Lectures on Algebraic Geometry

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Lecture 21 Weighted projective spaces

Introduction:

Recall, $\mathbb{P}^n$ is defined as quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ given by

$$\lambda \cdot (b_0, \dots, b_n) = (\lambda b_0, \dots, \lambda b_n), \quad \lambda \in \mathbb{C}^*.$$

Every quasi-proj variety is embedded in $\mathbb{P}^n$.
Hypersurfaces are in particular relatively simple.
More generally, complete intersections behave well.

In this lecture we look at a more general kind of projective space which are often used to treat examples of varieties.
produce

Weighted projective spaces:

pick $r_0, \dots, r_n \in \mathbb{N}$.

Consider the group $G = \mathbb{Z}_{r_0} \times \dots \times \mathbb{Z}_{r_n}$.

Consider the action $G \circlearrowright \mathbb{P}^n$:

for $g = (g_0, \dots, g_n) \in G$, $x = (a_0 : \dots : a_n) \in \mathbb{P}^n$,

define $g \cdot x = (g_0 a_0, \dots, g_n a_n)$.

Here we consider g_j as in $\mathbb{C}$, $g_j = e^{2\pi i \, l_j / r_j}$.

Define $\mathbb{P}(r_0, \dots, r_n) = \mathbb{P}^n / G$.

Example: $\mathbb{P}(1, \dots, 1) = \mathbb{P}^n$ as G is trivial.

Example: $n = 1$, $\mathbb{P}(r_0, r_1) \simeq \mathbb{P}^1$ because $\mathbb{P}(r_0, r_1)$ is normal hence smooth (as $\dim = 1$) and because we have a finite morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}(r_0, r_1).$$

$\mathbb{P}(r_0, \dots, r_n)$ as quotient of $\mathbb{C}^{n+1} \setminus \{0\}$

Consider the action $\mathbb{C}^* \curvearrowright \mathbb{C}^{n+1} \setminus \{0\}$ given by

$$\lambda \cdot (b_0, \dots, b_n) = (\lambda^{r_0} b_0, \dots, \lambda^{r_n} b_n).$$

Consider

$$\begin{tikzcd}
\mathbb{C}^{n+1} \setminus \{0\} \arrow[r, "f"] \arrow[d, "g"', "/\mathbb{C}^*"] & \mathbb{C}^{n+1} \setminus \{0\} \arrow[dr, "/\mathbb{C}^*", "\rho"'] & \\
\mathbb{P}^n \arrow[r, "\pi"'] & \mathbb{P}(r_0, \dots, r_n) & V
\end{tikzcd}$$

where f is given by

$$f(a_0, \dots, a_n) = (a_0^{r_0}, \dots, a_n^{r_n}).$$

If $\pi(g(a_0, \dots, a_n)) = \pi(g(a_0', \dots, a_n'))$,

then one can check that

$$\rho(f(a_0, \dots, a_n)) = \rho(f(a_0', \dots, a_n')).$$

Thus as both g, π are quotients by group actions, we get a morphism

$$\mathbb{P}(r_0, \dots, r_n) \xrightarrow{h} V$$

giving

$$\begin{tikzcd}
\mathbb{C}^{n+1} \setminus \{0\} \arrow[r] \arrow[d, "/\mathbb{C}^*"] & \mathbb{C}^{n+1} \setminus \{0\} \arrow[dr, "/\mathbb{C}^*"] & \\
\mathbb{P}^n \arrow[r] & \mathbb{P}(r_0, \dots, r_n) \arrow[r, "h"] & V
\end{tikzcd}$$

on can check that h is an isomorphism.

That is,

$$\mathbb{P}(r_0, \dots, r_n) \simeq \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*.$$

$\mathbb{P}(r_0, \dots, r_n)$ as Proj of a graded ring

Recall,
$$\mathbb{P}^n = \text{Proj } \mathbb{C}[t_0, \dots, t_n].$$

one can show
$$\mathbb{P}(r_0, \dots, r_n) \simeq \text{Proj } \mathbb{C}[t_0^{r_0}, \dots, t_n^{r_n}].$$

In other words,
$$\mathbb{P}(r_0, \dots, r_n) \simeq \text{Proj } \mathbb{C}[u_0, \dots, u_n]$$

where u_i has degree r_i.

In particular, any $\varphi \in \mathbb{C}[u_0, \dots, u_n]$ that is homogeneous with respect to the degrees, defines a hypersurface in $\mathbb{P}(r_0, \dots, r_n)$.

Example: $\mathbb{P}(r_0, \dots, r_n) \simeq \mathbb{P}^n$ if $r_0 = r_1 = \dots = r_n$.

This follows from

$$\mathbb{P}(r_0, \dots, r_n) \simeq \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$$

as above.

Also follows from the Proj construction.

$\mathbb{P}(r_0, \dots, r_n)$ is a Fano Variety

Recall we have a finite quotient morphism

$$\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}(r_0, \dots, r_n).$$

By Riemann-Hurwitz formula

$$K_{\mathbb{P}^n} = \pi^* K_{\mathbb{P}(r_0, \dots, r_n)} + R$$

where $R \geq 0$.

Since $K_{\mathbb{P}^n} = -(n+1) H$, H hyperplane, we see $\pi^* K_{\mathbb{P}(r_0,\dots,r_n)} \sim -dH$ for some $d>0$.

Therefore $-K_{\mathbb{P}(r_0,\dots,r_n)}$ is ample, so $\mathbb{P}(r_0,\dots,r_n)$ is Fano.

Local description & singularities

Recall, $\mathbb{P}(r_0,\dots,r_n) = \mathbb{C}^{n+1}\setminus\{0\} / \mathbb{C}^*$.

Let
$$U_j = \{ (b_0,\dots,b_n) \mid b_j \neq 0 \} \subseteq \mathbb{C}^{n+1}\setminus\{0\}$$

and let V_j = image of U_j on $\mathbb{P}(r_0,\dots,r_n)$.

The V_j are open subsets covering $\mathbb{P}(r_0,\dots,r_n)$.

Consider
$$S = \{ (1, b_1, \dots, b_n) \} \subseteq \mathbb{C}^{n+1}\setminus\{0\}.$$

Each $c \in U_0$ has a point $b \in S$ in its orbit.

If $\lambda \cdot b = (\lambda^{r_0}, \lambda^{r_1} b_1, \dots, \lambda^{r_n} b_n) \in S$, then $\lambda^{r_0} = 1$.

So $V_0 \simeq S/\mathbb{Z}_{r_0} \simeq \mathbb{C}^n/\mathbb{Z}_{r_0}$

where $g \in \mathbb{Z}_{r_0}$ acts on $\mathbb{C}^n$ by

$$g \cdot (b_1, \dots, b_n) = (g^{r_1} b_1, \dots, g^{r_n} b_n).$$

Similarly, one shows $V_j \simeq \mathbb{C}^n/\mathbb{Z}_{r_j}$

where $\mathbb{Z}_{r_j} \circlearrowright \mathbb{C}^n$ is similarly defined.

In particular, the singularities of $\mathbb{P}(r_0, \dots, r_n)$

of a special kind: cyclic quotient singularities.

<u>Example</u>: $\mathbb{P}(1,1,2)$.

This is covered by the open sets

$V_0 \simeq \mathbb{C}^2/\mathbb{Z}_1 \simeq \mathbb{C}^2$

$V_1 \simeq \mathbb{C}^2/\mathbb{Z}_1 \simeq \mathbb{C}^2$

$V_2 \simeq \mathbb{C}^2/\mathbb{Z}_2$ with one singular point (image of $(0,0)$).

As we saw in previous lecture, this singularity is locally the same as the singularity

$$V(xy-z^2) \subseteq \mathbb{C}^3.$$

Example: $\mathbb{P}(1,1,r)$.

Singularities: Similar to $\mathbb{P}(1,1,2)$, we have

$$U_0 = \mathbb{C}^2 \xrightarrow{/\mathbb{Z}_1} V_0 \simeq \mathbb{C}^2 \subseteq \mathbb{P}(1,1,r)$$

$$U_1 = \mathbb{C}^2 \xrightarrow{/\mathbb{Z}_1} V_1 \simeq \mathbb{C}^2 \subseteq \mathbb{P}(1,1,r)$$

$$U_2 = \mathbb{C}^2 \xrightarrow{/\mathbb{Z}_r} V_2 \subseteq \mathbb{P}(1,1,r).$$

So $\mathbb{P}(1,1,r)$ has only one singular point, the image of $(0,0)$ in V_2.

Ramifications: We can also see that $\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}(1,1,r)$

is ramified only along the hyperplane

$$H = V(t_2) \subseteq \mathbb{P}^2 = \mathrm{Proj}\ \mathbb{C}[t_0, t_1, t_2],$$

with ramification r.

So by Riemann-Hurwitz formula

$$K_{\mathbb{P}^2} = \pi^* K_{\mathbb{P}(1,1,r)} + (r-1) H.$$

Volume of $-K$: Since $\deg(\pi) = r$, we get

$$\begin{aligned}(-K_{\mathbb{P}(1,1,r)}) \cdot (-K_{\mathbb{P}(1,1,r)}) &= (-K_{\mathbb{P}^2} + (r-1)H)^2 / r \\ &= (3H + (r-1)H)^2 / r \\ &= (r+2)^2 / r.\end{aligned}$$

The number $(-K_{\mathbb{P}(1,1,r)})^2$ reflects global properties of $\mathbb{P}(1,1,r)$. Now it goes to ∞ when r goes to ∞.

Unboundedness: This in particular shows that

$$\{ \mathbb{P}(1,1,r) \mid r \in \mathbb{N} \}$$

is not bounded, i.e., this set can not be parametrised by finitely many varieties.

Interpretation as cone: $\mathbb{P}(1,1,r)$ can be interpreted as a cone over $\mathbb{P}^1$.

First recall, $\exists$ embedding

$$C = \mathbb{P}^1 \hookrightarrow \mathbb{P}^r$$
$$(a_0 : a_1) \longmapsto (a_0^r : a_0^{r-1} a_1 : \cdots : a_0 a_1^{r-1} : a_1^r).$$

$\mathbb{P}^1$ has degree r with respect to this embedding, i.e. if $L \subseteq \mathbb{P}^r$ is a hyperplane, then

$$L \cdot C = r.$$

On the other hand, the cone over C is

$$Y_C = \{ (a_0^r : a_0^{r-1} a_1 : \cdots : a_1^r : b) \in \mathbb{P}^{r+1},\ a_0, a_1, b \in \mathbb{C} \}.$$

Now consider the morphism

$$\begin{aligned} \mathbb{C}^3 \setminus \{0\} &\xrightarrow{q} Y_C \\ (e_0, e_1, e_2) &\longmapsto (e_0^r : e_0^{r-1} e_1 : \cdots : e_1^r : e_2) \end{aligned}$$

Note $q(e_0, e_1, e_2) = q(\lambda e_0, \lambda e_1, \lambda^r e_2)$, $\forall \lambda \in \mathbb{C}^*$.

So we have

$$\begin{tikzcd} \mathbb{C}^3 \setminus \{0\} \arrow[rr, "q"] \arrow[dr] & & Y_C \\ & \mathbb{P}(1,1,r) \arrow[ur, "u"'] & \end{tikzcd}$$

and one can check that u is an isomorphism.

Y_C has only one singular point: $(0 : \cdots : 0 : 1)$.

Blowing up $\mathbb{P}^{r+1}$ at this point induces a resolution

$$\varphi : W \longrightarrow Y_C$$

with only one exceptional curve E with

$$E \simeq \mathbb{P}^1 \quad \& \quad E \cdot E = r.$$

Rationality: We saw that $\mathbb{P}(1,1,r)$ has an open subset $V_0 \cong \mathbb{C}^2$. This implies that $\exists$ birational isomorphism

$$\mathbb{P}(1,1,r) \dashrightarrow \mathbb{P}^2$$

So $\mathbb{P}(1,1,r)$ is a <u>rational</u> variety.

More generally, $\mathbb{P}(r_0, \dots, r_n)$ is a rational variety as it is a <u>toric</u> variety.

A hypersurface:

Recall, we can consider variables u_0, u_1, u_2 on $\mathbb{P}(1,1,r)$ where $\deg u_0 = \deg u_1 = 1$, $\deg u_2 = r$.

If $h \in \mathbb{C}[u_0, u_1]$ is homogeneous of deg $2r$, then $f = u_2^2 - h$ is "weighted" homogeneous of deg $2r$.

Let $T \subseteq \mathbb{P}(1,1,r)$ be the hypersurface defined by f.

Note the singular point $(0:0:1)$ of $\mathbb{P}(1,1,r)$ is not in T.

If h is "general", then T is a smooth curve.

It is known that $\text{genus}(T) = r-1$.

[this is derived from $K_T = (K_{\mathbb{P}(1,1,r)} + T)|_T$]

<u>Remark</u>: <u>Hypersurfaces</u> in weighted projective spaces and complete intersection

Produce lots of interesting examples of varieties, such as Fano and Calabi-Yau varieties.

These are especially used in conjunction with mirror symmetry.

<u>References</u>: A. Iano-Fletcher, Working with weighted complete intersections.

I. Dolgachev, Weighted projective varieties.