Lectures on Algebraic Geometry

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Lecture 1: singularities

Conventions:

We work over the complex numbers $\mathbb{C}$.

A <u>variety</u> means a quasi-projective algebraic variety

Introduction:

classical algebraic geometry is mainly about classification of smooth varieties.

Smooth varieties are complex manifolds.

They behave well in many ways.

Theorem (Hironaka)

Any variety X has a resolution of singularities:

$\exists$ projective birational morphism

$$\varphi : W \longrightarrow X$$

where W is smooth.

So it is not a surprise that smooth varieties are at the centre of attention.

But singularity theory is central to modern algebraic geometry.

singular varieties help to better understand smooth varieties.

Also singular varieties exhibit very interesting behaviour.

Normal varieties and divisors:

A variety X is normal if its local rings $\mathcal{O}_x$ are normal $\forall x \in X$ ($\mathcal{O}_x$ normal means it is algebraically closed in its fraction field).

Fact: X normal $\Longrightarrow$ $\dim X_{sing} \leq \dim X - 2$.

A $\mathbb{Q}$-divisor on a normal variety X is as

$$B = \sum_{\text{finite}} b_i B_i \quad , \quad b_i \in \mathbb{Q} \ , \ B_i \text{ prime divisor}$$

we say B is $\mathbb{Q}$-Cartier if mB is Cartier for some $m \in \mathbb{N}$ (Cartier means it is locally defined by one equation).

If B is $\mathbb{Q}$-Cartier and $C \subseteq X$ is a projective curve, define the intersection number

$$B \cdot C = \deg B_{|C} = \text{sum of coefficients of the divisor } B_{|C}.$$

If B and D are $\mathbb{Q}$-Cartier divisors on a projective X, can define $B \cdot D$ by linearity.

Recall, the canonical divisor K_X of a normal variety is the divisor of a top degree rational differential form.

Fact (Adjunction):

X smooth variety, $S \subseteq X$ smooth prime divisor, then $K_S = (K_X + S)_{|S}$.

Singularities of Pairs

X normal variety, $B = \sum b_i B_i \geq 0$ s.t. $K_X + B$ is $\mathbb{Q}$-Cartier. } (X, B) a **Pair**

$\varphi : W \longrightarrow X$ a resolution of singularities of (X, B)

Can write

$$K_W + B_W = \varphi^*(K_X + B)$$

We say

$$(X,B) \text{ is } \begin{cases} lc & \text{if every coefficient of } B_W \text{ is } \leq 1 \\ klt & \text{..................................... } < 1. \end{cases}$$

Example: $(X, B = \sum b_i B_i)$ log smooth, i.e. X is smooth, B_i are smooth and intersect transversally

(X,B) is $lc \iff b_i \leq 1, \ \forall i$

(X,B) is $klt \iff b_i < 1, \ \forall i$

Example: $X = \mathbb{A}^2$, $B = B_1 \subseteq X$ curve given by the equation

$$y^2 - x^2(x+1) = 0.$$

Let $\varphi : W \longrightarrow X$ be the blowup of X at $(0,0)$.

write $K_W + B_W = \varphi^*(K_X + B)$.

Then $B_W = B^\sim + E$

where $B^\sim$ is the birational transform of B

and E is the exceptional curve of the blowup.

$E \cong \mathbb{P}^1$

Indeed, we know $B_W = B^\sim + eE$,

and $(K_W + B_W) \cdot E = 0$, so

$$(K_W + B^\sim + eE) \cdot E = 0,$$

$$\| $$

$$K_W \cdot E + B^\sim \cdot E + eE \cdot E$$

$$\|$$

$$-1 + 2 - e \quad \Longrightarrow \quad e = 1.$$

Therefore, (X, B) is lc but not klt.

Example: $X \subseteq \mathbb{A}^3$ defined by $z^2 - xy = 0$, $B = 0$.

Let $\psi: V \longrightarrow \mathbb{A}^3$ be blowup at $(0,0,0)$,
W = birational transform of X,
and $\varphi: W \longrightarrow X$ the induced morphism.

Then φ is a resolution of singularities.

Let $\begin{cases} F = \text{exceptional divisor of } \psi \\ E = W \cap F = \text{exceptional divisor of } \varphi. \end{cases}$

We want to calculate $E \cdot E$ on W:

$$E \cdot E = \deg E|_E = \deg F|_E = F \cdot E \quad \text{on } V.$$

But $F \cong \mathbb{P}^2$ and $E \subseteq F$ given by $z^2 - xy$.
Moreover, if $H \subseteq \mathbb{A}^3$ is a plane through $(0,0,0)$,
then $\psi^* H = H' + F$ and $(\psi^* H) \cdot E = 0$,

so $F \cdot E = -H'|_F \cdot E = -2$

because $H'|_F \subseteq F$ is just a line.

Thus $E \cdot E = -2$.

Now writing $K_W + B_W = \varphi^* K_X$,
$B_W = eE$ for some e,
and $(K_W + eE) \cdot E = 0$.
Also $(K_W + E) \cdot E = \deg K_E = -2$
by adjunction. Therefore,

$$e = 0 \quad \text{and} \quad K_W = \varphi^* K_X.$$

So $(X, 0)$ is klt.

Example: Assume X is a surface, $\varphi: W \longrightarrow X$ a resolution s.t. φ has one exceptional curve E with $E \cdot E = -n$, $n \geq 2$, $n \in \mathbb{N}$.

| K_X $\mathbb{Q}$-Cartier

Then
$$K_W + eE = \varphi^* K_X$$

and using adjunction as in previous example, can show
$$e = \frac{n-2}{n}.$$

Such X exist: it is the cone over a rational curve of deg n.

Example: $X = \mathbb{A}^2$, $B = 1/2\,(B_1 + B_2 + B_3 + B_4)$

B_i distinct line through $(0,0)$.

Can calculate

$$K_W + \tfrac{1}{2} B_1^\sim + 1/_2\, B_2^\sim + 1/_2\, B_3^\sim + 1/_2\, B_4^\sim + E = \varphi^*(K_X + B).$$

So (X, B) is lc but not klt.

Theorem: Assume (X,B) is Klt of dimension 2, and $W \xrightarrow{\varphi} X$ resolution s.t. writing

$$K_W + B_W = \varphi^*(K_X + B)$$

we have $B_W \geq 0$.

Then $E \simeq \mathbb{P}^1$ for every exceptional curve E of φ.

Proof:

[In fact such φ always exists in dimension 2 when $B=0$]

Let $e =$ coefficient of E in B_W.

(X,B) is Klt, so $e < 1$.

Now

$$(1-e)\,E\cdot E = (1-e)E\cdot E + (K_W+B_W)\cdot E$$

$$= (K_W+E)\cdot E + \underbrace{(B_W - eE)\cdot E}_{\geq 0}.$$

Since E is exceptional, $E\cdot E^2 < 0$:

we can see this by taking $H \geq 0$ $\mathbb{Q}$-Cartier on X passing through $\varphi(E)$ and then noting

$$\varphi^* H \cdot E = 0$$

and that $\varphi^* H = tE + D$ for some $t>0$ and some divisor $D \geq 0$ intersecting E but not containing E.

Therefore $\underset{\deg K_E}{\overset{\shortparallel}{(K_W + E)\cdot E}} < 0$

but $\deg K_E = 2g-2$ where $g =$ genus of X, by Riemann-Roch theorem.

So $g=0$ and $E \simeq \mathbb{P}^1$.

Classification of surface klt singularities:

Given a surface X and a "minimal" resolution

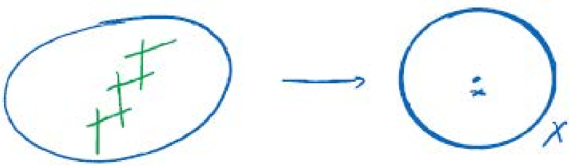
$$\varphi: W \longrightarrow X$$

with exceptional divisors $E_1, \dots, E_r$,

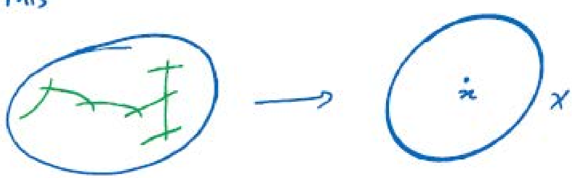
we can understand its singularities by knowing the numbers $E_i \cdot E_j \quad \forall\ i, j$.

The configuration of the E_i takes only some simple forms.

For example like this



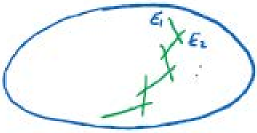
or like this



As an example, it is well-known that the resolution of the singularity

$$X \subseteq \mathbb{A}^3 \text{ given by } x^2+y^2+z^{n+1}=0$$

is as



$E_1, \dots, E_n$

$E_i \cdot E_i = -2.$

Klt pairs (X,B) can be classified similarly but we need to take into account the configuration of

$$B^{\sim} \cup (\cup E_i).$$

The smaller the coefficients of B the more complicated the configurations can be.

Although it is helpful to know examples of singularities but often it is their formal properties which are very helpful in proofs and inductive arguments.

Example: $X \subseteq \mathbb{A}^4$ given by $xy - zt = 0$.

X is singular at $(0,0,0,0)$.

Let $\psi: V \longrightarrow \mathbb{A}^4$ be the blowup at $(0,0,0,0)$,

W birational transform of X, and

$\varphi: W \longrightarrow X$ the induced morphism.

Let F = exceptional divisor of ψ

and $E = W \cap F$.

Then $F \simeq \mathbb{P}^3$ and $E \subseteq \mathbb{P}^3$ is given by

the equation $xy - zt$.

In particular, we can see that W is smooth,

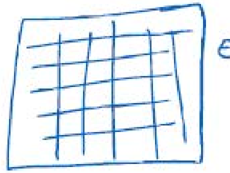
and φ is a resolution.

Now we have an isomorphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow E \subseteq F = \mathbb{P}^3$$

$$(a:b),(c:d) \longmapsto (ac : ad : bc : bd)$$

Segre map



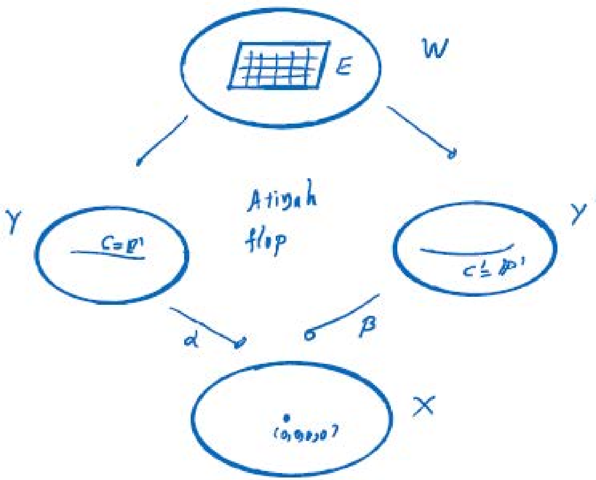
We have projection maps

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

They induce maps

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ Y \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

where both Y and Y' are smooth.



An interesting feature of this example is that both $Y \longrightarrow X$ and $Y' \longrightarrow X$ are resolutions that do not have exceptional divisors.

Both Y and Y' can claim to be "minimal" resolutions.

This is quite different from surface case where there is only one "minimal" resolution.

Another interesting feature is that there are divisors on X which are not $\mathbb{Q}$-Cartier.

Indeed, let A be a divisor on Y s.t.

$$A \cdot C \neq 0,$$

and let $H = \alpha_* A$.

Then H is not $\mathbb{Q}$-Cartier:

if H is $\mathbb{Q}$-Cartier, then

$A = \alpha^* H \implies A \cdot C = 0,$

a contradiction.