (Hölder-)Besov space with exponent $\alpha \in \mathbb{R}$.
- The linear SPDE ($d = 1$): (Schauder effect)
  $$\partial_t h = \frac{1}{2}\partial_x^2 h + \dot{W}(t,x), \quad x \in \mathbb{T}$$
  obtained by dropping the nonlinear term has a solution $h \in C^{\frac{1}{4}-,\frac{1}{2}-}([0,\infty)\times\mathbb{T}) := \bigcap_{\delta>0} C^{\frac{1}{4}-\delta,\frac{1}{2}-\delta}([0,\infty)\times\mathbb{T})$ a.s.
  (This will be discussed later $\to$ Lecture No 2).
- Therefore, no way to define the nonlinear term $(\partial_x h)^2$ in (1) in a usual sense.
- Actually, it requires a renormalization. The following Renormalized KPZ equation with compensator $\delta_x(x)$ ($= +\infty$) would have a meaning (cf. Cole-Hopf solution):
  $$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x).$$

5. Cole-Hopf solution, Multiplicative linear stochastic heat equation, Itô's formula

- Recall classical Cole-Hopf (Hopf-Cole) transformation: Let u be a solution of viscous Burgers equation:

$$\partial_t u = \tfrac{1}{2}\partial_x^2 u + \tfrac{1}{2}\partial_x u^2 + \partial_x \zeta(t,x),$$

with smooth ζ. Then, $Z(t,x) := e^{\int_{-\infty}^x u(t,y)dy}$ solves the linear heat equation

$$\partial_t Z = \tfrac{1}{2}\partial_x^2 Z + Z\zeta.$$

- In fact,

$$\begin{aligned}\partial_t Z &= Z \cdot \int_{-\infty}^x \partial_t u(t,y)dy \\ &= Z \cdot (\tfrac{1}{2}\partial_x u + \tfrac{1}{2}u^2 + \zeta),\end{aligned}$$

while

$$\begin{aligned}\partial_x^2 Z &= \partial_x(uZ) = \partial_x u \cdot Z + u \cdot \partial_x Z \\ &= \partial_x u \cdot Z + u^2 \cdot Z.\end{aligned}$$

- This leads to the above heat equation for Z.

- Motivated by this and regarding $u = \partial_x h$, consider the (multiplicative) linear stochastic heat equation (SHE) for $Z = Z(t, x, \omega)$:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x), \quad x \in \mathbb{R}, \tag{4}$$

  with a multiplicative noise (defined in Itô's sense).
- The solution $Z(t)$ of (4) can be defined in a generalized functions' sense or in a mild form (Duhamel's formula):

$$Z(t,x) = \int_{\mathbb{R}} p(t,x,y)Z(0,y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x,y)Z(s,y)dW(s,y),$$

  where $p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)}$ is the heat kernel.
- (4) in Itô's sense is well-posed ($\rightarrow$ see next page)

- SHE (4) defined in Stratonovich sense:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z \circ \dot{W}(t,x)$$

  is ill-posed. ($\rightarrow$ see below)

- These two notions of solutions (in generalized functions or mild) are equivalent, and $^{\exists}$unique solution s.t. $Z(t) \in C([0,\infty), \mathcal{C}_{\mathrm{tem}})$ a.s., where

$$\mathcal{C}_{\mathrm{tem}} = \{Z \in C(\mathbb{R},\mathbb{R}); \|Z\|_r < \infty, ^{\forall} r > 0\},$$
$$\|Z\|_r = \sup_{x\in\mathbb{R}} e^{-r|x|}|Z(x)|.$$

- (Strong comparison) If $Z(0,x) \geq 0$ for $^{\forall}x \in \mathbb{R}$ and $Z(0,x) > 0$ for $^{\exists}x \in \mathbb{R}$, then $Z(t) \in C((0,\infty), \mathcal{C}_+)$ a.s., where $\mathcal{C}_+ = C\big(\mathbb{R}, (0,\infty)\big)$.
- Therefore, we can define the Cole-Hopf transformation:

$$h(t,x) := \log Z(t,x). \tag{5}$$

Heuristic derivation of the KPZ eq (with renormalization factor $\delta_x(x)$) from SHE (4) under the Cole-Hopf transformation (5):

- (Finite-dimensional) Itô's formula:

$$df(X_t) = f'(X_t)dX_t + \tfrac{1}{2}f''(X_t)(dX_t)^2$$

  for example, for $X_t = B_t$, $(dB_t)^2 = dt$.
- In infinite-dimensional setting,

$$dW(t,x)dW(t,y) = \delta(x-y)dt \; (= \delta_x(y)dt)$$

- By Itô's formula, taking $f(z) = \log z$ under the C-H transformation (5), we have

$$dh(t,x) = f'(Z(t,x))dZ(t,x) + \tfrac{1}{2}f''(Z(t,x))(dZ(t,x))^2.$$

- Note $f'(z) = (\log z)' = z^{-1}$, $f''(z) = (\log z)'' = -z^{-2}$.
- Note also from SHE (4),

$$(dZ(t,x))^2 = (Z(t,x)dW(t,x))^2 = Z^2(t,x)\delta_x(x)dt.$$

- Therefore, writing $\partial_t h$ for $\frac{dh(t,x)}{dt}$, we obtain

$$\begin{aligned}\partial_t h &= Z^{-1}\partial_t Z - \tfrac{1}{2}Z^{-2}Z^2\delta_x(x)\\ &= Z^{-1}\left(\tfrac{1}{2}\partial_x^2 Z + Z\dot{W}\right) - \tfrac{1}{2}\delta_x(x) \quad \text{(by SHE (4))}\\ &= \tfrac{1}{2}Z^{-1}\partial_x^2 Z + \dot{W} - \tfrac{1}{2}\delta_x(x).\end{aligned}$$

- However, since $h = \log Z$, a simple computation (as we already saw for $u = \partial_x h$) shows

$$Z^{-1}\partial_x^2 Z = \partial_x^2 h + (\partial_x h)^2 \qquad (= \partial_x u + u^2).$$

- This leads to the KPZ eq with renormalization factor:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \dot{W}(t,x). \qquad (6)$$

- The function $h(t,x)$ defined by (5) is meaningful and called the Cole-Hopf solution of the KPZ equation, although the equation (1) does not make sense.
- Problem: To introduce approximations for (6), in particular, well adapted to finding invariant measures. ($\rightarrow$ F-Quastel, Lecture No 3)
- Hairer gave a meaning to (6) without bypassing SHE.

- Itô's formula for Stratonovich integral has no Itô correction term (i.e. the term with $\frac{1}{2}$). If SHE defined in Stratonovich sense were well-posed, we would obtain well-posed KPZ equation. But, this is not true.

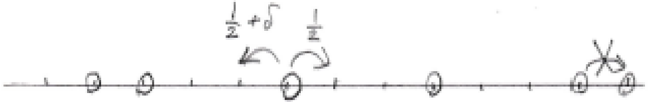
6. KPZ equation from interacting particle systems

- One of our interests is to derive KPZ(-Burgers) equation from microscopic particle systems.
- Bertini-Giacomin (1997): Derivation of Cole-Hopf solution of KPZ equation from WASEP (weakly asymmetric simple exclusion process)
- For WASEP, Cole-Hopf transformation works even at microscopic level (Gärtner).

6.1 WASEP (weakly asymmetric simple exclusion process)

- WASEP (on $\mathbb{Z}$) is a collection of infinite particles on $\mathbb{Z}$.
- Each particle performs simple random walk with jump rates $\frac{1}{2}$ to the right and $\frac{1}{2} + \delta$ to the left, under the exclusion rule that at most one particle can occupy each site, where $\delta > 0$ is a small parameter (weak asymmetry).
- Configuration space: $\mathcal{X} = \{+1, -1\}^{\mathbb{Z}}$
- $\sigma = \{\sigma(x)\}_{x \in \mathbb{Z}} \in \mathcal{X}$ and

$$\sigma(x) = \left.\begin{matrix} +1 \\ -1 \end{matrix}\right\} \iff \begin{cases} \exists \text{ particle at } x \\ \text{no particle at } x \end{cases}$$

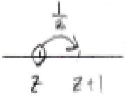


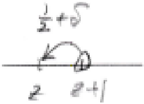
- $\sigma^{x,y} \in \mathcal{X}$ denotes a new configuration after exchanging variables at x and y (i.e., if there is a particle at x and no particle at y, $\sigma^{x,y}$ is the configuration after the particle at x jumps to y. Or a particle at y jumps to x if x is vacant.)

$$\sigma^{x,y}(z) = \begin{cases} \sigma(y), & \text{if } z = x, \\ \sigma(x), & \text{if } z = y, \\ \sigma(z), & \text{otherwise.} \end{cases}$$

- (Infinitesimal) rate of transition $\sigma \mapsto \sigma^{z,z+1}$, when the whole configuration is σ, is given by

$$c_{z,z+1}(\sigma) = \frac{1}{2}1_{\{\sigma(z)=1,\sigma(z+1)=-1\}} + (\frac{1}{2}+\delta)1_{\{\sigma(z)=-1,\sigma(z+1)=1\}}.$$





- Generator: For a function f on $\mathcal{X}$,

$$Lf(\sigma) = \sum_{z\in\mathbb{Z}} c_{z,z+1}(\sigma)\{f(\sigma^{z,z+1}) - f(\sigma)\}.$$

- The rate $c_{z,z+1}$ can be decomposed as follows.
- The rate that a particle makes a jump:
  $\lambda = 1 + \delta \left(= \frac{1}{2} + \left(\frac{1}{2} + \delta\right)\right)$
- When a jump occurs,

$$p_+ = \frac{\frac{1}{2}}{1+\delta} \quad : \quad \text{probability of jump to the right}$$
$$p_- = \frac{\frac{1}{2}+\delta}{1+\delta} \quad : \quad \text{probability of jump to the left}$$

Note that $p_+ + p_- = 1$ (i.e., $p_\pm$ is a probability), by normalizing $c_{z,z+1}$ by λ.

6.2 Construction of interacting particle systems (in general)

- Particle system is a continuous-time (jump) Markov process $\sigma_t \equiv \sigma_t(\omega)$ on a configuration space $\mathcal{X}$ of particles.
- Once infinitesimal rate $c(\sigma)$ governing the random motion of particles is given, one can construct σ_t as follows.
- [Distributional construction]
  - $c(\sigma)$ determines the generator of Markov process L
  - We can construct corresponding semigroup e^{tL} on $C(\mathcal{X})$.
  - By Markov property, e^{tL} determines finite-dimensional distributions (joint distributions of Markov process at finitely many times).
  - By Kolmogorov's extension theorem+regularization of paths, this determines the distribution of the Markov process on the path space $D([0,\infty),\mathcal{X})$, which denotes the Skorohod space allowing jumps of functions.

Liggett, Interacting Particle Systems, Springer, 1985.

- [Pathwise construction]
  - Each particle has its own "bell". Bells are independent and ring according to the exponential holding time:
    $$P(T > t) = e^{-\lambda t}, \quad t \geq 0,\ \lambda > 0.$$
    Since $E[T] = \frac{1}{\lambda}$, "large λ" means that the bell rings quickly. We write $T \overset{d}{=} \exp(\lambda)$.
  - λ for each particle is determined from infinitesimal rate $c(\sigma)$. (For WASEP, $\lambda = 1 + \delta$)
  - When first bell rings, the corresponding particle makes a jump to a place chosen by certain probability $\{p\}$. (For WASEP, $\{p_\pm\}$)
  - After this jump, whole system refreshes with all bells, and repeats the procedure.
  - We usually consider infinite particle system, and this requires careful construction of the system.

6.3 Hydrodynamic limit (LLN)

- WASEP $\sigma_t = (\sigma_t(x))_{x\in\mathbb{Z}}$ is constructed by the above recipe from $c_{z,z+1}(\sigma)$ with weak asymmetry δ.
- We first study the hydrodynamic limit (HDL) for the WASEP σ_t taking $\delta = \varepsilon$, where ε is the ratio of microscopic/macroscopic spatial sizes.
- As we will see, scalings in δ are different for HDL/KPZ.
- Consider the macroscopic empirical measure of σ_t defined by small-mass and space-time-diffusive scaling:

$$X_t(du) = \varepsilon \sum_{x\in\mathbb{Z}} \sigma_{\varepsilon^{-2}t}(x)\delta_{\varepsilon x}(du), \quad u \in \mathbb{R},$$

or equivalently, for a test function $\varphi \in C_0^\infty(\mathbb{R})$,

$$\langle X_t, \varphi\rangle = \varepsilon \sum_{x\in\mathbb{Z}} \sigma_{\varepsilon^{-2}t}(x)\varphi(\varepsilon x).$$

Theorem 1

$$X_t(du) \underset{\varepsilon \downarrow 0}{\longrightarrow} \alpha(t,u)du \quad \textit{(in prob)},$$

where $\alpha(t,u)$ is a solution of viscous Burgers equation:

$$\partial_t \alpha = \tfrac{1}{2}\partial_u^2 \alpha + \tfrac{1}{2}\partial_u (1-\alpha^2).$$

If $\alpha = \partial_u m$, the equation for m is

$$\partial_t m = \tfrac{1}{2}\partial_u^2 m + \tfrac{1}{2}(1-(\partial_u m)^2).$$

(KPZ type but without noise)

F-Sasada, CMP **299**, 2010

F, Lectures on Random Interfaces, SpringerBriefs, 2016, Theorem 2.7 for relation to Vershik curve (introducing boundary).

Heuristic derivation of the limit equation

- To show this theorem, we use Dynkin's formula (→ Lecture No 2):

$$\langle X_t, \varphi \rangle = \langle X_0, \varphi \rangle + \int_0^t \varepsilon^{-2} \cdot \varepsilon \sum_x (L\sigma)_{\varepsilon^{-2}s}(x) \varphi(\varepsilon x) ds + M_t^\varepsilon(\varphi).$$

- ε^{-2} comes from the time change.
- The contribution of the martingale term $M_t^\varepsilon(\varphi)$ vanishes in the limit as $\varepsilon \downarrow 0$. (In Lecture No 2, we will explain martingale.)

- For the term with integral, we can compute as

$$\begin{aligned}
&\varepsilon^{-1}\sum_x L\sigma(x)\varphi(\varepsilon x)\\
&=\frac{\varepsilon^{-1}}{2}\sum_x \sigma(x)\Big[\Big\{\varphi(\varepsilon(x+1))-\varphi(\varepsilon x)\Big\}-\Big\{\varphi(\varepsilon x)-\varphi(\varepsilon(x-1))\Big\}\Big]\\
&\quad-\varepsilon^{-1}\cdot 2{\color{red}\varepsilon}\sum_x 1_{\sigma(x+1)=1,\sigma(x)=-1}\Big\{\varphi(\varepsilon(x+1))-\varphi(\varepsilon x)\Big\}\\
&=\frac{\varepsilon^{-1}}{2}\sum_x \sigma(x)\,\varepsilon^2\big(\varphi''(\varepsilon x)+O(\varepsilon)\big)\\
&\quad-\varepsilon^{-1}\cdot 2{\color{red}\varepsilon}\sum_x 1_{\sigma(x+1)=1,\sigma(x)=-1}\,\varepsilon\big(\varphi'(\varepsilon x)+O(\varepsilon)\big).
\end{aligned}$$

- Red ${\color{red}\varepsilon}$ was originally δ. Other ε's are from the definition of X_t.
- Note that the RHS is now $O(1)$ in ε, though it still contains nonlinear microscopic function.
- This is called the gradient property of the model.

- From the above computation, the drift term is rewritten as
$$\frac{1}{2}\langle X_t, \varphi''\rangle - \varepsilon \sum_x A_x(\sigma_{\varepsilon^{-2}t})\varphi'(\varepsilon x) + O(\varepsilon),$$
where $A_x(\sigma) = 2 1_{\sigma(x+1)=1, \sigma(x)=-1}$.
- By the assumption of the local equilibrium, we can expect $\sigma_{\varepsilon^{-2}t}(\cdot) \overset{\text{law}}{=} \nu_{\alpha(t,u)}$ asymptotically as $\varepsilon \downarrow 0$, where ν_α is the Bernoulli measure on $\{\pm 1\}^{\mathbb{Z}}$ with mean $\alpha \in [-1, 1]$.
- In particular, $\nu_\alpha(\sigma(0) = 1) = \frac{\alpha+1}{2}$, $\nu_\alpha(\sigma(0) = -1) = \frac{1-\alpha}{2}$.
- Bernoulli product measures are invariant (and reversible) measures of the leading SEP of WASEP (or its symmetrization).
- Thus, by assuming local ergodicity, one can replace $A_x(\sigma)$ by its local average with proper α:
$$E^{\nu_\alpha}[A_x] = 2 \cdot \frac{\alpha+1}{2} \cdot \frac{1-\alpha}{2} = \frac{1}{2}(1-\alpha^2).$$
- We obtain the HD equation (closed equation) for $\alpha(t, u)$
$$\partial_t \alpha = \frac{1}{2}\alpha'' + \frac{1}{2}(1-\alpha^2)'.$$

6.4 Equilibrium linear fluctuation (CLT)

- We consider the fluctuation of WASEP with asymmetry $\delta = \varepsilon$ (same as HDL) under the global equilibrium ν_α around its mean α:

$$Y_t^\varepsilon(du) = \sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} \big(\sigma_{\varepsilon^{-2}t}(x) - \alpha \big) \delta_{\varepsilon x}(du),$$

- Non-equilibrium fluctuation: F-Sasada-Sauer-Xie, SPA **123**, 2013.

Theorem 2

$Y_t^\varepsilon \to Y_t$ and Y_t is a solution of linear SPDE:

$$\partial_t Y = \frac{1}{2}\partial_u^2 Y - \alpha\partial_u Y + \sqrt{1-\alpha^2}\partial_u \dot{W}(t,u)$$

- Heuristically, this SPDE follows by observing

$$\sigma - \alpha = \sqrt{\varepsilon} Y \quad (\text{since } \sqrt{\varepsilon} = \tfrac{\varepsilon}{\sqrt{\varepsilon}} \text{ in } Y_t^\varepsilon)$$

$$E^{\nu_{\alpha+\sqrt{\varepsilon}Y}}[A] - E^{\nu_\alpha}[A] = \frac{1}{2}(1-(\alpha+\sqrt{\varepsilon}Y)^2) - \frac{1}{2}(1-\alpha^2)$$

$$\sim -\sqrt{\varepsilon}\alpha Y \quad (\to \text{fluctuation of drift term})$$

- Noise term is the same as KPZ as we will discuss.

6.5 KPZ limit (Nonlinear fluctuation)

- We consider the fluctuation of WASEP with asymmetry $\delta = \sqrt{\varepsilon}$ under the global equilibrium ν_α:

$$Y_t^\varepsilon(du) = \sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} \left(\sigma_{\varepsilon^{-2}t}(x) - \alpha\right) \delta_{\varepsilon x - c\varepsilon^{-1/2}t}(du),$$

- Fluctuation is observed under moving frame with macroscopic speed $c\varepsilon^{-1/2}$ (to cancel diverg. linear term).
- Choose $c = \alpha$.

Theorem 3

$Y_t^\varepsilon \to Y_t$ and Y_t is a solution of KPZ-Burgers equation:

$$\partial_t Y = \tfrac{1}{2}\partial_u^2 Y - \tfrac{1}{2}\partial_u Y^2 + \sqrt{1-\alpha^2}\,\partial_u \dot{W}(t,u).$$

If h_t is determined as $Y_t = \partial_u h_t$, then h_t satisfies the KPZ equation (more precisely, its Cole-Hopf solution)

$$\partial_t h = \tfrac{1}{2}\partial_u^2 h - \tfrac{1}{2}(\partial_u h)^2 + \sqrt{1-\alpha^2}\,\dot{W}(t,u).$$

- By the similar computation to above, we have

$$\begin{aligned}\langle Y_t, \varphi \rangle =& \langle Y_0, \varphi \rangle + \int_0^t \varepsilon^{-2} \cdot \sqrt{\varepsilon} \sum_x (L_{\sqrt{\varepsilon}} \sigma)_{\varepsilon^{-2} s}(x) \varphi(\varepsilon x - c\varepsilon^{-1/2} s) ds \\ & - \int_0^t c \sum_x \left(\sigma_{\varepsilon^{-2} s}(x) - \alpha \right) \varphi'(\varepsilon x - c\varepsilon^{-1/2} s) ds + M_t^\varepsilon(\varphi),\end{aligned}$$

  where $M_t^\varepsilon(\varphi)$ is a martingale different from that in HDL (but asymptotically the same as that appears in linear fluctuation).
- For the martingale M_t^ε, under the equilibrium ν_α,

$$E[M_t^\varepsilon(\varphi)^2] \sim \varepsilon t(1-\alpha^2) \sum_x \varphi'(\varepsilon x)^2 \sim t(1-\alpha^2) \|\varphi'\|_{L^2(\mathbb{R})}^2.$$

  ($\rightarrow$ see Lecture No 2 for quadratic variation of M)
- This means $M_t^\varepsilon \rightarrow \sqrt{1-\alpha^2} \partial_u W(t,u)$.
- $W(t,u)$ is an integral of $\dot{W}(t,u)$ in t.

- The first term in the drift is

$$\begin{aligned}
&\varepsilon^{-2} \cdot \sqrt{\varepsilon} \sum_x L_{\sqrt{\varepsilon}} \sigma(x) \varphi(\varepsilon x - c\varepsilon^{-1/2} t) \\
&\qquad = \varepsilon^{-2} \cdot \frac{\sqrt{\varepsilon}}{2} \sum_x \sigma(x)\, \varepsilon^2 \Big(\varphi''(\varepsilon x - c\varepsilon^{-1/2} t) + O(\varepsilon) \Big) \\
&\qquad\quad - \varepsilon^{-2} \cdot \sqrt{\varepsilon} \cdot \color{red}{\sqrt{\varepsilon}} \sum_x A_x(\sigma)\, \varepsilon \Big(\varphi'(\varepsilon x - c\varepsilon^{-1/2} t) + O(\varepsilon) \Big).
\end{aligned}$$

- Red $\color{red}{\sqrt{\varepsilon}} = \delta$ originally. Other $\sqrt{\varepsilon}$ comes from that in the definition of Y_t^ε.
- The first term is $\frac{1}{2}\langle Y_t, \varphi'' \rangle$ by noting that $\sum_x \alpha \Delta \varphi = 0$.

- The second term (after all ε cancel) is still diverging. But, we can expect by the local ergodicity (Boltzmann-Gibbs principle= combination of local averaging due to local ergodicity and Taylor expansion)

$$
\begin{aligned}
&A_x(\sigma) \sim E^{\nu_{\alpha+\sqrt{\varepsilon}Y_t(\varepsilon x - c\varepsilon^{-1/2}t)}}\Big[A_x(\sigma)\Big] \\
&= \tfrac{1}{2}\Big(1-(\alpha+\sqrt{\varepsilon}Y_t(\varepsilon x - c\varepsilon^{-1/2}t))^2\Big) \\
&= \tfrac{1}{2}(1-\alpha^2) - \alpha\sqrt{\varepsilon}Y_t(\varepsilon x - c\varepsilon^{-1/2}t) - \tfrac{1}{2}\varepsilon Y_t^2(\varepsilon x - c\varepsilon^{-1/2}t).
\end{aligned}
$$

- Thus, one can expect that this term behaves as

$$
\varepsilon^{-\frac{1}{2}}\alpha Y_t(\varphi') + \tfrac{1}{2}\langle Y_t^2, \varphi'\rangle
$$

since $\sum_x \frac{1}{2}(1-\alpha^2)\varphi' = 0$.

- The first term cancels with the second term in the drift $\simeq -\varepsilon^{-\frac{1}{2}}cY_t(\varphi')$ (originally from moving frame) if we choose the frame speed $c = \alpha$, and one would obtain $\frac{1}{2}\langle Y_t^2, \varphi'\rangle$ in the limit.

- Therefore, in the limit we would have the KPZ-Burgers equation
$$\partial_t Y = \frac{1}{2}\partial_u^2 Y - \frac{1}{2}\partial_u Y^2 + \sqrt{1-\alpha^2}\partial_u \dot{W}(t,u).$$
- Note: For Y, renormalization is unnecessary, since one would have $\partial_u\{\delta_u(u)\} = \partial_u\{\text{const}\} = 0$.
- The above derivation is heuristic.
- Bertini-Giacomin relied on microscopic Cole-Hopf transformation for the proof.
- Roughly, consider the process
$$\zeta_t^\varepsilon(x) := \exp\Big\{ -\gamma_\varepsilon \sum_{y=x_0(t)}^{x} \sigma_t(y) - \lambda_\varepsilon t \Big\}$$
and show that ζ_t^ε converges to the solution Z_t of SHE in a proper scaling. $x_0(t)$ is a properly chosen point defined by the position of a tagged particle. See F, Lectures on Random Interfaces, p.56 for this transformation.
- $\sum_{x_0(t)}^{x} \sigma(y)$ corresponds to the height process.

6.6 Other models

Derivation of scalar KPZ (-Burgers) equation

- Bertini-Giacomin (as discussed above): Derivation from WASEP (weakly asymmetric simple exclusion process), Cole-Hopf transformation (even at microscopic level).
- Goncalves-Jara, Goncalves-Jara-Sethuraman: Derivation from general WAEP with speed change of gradient type and with Bernoulli invariant measures, or from WA zero-range process (of gradient type).
- Method: 2nd order Boltzmann-Gibbs principle, martingale formulation (called energy solutions).
- Gubinelli-Perkowski: Uniqueness of stationary energy solutions (satisfying Yaglom reversibility, i.e., - (nonlinear drift term) for time reversed process).

Derivation of coupled KPZ (-Burgers) equation

- We will discuss later.

7. Quick overview of the course

1. Introduction
2. Supplementary materials

   Brownian motion, Space-time Gaussian white noise, (Additive) linear SPDEs, (Finite-dimensional) SDEs, Martingale problem, Invariant/reversible measures for SDEs, Martingales
3. Invariant measures of KPZ equation (F-Quastel)
4. Coupled KPZ equation by paracontrolled calculus (F-Hoshino)
5. Coupled KPZ equation from interacting particle systems (Bernardin-F-Sethuraman)

   5.1 Independent particle systems

   5.2 Single species zero-range process

   5.3 n-species zero-range process

   5.4 Hydrodynamic limit, Linear fluctuation

   5.5 KPZ limit=Nonlinear fluctuation