Lectures on Algebraic Geometry

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Lecture 3: Quotient varieties

Introduction

Group actions commonly occur in maths.

A group G acting on a set X is defined to be a function

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g.x \end{aligned} \qquad \Big|\ G \curvearrowright X$$

such that

$$\begin{cases} 1.x = x & \forall x \\ g.(h.x) = (gh).x, & \forall g, h, x. \end{cases}$$

So each $g \in G$ gives a bijection

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto g.x \end{aligned}$$

(often we identify this with g)

On the other hand, for each $x \in X$, we have the orbit

$$[x] = \{ g.x \mid g \in G \}.$$

The orbits define an equivalence relation on X,

so we get a quotient space and map:

$$\begin{aligned} X &\xrightarrow{\pi} X/G. \\ x &\longmapsto [x] \end{aligned}$$

Example: G a group, $G \curvearrowright G$ by

$$G \times G \longrightarrow G$$
$$(g,h) \longmapsto g\cdot h$$

$\pi: G \longrightarrow G/G$ is constant. $[1] = G$.

Example: $G = \mathbb{Z} \curvearrowright X = \mathbb{R}$ by

$$G \times X \longrightarrow X$$
$$(g,x) \longmapsto g+x.$$

we can identify X/G with S^1 = circle.

$X/G = \pi([0,1))$

Quotients in algebraic geometry:

G group, X variety, $G \curvearrowright X$.

we like X/G to be a variety and

$\pi: X \longrightarrow X/G$ to be a morphism.

For each $g \in G$ we will assume

$$X \longrightarrow X$$
$$x \longmapsto g\cdot x$$

is an isomorphism

Example $G = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $X = \mathbb{C}$, $G \curvearrowright X$

by
$$G \times X \longrightarrow X$$
$$(g,x) \longmapsto gx \quad \text{(product in } \mathbb{C}).$$

we see
$$[0] = \{g\cdot 0 \mid g \in G\} = \{0\}.$$

$x \neq 0$, $[x] = \{g\cdot x \mid g \in G\} = \mathbb{C}^*$.

so X/G has two points.

Then $\pi: X \longrightarrow X/G$ cannot be a morphism

of varieties because X is connected but X/G is not.

Theorem: G finite group, X variety, $G \curvearrowright X$.

Then $\pi: X \longrightarrow X/G$ is a finite surjective morphism of varieties.

Proof:

For simplicity, we assume X is affine.

Let $A = \mathbb{C}[X]$ be the coordinate ring of X.

Then A is a finitely generated $\mathbb{C}$-algebra:

if $X \subseteq \mathbb{C}^n$, then $\mathbb{C}[X] \simeq \dfrac{\mathbb{C}[t_1, \dots, t_n]}{I}$

where I is the ideal of X.

G naturally acts on A: $g \in G$ gives $X \xrightarrow[x \mapsto g \cdot x]{\alpha_g} X$ which determines $\begin{array}{l} A \longrightarrow A \\ f \mapsto f.g = f \circ \alpha_g \end{array}$

let $A^G = \{ f \in A \mid f.g = f, \ \forall g \in G \}$. $\qquad \big((f.g)(x) = f(g.x) \big)$

Assume A^G is a finitely generated $\mathbb{C}$-algebra.

Then A^G is the coordinate ring of some variety Y. $\left(\text{as } A^G \simeq \dfrac{\mathbb{C}[s_1, \dots, s_\ell]}{J} \text{ for some } s_1, \dots, s_\ell, J \right)$

The inclusion $A^G \hookrightarrow A$ corresponds to a morphism

$$\lambda: X \longrightarrow Y.$$

claim: $\lambda(x) = \lambda(x') \iff [x] = [x']$.

$\Leftarrow$ $x' = g \cdot x$ for some $g \in G$, so $\forall f \in A^G$ we have $f(x') = f(g.x) = (f.g)(x) = f(x)$,

so $\lambda(x) = \lambda(x')$.

$\Rightarrow$ $\begin{cases} x \longleftrightarrow P \trianglelefteq A \\ x' \longleftrightarrow P' \trianglelefteq A \end{cases}$

$\lambda(x) = \lambda(x')$ means $P \cap A^G = P' \cap A^G$.

if $f \in P'$, then

$$\prod_{g \in G} f.g \in P' \cap A^G = P \cap A^G \subseteq P.$$

$g \in G$

so $f \cdot g \in P$ for some $g \in G$.

Then $f \in P \cdot g^{-1}$.

so $P' \subseteq \bigcup_{g \in G} P \cdot g^{-1}$.

But then by a result in algebra

we have $P' \subseteq P \cdot g^{-1}$ for some $g \in G$, (Exercise)

and this implies $P' = P \cdot g^{-1}$ for some g,

so $x' = g \cdot x$, hence $[x] = [x']$.

Therefore, $Y = X/G$ and $\lambda = \pi$.

Now π is a finite morphism because

A is integral over A^G: indeed, if $f \in A$,

then f is a root of

$$\prod_{g \in G} (t - f \cdot g) \in A^G[t] \subseteq A[t].$$

Note: A is a finitely generated A^G-algebra.

Finally, A^G is indeed a finitely generated

$\mathbb{C}$-algebra. For example see Appendix 4 of

Shafarevich: Basic Algebraic Geometry.

Exercise: G finite group acting on a variety X. If X is normal, then X/G is normal.

Remark: Assume $\pi: X \longrightarrow Y$ is a finite surjective morphism, X smooth, Y normal, K_Y $\mathbb{Q}$-Cartier. The <u>Riemann-Hurwitz formula</u> says that we can write

$$K_X = \pi^* K_Y + \sum_{D \text{ prime div}} (r_D - 1) D ,$$

r_D = ramification index of π along D.

For curves, see Hartshorne book, page 301.

Example: $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $M \cdot M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$G = \langle M \rangle \simeq \mathbb{Z}_2$.

$G \curvearrowright X = \mathbb{P}^1$ determined by

$M \cdot (a:b) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$.

Then $X \xrightarrow{\pi} X/G$ is a finite morphism.

As X/G is normal and $\dim(X/G) = 1$, X/G is smooth. (and projective).

But $g(\mathbb{P}^1) = 0$, so $g(X/G) = 0$ by using the Riemann-Hurwitz formula.

So $X/G \simeq \mathbb{P}^1$.

Example: X = elliptic curve.

X itself is an abelian group.

consider $\sigma: X \longrightarrow X$ ($-x$ in the group law of X).
$x \longmapsto -x$

Then σ^2 = identity,

so $G = \langle \sigma \rangle \simeq \mathbb{Z}_2$.

Now $X \xrightarrow{\pi} X/G$ is a finite morphism and

X/G is smooth projective.

It is well-known that $\exists\, x \in X$ s.t.

$x = -x$, so $[x] = \{x\}$.

Thus π is ramified at such x, so

$\deg K_Y < \deg K_X = 0$ by the

Riemann-Hurwitz formula, so $Y \simeq \mathbb{P}^1$.

$Y = X/G$

Example: $X = \mathbb{C}^2$, $\sigma: X \longrightarrow X$
$(a,b) \longmapsto (-a,-b)$.

$G = \langle \sigma \rangle \subseteq \mathrm{Aut}(X)$.

$A := \mathbb{C}[s,t] := \mathbb{C}[X]$.

$A^G = \mathbb{C}[X/G]$.

Not difficult to see

$$A^G = \mathbb{C}[s^2, st, t^2].$$

Define

$$\mathbb{C}[\alpha, \beta, \gamma] \xrightarrow{L} A^G$$

$$\alpha \longmapsto s^2$$
$$\beta \longmapsto st$$
$$\gamma \longmapsto t^2$$

One can see $\ker(L) = \langle \alpha\gamma - \beta^2 \rangle$.

So $\mathbb{C}[X/G] \simeq \mathbb{C}[\alpha, \beta, \gamma] / \langle \alpha\gamma - \beta^2 \rangle$

hence X/G is the hypersurface

$$X/G = V(\alpha\gamma - \beta^2) \subseteq \mathbb{C}^3.$$

We saw in lecture 1 that this has one singular point, $(0,0,0)$, and blowing up this point on $\mathbb{C}^3$ resolves the singularity:

$$\varphi: W \longrightarrow X/G$$

Moreover, $K_W = \varphi^* K_{X/G}$

Example: X = an abelian surface, $\left(\text{e.g. } X = E \times E \quad E = \text{elliptic curve} \right)$

$$\iota : X \longrightarrow X$$
$$x \longmapsto -x$$

$G = \langle \iota \rangle \leq \mathrm{Aut}(X)$, $\qquad \pi : X \longrightarrow X/G$.

Fact: $S = \{ x \in X \mid x = -x \}$ has 16 points.

Now $\pi(x)$ is singular $\iff$ $x \in S$:

The action of G on X in some small analytic neighbourhood of $x \in S$ looks like $G \curvearrowright \mathbb{C}^2$ of the previous example.

If $x \notin S$, π is an isomorphism in some small analytic neighbourhood of x.

In particular,

- $K_X = \pi^* K_{X/G}$, so $K_{X/G} \sim_{\mathbb{Q}} 0$,
- $\exists$ resolution $W \xrightarrow{\varphi} X/G$ s.t.

$$K_W = \varphi^* K_{X/G}, \quad \text{so} \quad K_W \sim_{\mathbb{Q}} 0.$$

Such W are called Kummer surfaces which are examples of $K3$ surfaces.

Remark: G finite group acting on smooth variety X.

$\pi: X \to X/G$.

(1) The singularities on such X/G are <u>quotient singularities</u>.

Fact: Any divisor L on X/G is $\mathbb{Q}$-Cartier

This shows that the variety

$$V(st-uw) \subseteq \mathbb{C}^4$$

is not finite quotient of any smooth variety.

(2)

In the Riemann-Hurwitz formula for $X \xrightarrow{\pi} X/G$,

$$K_X = \pi^* K_{X/G} + \sum_{D \text{ prime div}} (r_D - 1) D$$

one can check that the part $\sum$ is G-invariant,

so $\exists$ $\mathbb{Q}$-div $B \geq 0$ on X/G s.t.

$$K_X = \pi^*(K_{X/G} + B).$$

If X is a projective Calabi-Yau variety ($K_X \equiv 0$),

then $(X/G, B)$ is a Calabi-Yau pair.