2021 - 11 - 03 Kähler geometry

Donaldson–Fujiki picture

Scalar curvature and moment map.

"GIT–symplectic" stability

M compact Kähler manifold.

ω_0 fixed Kähler form.

$$\Omega = \{ \omega = \omega_0 + i\partial\bar\partial\varphi \mid \omega \text{ positive} \}.$$

$$\Uparrow$$

$$[\omega_0].$$

$\omega \qquad \omega'$

$g \qquad g'$

Moser's theorem

Let M be a compact symplectic mfd, and ω_t be a path by symplectic forms joining ω and ω'. Then $\exists$ diffeo $f : M \to M$ such that $\omega = f^*\omega'$

We apply this to two Kähler forms in a fixed cohomology class.

$$(M, \omega, J), \quad (M, \omega', J) \qquad J'$$

$$f^*(M, \omega', J) = (M, \omega, f^*J)$$

Ordinary Kähler geometry (Calabi's setup) ②

J fixed, ω varying.

Donaldson–Fujiki picture (using Moser's idea)

ω fixed, J varying.

M compact symplectic mfd.

$\dim_{\mathbb{R}} = n = 2m$

ω fixed symplectic form.

$Z = \{ J \in \Gamma(\mathrm{End}(TM)) \mid \underline{\omega\text{-compatible}}$ integrable almost complex structure $\}$

$\overset{\shortparallel}{N}$ before.

<u>Def</u> ω-compatible

$\underset{\text{def}}{\iff}$
- $\omega(JX, JY) = \omega(X, Y)$
- $J^2 = -\mathrm{id}$
- $g_J(X, Y) = \omega(X, JY)$ is a Kähler metric.

$\iff (M, \omega, J)$ is a Kähler mfd.

$J \in Z$ fixed.

$$T^*M \otimes \mathbb{C} = T_J^{*\prime}M \oplus T_J^{*\prime\prime}M$$

i-eigensp $(-i)$-eigensp of J.

$$T_J^{*\prime\prime}M = \overline{T_J^{*\prime}M}.$$

$J' \in Z$ another complex structure

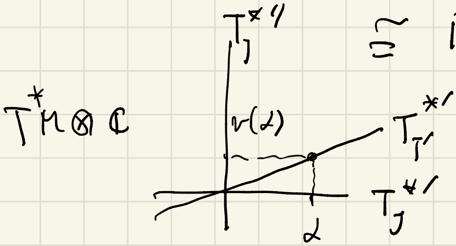
$$T_J^{*}M \otimes \mathbb{C} = T_{J'}^{*\prime}M \oplus T_{J'}^{*\prime\prime}M$$

If J' is sufficiently close to J, then $T_{J'}^{*\prime}M$ is expressed as a graph over $T_J^{*\prime}M$ as

$$T_{J'}^{*\prime}M = \{\alpha + v(\alpha) \mid \alpha \in T_J^{*\prime}M\}$$

where $v \in \Gamma(\mathrm{End}(T_J^{*\prime}M, T_J^{*\prime\prime}M))$

$$\cong \Gamma(T_J' M \otimes T^{*\prime\prime}_J M)$$



$$v \in \Gamma(T_J'M \otimes T_J^{\prime\prime *}M) \underset{\text{metric}}{\cong} \Gamma(T_J'M \otimes T_J'M)$$

Lemma $v \in \mathrm{Sym}(T'_J M \otimes T'_J M)$

symmetric part.

Proof A symplectic form ω on $T'_J M$ defines a symplectic form ω^{-1} on $T^{*\prime}_J M$.

If $\alpha, \beta \in T^{*\prime}_J M$, then by J-invariance of ω^{-1}

$$\omega^{-1}(\alpha, \beta) = \omega^{-1}(J\alpha, J\beta) = \omega^{-1}(i\alpha, i\beta)$$

$$= -\omega^{-1}(\alpha, \beta)$$

$$\therefore \omega^{-1}(\alpha, \beta) = 0.$$

Similarly

$$\omega^{-1}(v\alpha, v\beta) = 0.$$

Since ω^{-1} is also J'-invariant

$$\omega^{-1}(\alpha + v(\alpha), \beta + v(\beta)) = 0.$$

Thus

$$\omega^{-1}(\alpha, v(\beta)) = -\omega^{-1}(v(\alpha), \beta)$$

$$= \omega^{-1}(\beta, v(\alpha))$$

$$v^{ij} \leftarrow \underbrace{g^{j\bar{k}}}\,\alpha_j\, \underbrace{v^i{}_{\bar{k}}}\,\beta_i = \underbrace{g^{i\bar{k}}}\,\beta_i\, \underbrace{v^j{}_{\bar{k}}}\,\alpha_j. \rightarrow v^{ji}$$

$v^{ij} = v^{ji}$

$S_0, T_J Z \subset C^\infty(\mathrm{Sym}(T_J' M \otimes T_J' M))$

L^2-inner product.
i complex str.
$\Rightarrow$ Z : Kähler str.

K = the group of all Hamiltonian symplectomorphisms of (M, ω)

$\mathfrak{k} = C_0^\infty(M) = \{ u \in C^\infty(M) \mid \int_M u\, \omega^m = 0 \}$

$i(X)\omega = -du_X$.

= "normalized Hamiltonian functions"

K acts on Z as holomorphic isometries.

$f \in K$, $f : (M, \omega, J) \longrightarrow (M, \omega, J')$, $J' = (f^{-1})^* J$

<u>Theorem</u> (Donaldson-Fujiki)

$$\mu : Z \longrightarrow C^\infty(M)/\mathbb{R} = \mathfrak{k}^*$$

$$J \longmapsto (S(J), \cdot)_{L^2}$$

L^2-dual of scalar curvature $S(\omega, J) =: S(J)$.

is an equivariant moment map.

Proof omitted.

We want to show:

$$\mu^{-1}(0) = \left\{ J \;\middle|\; \begin{array}{l} (M, \omega, J) \text{ has a constant} \\ \text{scalar curvature} \end{array} \right\}$$

$N =$ the space of Kähler forms $\left| \begin{array}{l} L \to M \\ c_1(L) \end{array} \right.$

$\underset{\text{Moser}}{\simeq}$ the space of almost complex strs.

$\mu(\omega) =$ scalar curvature as L^2-dual.

Prop $\boxed{\mu(\omega) = 0 \iff \text{scal} = \text{const.}}$

Proof

$$\mathfrak{k} = \left\{ u_X \in C^\infty(M) \,\middle|\, X \text{ holo on } M, \ \int_M u_X \, \omega^n = 0 \right\}$$

$$\mu(\omega) = \text{Scal} \in \mathfrak{k}^*$$

$$\langle \text{Scal}, u \rangle_{L^2} = \int_M \text{Scal} \cdot u \, \omega^n$$

If Scal $=$ const

$$\Rightarrow \text{RHS} = \text{Scal} \int_M u_X \, \omega^n = 0.$$

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