On the geometry of the growing ball of first-passage percolation

(Based on joint works with M. Damron, J. Gold, J. Hanson, X. Shen)

First-passage percolation

Consider $\mathbb{Z}^d$ $(d \geq 2)$ with nearest-neighbor edges $\mathcal{E}^d$,

Let $(t_e)_{e \in \mathcal{E}^d}$ be iid nonnegative weights (passage times)

For a path γ, define $T(\gamma) = \sum_{e \in \gamma} t_e$

For $x, y \in \mathbb{Z}^d$, define

$$T(x,y) = \inf \{ T(\gamma) : \gamma \text{ is a path from } x \text{ to } y \}$$

Easy to verify: T is a pseudometric.

Geometry of the metric space $(\mathbb{Z}^d, T)$?

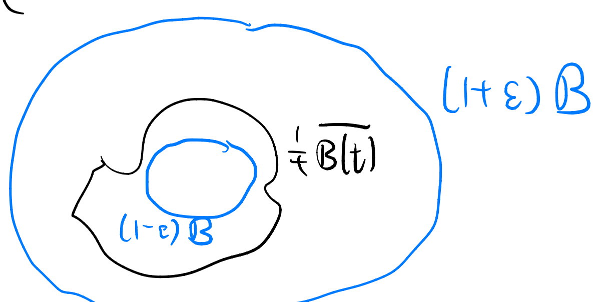
$$B(t) = \{x \in \mathbb{Z}^d ;\ T(0,x) \leq t\}.$$

Critical threshold for Bernoulli percolation

<u>Shape theorem</u> (Cox – Durrett).

Define $\overline{B(t)} = B(t) + [0,1)^d$. If $\mathbb{P}(t_e = 0) < p_c$ and $\mathbb{E}Y^d < \infty$, where $Y = \min\{t_1, \dots, t_{2d}\}$ and t_i's are iid copies of t_e, then $\exists$ a nonrandom, compact, convex set $\mathbb{B}$ with nonempty interior s.t. $\forall \varepsilon > 0$,

$$\mathbb{P}\left((1-\varepsilon)\mathbb{B} \subseteq \frac{1}{t}\overline{B(t)} \subseteq (1+\varepsilon)\mathbb{B} \quad \forall \text{large } t \right) = 1.$$



A weaker version (Kesten):

If $P(t_e = 0) < p_c$, then $\exists\, \mathcal{B}$ with the above properties s.t.

a.s., $\quad \mathrm{Vol}\left(\frac{1}{t}\overline{B(t)} \,\Delta\, \mathcal{B}\right) \to 0$ as $t \to \infty$.

Not too much is known about $\mathcal{B}$.

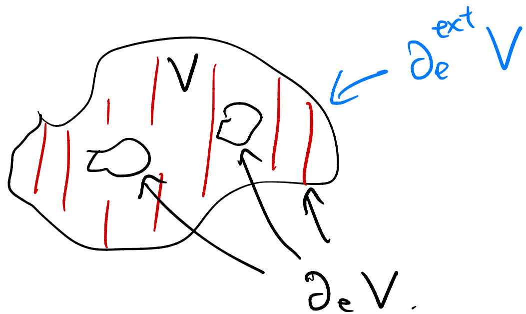
Also interesting: The geometry of $B(t)$ for large t?

We will focus on the boundary and holes of $B(t)$.

<u>Definition</u> V vertex set.

$$\partial_e V = \{\{x,y\} \in \mathcal{E}^d : x \in V,\ y \notin V\}.$$

$$\partial_e^{ext} V = \{\{x,y\} \in \mathcal{E}^d : x \in V,\ y \notin V,\ y \leftrightarrow \infty \text{ without using a vertex in } V\}$$



Want to study: $\partial_e B(t)$, $\partial_e^{ext} B(t)$.

What is the size of $\partial_e B(t)$? $\partial_e^{ext} B(t)$?

Expect: the order should be $\sim t^{d-1}$.

<u>Theorem</u> (Damron - Hanson - L., '18). Suppose that $\mathbb{P}(t_e = 0) < p_c$.

Upper bound: $\exists C > 0$ s.t. a.s.,

$$\limsup_{t\to\infty} \frac{\mathrm{Leb}\left(\{s \in [0,t] : \#\partial_e B(s) \geq a\, s^{d-1}\, \mathbb{E}[Y \wedge s]\}\right)}{t} \leq \frac{C}{a}$$

$$\limsup_{t\to\infty} \frac{\mathrm{Leb}(\{s\in[0,t] : \#\partial_e^{ext} B(s) \geq a s^{d-1}\})}{t} \leq \frac{C}{a}.$$

Lower bound:

$\exists c>0$ s.t. a.s.,

$$\#\partial_e B(t) \geq c \max\{t\,\mathbb{P}(Y>t), 1\}\, t^{d-1} \quad \forall \text{ large } t.$$

<u>Explanations and remarks</u>

- $Y = \min\{t_1, \dots, t_{2d}\}$, t_i's iid copies of t_e.
- Upper bounds say that for <u>most</u> times t,

$$\#\partial_e B(t) \leq C t^{d-1}\mathbb{E}[Y\wedge t], \quad \#\partial_e^{ext} B(t) \leq C t^{d-1}$$

- By isoperimetric inequality, $\#\partial_e^{ext} B(t) \geq c t^{d-1}$

So the exterior boundary of $B(t)$ is "smooth".

- $\mathbb{E}[Y \wedge t]$? $t\mathbb{P}(Y>t)$?
  - If $\mathbb{E}Y < \infty$, then $\mathbb{E}[Y\wedge t] \leq \mathbb{E}Y$

    $\Rightarrow \#\partial_e B(t) \leq C t^{d-1}$ for most times t.

    $\max\{t\,\mathbb{P}(Y>t), 1\} \geq c$

    $\Rightarrow \#\partial_e B(t) \geq c t^{d-1}$ for all large t
  - If $\mathbb{P}(Y>t) \asymp \frac{1}{t^{1-\alpha}}$, $\alpha \in (0,1)$, then

    $$c\, t^{d-1+\alpha} \underset{\text{all}}{\leq} \#\partial_e B(t) \underset{\text{most}}{\leq} C\, t^{d-1+\alpha}$$

    Can be $\gg t^{d-1}$!

    But $\#\partial_e^{ext} B(t) \asymp t^{d-1}$

  $\to$ Many / large holes when Y has a heavy tail.

<u>Theorem</u> (DHL, '18). If we assume that $\mathcal{B}$ is uniformly curvature (not proved, but widely believed to hold for a large class of distributions), and t_e has an exponential moment, then $\exists C_1, C_2 > 0$ s.t. a.s. $\forall$ large t,

$$\# \partial_e B(t) \le C_1 t^{d-1} (\log t)^{C_2}.$$

Holes in $B(t)$

$M(t)$ = size of the largest hole in $B(t)$

$N(t)$ = number of holes in $B(t)$.

<u>Theorem</u> (Damron – Gold – L. – Shen, '22).

Suppose that $\mathbb{P}(t_e = 0) < p_c$ and dist. of t_e is nontrivial.

$\exists c > 0$ s.t.

$$\mathbb{P}(M(t) \geq c \log t \ \ \forall \text{ large } t) = 1, \quad \mathbb{P}(N(t) \geq c t^{d-1} \ \ \forall \text{ large } t) = 1.$$

Are these sharp?

<u>Remark</u> $N(t) \leq \# \partial_e B(t)$

If we assume that $\mathbb{B}$ is uniformly curved and t_e has exp. moment,

then $N(t) \leq C_1 t^{d-1} (\log t)^{C_2}$ $\forall$ large t.

<u>Theorem</u> (DGLS, '22).

Suppose $d=2$.

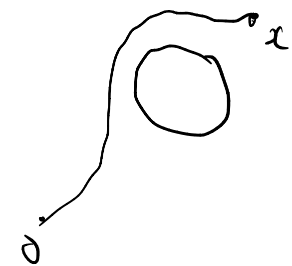
- If $\mathbb{P}(t_e=0)<p_c$, $\mathbb{B}$ is uniformly curved, t_e has exp. moments.

  then $\exists\, c>0$ s.t.

$$\mathbb{P}(M(t) \leq (\log t)^C \ \forall \text{ large } t) = 1.$$

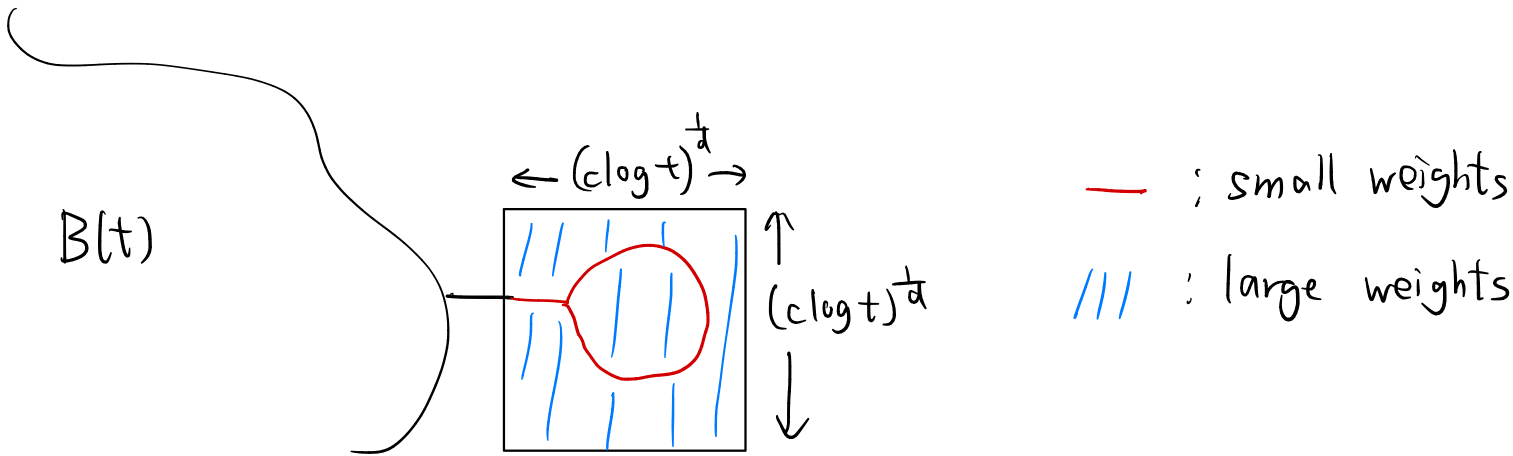
- Without curvature, $\exists\, C>0$ s.t.

$$\mathbb{P}(M(t) \leq Ct\log t \ \forall \text{ large } t) = 1.$$



Sketch of proof of $M(t) \geq c \log t$

Consider the configuration



Conditional on $B(t)$, if we are able to create such a configuration, $\exists$ a hole of a reasonable size at time $t+K$.

Difficulties: – If the edge weights are bounded, need to define the weights in the box carefully.

- If the edge weights are unbounded, the connecting edge can possibly have large weight.

$\# \partial_e^{ext} B(t) \gtrsim c t^{d-1} \Rightarrow \sim \dfrac{t^{d-1}}{(\log t)^{\frac{1}{d}}}$ positions to put disjoint configurations in $B(t)^c$.

Each configuration causes a prob. factor $\varepsilon^{c \log t}$.

$$\mathbb{P}(\exists \text{ such a configuration at time } t) = 1 - \left(1 - \varepsilon^{c\log t}\right)^{\frac{t^{d-1}}{(\log t)^{\frac{1}{d}}}}$$

$$= 1 - \left(1 - t^{c \log \varepsilon}\right)^{\frac{t^{d-1}}{(\log t)^{\frac{1}{d}}}}$$

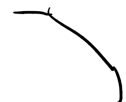
> 0 if c is small enough

A more careful argument allows us to conclude.

$d=2$

$\mathbb{P}(t_e = 1) > \vec{p}_c$

$\mathrm{supp}(t_e) \subseteq [1,\infty)$.



Häggstörm, Meester

Given any B, then $\exists$ dist. (t_e) ergodic, stationary

s.t. the limit shape is B.