Lectures on Algebraic Geometry

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Lecture 2: Birational geometry of surfaces

Introduction: X smooth projective surface.

The aim of birational classification is to find

$$X \overset{\text{bir}}{\dashrightarrow} Y \quad \text{proj}$$

s.t. Y is as simple as posible.

Next we want to classify such Y,

e.g. construct their moduli spaces.

In this lecture we discuss the following:

Theorem: X smooth projective surface.

Then we have a sequence of birational transformations

$$X = X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n = Y$$

such that $\begin{cases} K_Y \text{ is nef} \\ \quad \text{or} \\ \exists \text{ Fano fibration } Y \longrightarrow V. \end{cases}$

Some definitions:

X normal projective variety,

L $\mathbb{Q}$-Cartier divisor on X.

We say L is <u>nef</u> if $L \cdot C \geq 0$ $\forall$ curve $C \subseteq X$.

We say L is <u>ample</u> if

$$\exists \underbrace{X \subseteq \mathbb{P}^n}_{\text{embedding}}, \underbrace{H \subseteq \mathbb{P}^n}_{\text{hyperplane}}, X \nsubseteq H,$$

s.t. $mL \sim H|_X$ for some $m \in \mathbb{N}$.

Note: L ample $\Rightarrow L \cdot C > 0 \Rightarrow L$ nef.

Example:

X smooth projective curve, L $\mathbb{Q}$-divisor on X.

If $\deg L \geq 0$, then L is nef.

If $\deg L > 0$, then L is ample and nef and big.

If $L = x - y$, $x \neq y$ points, then L is nef but

not ample and not nef and big.

Base point freeness and extremal contractions

One of the cornerstones of birational geometry is the following result which we use without proof.

Theorem:

(X, B) a projective klt pair,

A an ample $\mathbb{Q}$-divisor on X,

$K_X + B + A$ is nef.

Then K_X+B+A is <u>semi-ample</u>, that is

$\exists \begin{cases} \text{projective morphism } f: X \longrightarrow Z \\ \quad \text{and} \\ \text{ample } \mathbb{Q}\text{-divisor } H \text{ on } Z \end{cases}$

s.t. $K_X+B+A \sim_{\mathbb{Q}} f^* H$.

Reference: Kollár – Mori: Birational geometry of algebraic varieties,

Birkar: Lectures on birational geometry (arxiv).

Corollary (Extremal contractions)

(X, B) projective klt pair,

$K_X + B$ not nef.

Then $\exists$ (K_X+B)-negative <u>extremal contraction</u>.

Sketch of proof:

Pick an ample divisor L s.t. K_X+B+L is ample.

Let
$$t := \inf\left\{ 0\leq u \in \mathbb{R} \;\middle|\; K_X+B+uL \text{ is nef} \right\}.$$

Fact: $t \in \mathbb{Q}$.

Then by the base point freeness theorem,

K_X+B+tL is semi-ample, so $\exists$ projective morphism

$f: X \longrightarrow Z$ and ample H s.t.

$$K_X+B+tL \sim_{\mathbb{Q}} f^*H.$$

Taking the Stein factorisation we can assume

f is a <u>contraction</u>, that is, it has connected fibres.

Also replacing L with a suitable ample divisor

we can assume f is <u>extremal</u>, that is,

if
$$\begin{tikzcd} X \arrow[r, "\text{contraction}"] \arrow[dr] & V \arrow[d] \\ & Z \end{tikzcd} \qquad \left(\begin{array}{c} X \to V \\ \text{not isom} \end{array} \right)$$

then $V \to Z$ is an isomorphism.

Finally note that if $C \subseteq X$ is a curve

contracted by f ($f(C)=\text{pt.}$), then

$$(K_X+B)\cdot C = -tL\cdot C < 0.$$

$\square$

Example:

Z smooth projective variety, $z \in Z$.

$f: X \longrightarrow Z$ blowup at z.

Then f is a K_X-negative extremal contraction.

Indeed, K_X is "negative" over Z, so

if H is a sufficiently ample divisor on Z,

then $L := -K_X + f^* H$ is ample.

Then $K_X + L = f^* H$ is semi-ample and

f is its associated contraction.

Applications to surfaces:

X smooth projective surface,

K_X not nef.

By the corollary, $\exists$ K_X-negative extremal

contraction $f: X \longrightarrow Z$.

Case 1: $\dim Z = 2$.

f is birational, contracting one curve E.

Then $E \cdot E < 0$ because if $H \geq 0$ contains

$f(E)$, then $f^* H = H^\vee + eE$, so $(e > 0)$

$$eE \cdot E = -H^\vee \cdot E < 0.$$

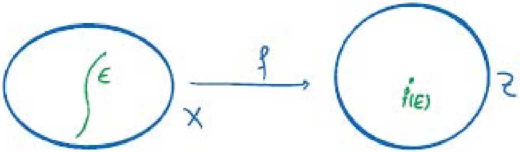
Also $K_X \cdot E < 0$ as f is K_X-negative.

so $(K_X + E) \cdot E < 0$.

But then a refinement of the adjunction formula (previous lecture) implies $E \simeq \mathbb{P}^1$ & $E \cdot E = -1$.

(such E are called (-1)-curves).

Fact: Z is smooth and f is just the blowup at the point $f(E)$.

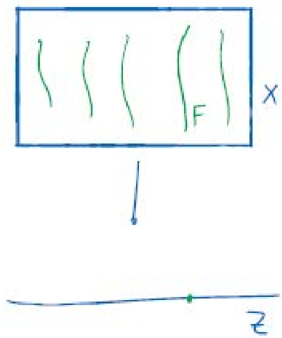


Case 2: $\dim Z = 1$.

Since X is smooth, most fibres F of f are smooth curves.

Then $\deg K_F = (K_X + F) \cdot F = K_X \cdot F < 0$,

by adjunction, so $F \simeq \mathbb{P}^1$.



Fact: All fibres of f are $\simeq \mathbb{P}^1$.

Case 3: $\dim Z = 0$.

so $K_X + A \sim_{\mathbb{Q}} 0$ for some ample divisor A.

so $-K_X$ is ample, that is, X is <u>Fano</u>.

Fact: $X \simeq \mathbb{P}^2$.

Minimal model program for surfaces: (MMP)

X smooth projective surface.

If K_X is nef, let $Y := X$.

If K_X is not nef, then $\exists$ extremal contraction

K_X-negative

$$X =: X_1 \longrightarrow X_2.$$

If $\dim X_2 \leq 1$, let $Y = X_1$, $V = X_2$.

If $\dim X_2 = 2$, then X_2 is smooth projective, and we can repeat the above with X_2 in place of X.

After <u>finitely</u> many steps, we get

$$X = X_1 \longrightarrow \cdots \longrightarrow X_n = Y \qquad \text{(minimal model)}$$

where $\begin{cases} K_Y \text{ is nef} \\ \text{or} \\ \exists \text{ Fano extremal contraction } Y \to V. \end{cases}$ $\begin{cases} \dim V = 1 \Rightarrow Y \to V \ \mathbb{P}^1\text{-fib} \\ \dim V = 0 \Rightarrow Y \simeq \mathbb{P}^2 \end{cases}$ (Mori fibre space)

$\dim V \leq 1$.

Fact: when K_Y is nef, it is semi-ample, so $\exists$ contraction $g: Y \longrightarrow T$ and ample divisor H s.t.

$$K_Y \sim_{\mathbb{Q}} g^* H.$$

(Calabi-Yau fibration if $\dim Y > \dim T$.)

A similar fact is expected to hold in higher dimension.

Conjecture (Abundance)

X projective variety with klt singularities.

If K_X is nef, then it is semi-ample.

MMP and singularities:

X surface with klt singularities.

$\varphi: W \longrightarrow X$ a resolution.

We can run MMP for W over X, similar to above. That is, $\exists$

$$W = W_1 \longrightarrow \cdots \longrightarrow W_n = Y \xrightarrow{\psi} X$$

where each $W_i \to W_{i+1}$ is blowup of a point and K_Y is nef over X: $K_Y \cdot C \geq 0$ if a curve $C \subseteq Y$ is contracted by $Y \to X$.

Such Y is a minimal resolution.

This implies: if $K_Y + B_Y = \psi^* K_X$, then $B_Y \geq 0$.

Applying adjunction as in previous lecture, shows every curve $C \subseteq Y$ contracted by $\psi: Y \longrightarrow X$ is $\simeq \mathbb{P}^1$.

References:

Kollár, Mori: Birational geometry of algebraic varieties

Matsuki: Introduction to Mori program

Birkar: Lectures on birational geometry

Hartshorne: Algebraic geometry