Lectures on Algebraic Geometry

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Lecture 5: Cremona groups

Introduction

A way to understand a variety X is to study the groups:

$$\mathrm{Aut}(X) = \{ X \xrightarrow{\text{isom}} X \}$$

$$\mathrm{Aut}(X) \leq \mathrm{Bir}(X).$$

$$\mathrm{Bir}(X) = \{ X \overset{\text{bir isom}}{\dashrightarrow} X \}$$

The group structures are given by composing maps.

These are especially important when $X = \mathbb{P}^n$.

A rational map

$$\mathbb{P}^n \overset{\varphi}{\dashrightarrow} \mathbb{P}^n$$

is given as

$$\varphi(x) = (F_0(x) : \cdots : F_n(x))$$

for some rational functions $F_0, \ldots, F_n$. Or we can choose the F_i to be homog. of same degree.

φ is bir if it has an inverse.

Definition: The Cremona group $C_n := \mathrm{Bir}(\mathbb{P}^n)$.

we can also define C_n in a purely algebraic way.

Let $t_1, \ldots, t_n$ be the coordinate functions on $\mathbb{A}^n$.

the function field

$$K(\mathbb{P}^n) = K(\mathbb{A}^n) = \mathbb{C}(t_1, \ldots, t_n).$$

We have a group isom

$$C_n \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(t_1, \ldots, t_n))$$

$$\mathbb{A}^n \overset{bir}{\dashrightarrow} \mathbb{A}^n$$

$$K(\mathbb{A}^n) \xleftarrow{\text{isom}} K(\mathbb{A}^n)$$

Example: $\mathrm{Aut}(\mathbb{P}^1) = PGL_2$ = group of 2×2 invertible matrices modulo multiplication by non-zero constant.

Indeed if $\mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^1$ is an isom, then

$\varphi = (F_0, F_1)$ where $\deg F_0 = \deg F_1 = 1$

and F_0, F_1 are linearly independent over $\mathbb{C}$,

so $F_0 = ax + by$, $F_1 = cx + dy$

giving $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PGL_2$.

On the other hand, since $\mathbb{P}^1$ is a smooth proj curve,

$$C_1 = \mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1) = PGL_2.$$

Note that some elements of C_1 have finite order,

e.g. $\varphi = (x : -y)$ but some have infinite order,

e.g. $\psi = (x+y : y)$.

Example: It is a well-known fact that

$$\mathrm{Aut}(\mathbb{P}^n) = PGL_{n+1}.$$

But

$$C_n = \mathrm{Bir}(\mathbb{P}^n) \neq \mathrm{Aut}(\mathbb{P}^n)$$

when $n \geq 2$.

Example: $\mathbb{P}^2 \overset{\varphi}{\dashrightarrow} \mathbb{P}^2$ given by

$$\varphi = (yz : xz : xy) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$$

is not an isom as it is not defined at $(1:0:0)$, $(0:1:0)$, $(0:0:1)$.

Fact: $C_2 = \mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and the φ above.

Example: $C_n = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(t_1, \dots, t_n))$, so naturally

$$S_n \leqslant C_n$$

given by permutation of the t_i.

Example: $(\mathbb{C}^*)^2 \subseteq \mathbb{A}^2 = \mathbb{C}^2$, so we have

$$\mathrm{Aut}((\mathbb{C}^*)^2) \leqslant C_2.$$

$\mathrm{Bir}((\mathbb{C}^*)^2) = \mathrm{Bir}(\mathbb{P}^2)$

Let t_1, t_2 be the coordinate functions on $\mathbb{A}^2$.

Then $(\mathbb{C}^*)^2 \xrightarrow{\varphi} (\mathbb{C}^*)^2$ given by

$$\varphi = \left(\frac{1}{t_1}, t_2\right)$$

is birational, so $\varphi \in C_2$.

Also $\psi = (t_1, t_1 t_2)$ is birational, so $\psi \in C_2$.

similarly, we can generate many elements of C_2 that are defined by "monomials".

In fact if $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any invertible matrix over $\mathbb{Z}$, then $\alpha_M : (\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^2$ given by

$$\alpha_M = (t_1^a t_2^c, t_1^b t_2^d) \in C_2 = Bir(\mathbb{P}^2)$$

as we can find an inverse for α_M which is $\alpha_{M^{-1}}$.

so we have

$$GL_2(\mathbb{Z}) \leqslant C_2.$$

Remark: The groups C_n are very large and complicated from a group theory point of view.

What makes them interesting is the connection with geometry.

A basic idea is to study its finite subgroups.

Theorem: C_n is Jordan, i.e. $\exists$ constant h (depending on n)

s.t. for any finite subgroup $G \leq C_n$,

$\exists$ normal abelian subgroup $A \trianglelefteq G$

with $|G| / |A| \leq h$.

This was proved by Serre for $n=2$, and in general

by Prokhorov–Shramov assuming boundedness of Fano's (Birkar).

I will discuss the proof for $n=2$ using a strategy

that works for every n.

Regularisations

Assume $G \leqslant C_2$ is a finite group.

For each smooth proj X bir to $\mathbb{P}^2$, naturally we have $G \leqslant \mathrm{Bir}(X)$:

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, dashed] \\
\mathbb{P}^2 \arrow[r, dashed, "\text{bir}", "\varphi"'] & \mathbb{P}^2
\end{tikzcd}$$

Fact: we can choose X s.t. $G \leqslant \mathrm{Aut}(X)$.

To see this, first find open $U \subseteq \mathbb{P}^2$ s.t. $G \circlearrowright U$.

Next compactify U to $\overline{U}$ s.t. $G \circlearrowright \overline{U}$, e.g. consider U/G and a compactification $\overline{U/G}$ from which we get $\overline{U}$.

Next take a resolution $X \longrightarrow \bar{U}$ compatible with the action of G, so that $G \circlearrowright X$.

G-MMP:

Assume X is a smooth proj surface, $G \leq \mathrm{Aut}(X)$ finite.

Recall the MMP from lecture 2.

Remember that we said $\exists$

$$X = X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_r = Y$$

where in each step we contract curves intersecting K negatively.

Since $G \circlearrowright X$, we can do a G-MMP similar to above where the maps respect the action of G.

So $G \circlearrowright X_i$ $\forall i$. In particular $G \circlearrowright Y$.

$\exists$ three possibilizies for Y:

$$Y = \begin{cases} G\text{-Fano} \\ \text{or} \\ \exists\ G\text{-Fano fib } Y \longrightarrow C, \quad \dim C = 1. \\ \text{or} \\ Y \text{ is } G\text{-minimal} \end{cases}$$

Sketch of proof of the theorem for $n=2$:

We want to show $C_2 = \mathrm{Bir}(\mathbb{P}^2)$ is Jordan. Pick finite $G \leq C_2$.

Step 1: By regularisation, $\exists$ smooth proj $X \overset{\text{bir}}{\dashrightarrow} \mathbb{P}^2$

s.t. $G \leq \mathrm{Aut}(X)$. So $G \circlearrowright X$.

Run the G-MMP on X:

$$X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots \longrightarrow X_r = Y.$$

Since $X \overset{\text{bir}}{\dashrightarrow} \mathbb{P}^2$, $\exists$ two possibilities for Y:

$$Y = \begin{cases} G\text{-Fano} \\ \quad \text{or} \\ \exists\ G\text{-Fano fib } Y \longrightarrow C, \qquad \dim C = 1. \end{cases}$$

Step 2: Case Y = G-Fano.

Fact: such Y form a bounded family.

i.e. $\exists$ fixed $n, d \in \mathbb{N}$ s.t.

we can find $Y \hookrightarrow \mathbb{P}^n$

s.t. $\deg_{\mathbb{P}^n} Y \leq d$.

Fact: we can find $Y \hookrightarrow \mathbb{P}^n$ so that $G \leq \mathrm{Aut}(Y)$

comes from $\mathbb{P}^n$, that is, $G \leq \mathrm{Aut}(\mathbb{P}^n) = PGL_{n+1}$.

Fact: PGL_{n+1} is Jordan.

So we are done in this case.

Step 3: Case G-Fano fib $Y \xrightarrow{f} C$.

f is compatible with the action of G. so $G \circlearrowright C$.

Thus we have a homomorphism $G \xrightarrow{\alpha} \mathrm{Aut}(C)$.

This gives a sequence

$$1 \longrightarrow \underset{\substack{\| \\ \text{Kernel of } \alpha}}{K} \longrightarrow G \longrightarrow \underset{\substack{\| \\ \text{Image of } \alpha}}{L} \longrightarrow 1$$

Now K acts trivially on C, so it acts on the fibres F of f.

But $F \simeq \mathbb{P}^1$, so $K \circlearrowright \mathbb{P}^1$. (F general fib).

On the other hand, you can show $C \simeq \mathbb{P}^1$, so $L \circlearrowright \mathbb{P}^1$.

Combining all these, one can show $\exists$ normal abelian

Combining all these, one can show $\exists$ normal abelian $A \trianglelefteq G$ s.t. $|G|/|A| \leq h$ for some fixed h.

Remark:

Finite subgroups of C_2 have been classified.

see [Dolgachev-Iskovskikh].

The non-abelian simple finite subgroups of C_2 are

$$A_5, \quad A_6, \quad PSL_2(7).$$

On the other hand, Prokhorov showed that the non-abelian simple finite subgroups of C_3 are

$$A_5, A_6, A_7, PSL_2(7), SL_2(8), PSp_4(3)$$

To prove this, Prokhorov uses a strategy similar to the proof of the theorem above.

Taking a non-abelian simple finite subgroup G, he reduces the statement to the case when $G \curvearrowright X$ for some Fano 3-fold X.

Remarks:

Cantat and Lamy showed that C_2 is not simple.

More recently, Blanc, Lamy and Zimmermann showed that C_n is not simple for $n \geq 3$.

References:

I. Dolgachev, V. Iskovskikh: Finite subgroups of the plane Cremona group.

S. Cantat: The Cremona group.

Yu. prokhorov, C. shramov: Jordan property for Cremona groups.