Lecture No 20 May 17, 2022 (Tue)

§22 Neumann problem and local time

- One can give a probabilistic representation of a solution of Neumann boundary value problem for general elliptic or parabolic PDEs by means of SDEs.
- Here, for simplicity, we consider only the heat equation in one-dimensional space.
- We set $\mathbb{R}_+ := (0,\infty), \bar{\mathbb{R}}_+ = [0,\infty)$.

22.1 Neumann boundary value problem

Let $B = (B_t)_{t\geq 0}$ be a 1-dimensional Brownian motion starting at $x \in \mathbb{R}$ in general. For a given $f \in C_b(\bar{\mathbb{R}}_+)$, set

$$u(t,x) := E_x[f(|B_t|)], \quad t \geq 0,\ x \in \bar{\mathbb{R}}_+. \tag{1}$$

Then, we see $u \in C^\infty((0,\infty)\times\mathbb{R}_+) \cap C([0,\infty)\times\bar{\mathbb{R}}_+)$ and u satisfies the following Cauchy problem (2)–(3) of the heat equation with the Neumann boundary condition (4).

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x), \quad t>0,\ x\in\mathbb{R}_+, \tag{2}$$

$$u(0,x) = f(x), \quad x\in\bar{\mathbb{R}}_+, \tag{3}$$

$$\partial^+ u(t,0) = 0, \quad t>0, \tag{4}$$

where ∂^+ denotes the right derivative at $x=0$.

⊙ First, we extend f to the whole line $\mathbb{R}$ as

$$\tilde{f}(x) := f(-x), \quad x\in(-\infty,0)$$

and set $\tilde{u}(t,x) := E_x[\tilde{f}(B_t)]$ for $^\forall x\in\mathbb{R}$. Note $\tilde{f}\in C_b(\mathbb{R})$. Then, $\tilde{u}(t,x) = u(t,x), x\in\bar{\mathbb{R}}_+$. By this, we see that u satisfies (2) and (3). Moreover, by the symmetry of $\tilde{f}$ and the Brownian motion B_t, we see that $\tilde{u}(t,x)$ is an even function in x and, in particular, we obtain (4). □

We call $|B| := (|B_t|)_{t\geq 0}$ a reflecting Brownian motion at 0. It is a diffusion process (strong Markov process) on $\bar{\mathbb{R}}_+$.

22.2 Local time

Since $|B| := (|B_t|)_{t \geq 0}$ is a submartingale, it has a Doob-Meyer decomposition: $|B_t| = M_t + L_t$, where M_t is a continuous martingale and L_t is a continuous increasing process.

In general, let us consider the reflection at $a \in \mathbb{R}$.

[Theorem 22.1] (Tanaka's formula) For ${}^\forall a \in \mathbb{R}$, there exists a continuous increasing process L_t^a such that

$$|B_t - a| = |B_0 - a| + \int_0^t \operatorname{sgn}(B_s - a)\, dB_s + L_t^a, \quad \forall t \geq 0, \text{ a.s.}, \tag{5}$$

where $\operatorname{sgn} x = 1 (x \geq 0), = -1 (x < 0)$. □

Note that, for $\varphi(x) := |x - a|$, its derivatives in the sense of generalized functions are given by

$$\varphi'(x) = \operatorname{sgn}(x - a), \varphi''(x) = 2\delta_a(x).$$

[Proof] Take an approximating sequence $\varphi_n(x) \in C^2(\mathbb{R})$ of $\varphi(x) := |x - a|$ such that, as $n \to \infty$,

- $\varphi_n(x)$ converges to $\varphi(x)$ uniformly on $\mathbb{R}$,
- $\varphi_n'(x)$ converges to $\mathrm{sgn}(x - a)$ for ${}^\forall x \neq a$,
- they satisfy $\sup_{n,x} |\varphi_n'(x)| < \infty$ and $\varphi_n''(x) \geq 0$.

Then, one can apply Itô's formula for $\varphi_n(B_t)$ and obtain

$$\varphi_n(B_t) = \varphi_n(B_0) + \int_0^t \varphi_n'(B_s)\, dB_s + \frac{1}{2} \int_0^t \varphi_n''(B_s)\, ds. \quad (6)$$

However, as $n \to \infty$, $\varphi_n(B_t)$ converges to $|B_t - a|$ uniformly in $t \in [0, T]$ in a.s. sense.

For the 2nd term of (6), by Doob's inequality and Itô isometry, and then applying Lebesgue's convergence theorem (noting $B_s \neq a$ a.s.), we have

$$E\left[\sup_{0\le t\le T}\left|\int_0^t \varphi_n'(B_s)\,dB_s - \int_0^t \mathrm{sgn}(B_s - a)\,dB_s\right|^2\right] \le 4\int_0^T E[|\varphi_n'(B_s) - \mathrm{sgn}(B_s - a)|^2]\,ds \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, the 2nd term converges to $\int_0^t \mathrm{sgn}(B_s - a)\,dB_s$ uniformly in $t \in [0, T]$ in probability sense.

In particular, regarding (6) as an equation determining the last term $L^a_{t,n} := \frac{1}{2}\int_0^t \varphi_n''(B_s)\,ds$, by taking subsequence $n' \to \infty$ (if necessary), each term converges uniformly in $t \in [0, T]$ in a.s. sense. Therefore, setting L^a_t the limit of $L^a_{t,n'}$, since $L^a_{t,n}$ is a continuous increasing process, L^a_t has the same property and (5) holds. $\square$

[Remark] (i) Recall that, for $\varphi(x) := |x - a|$, its derivatives in the sense of generalized functions are given by

$$\varphi'(x) = \mathrm{sgn}(x - a), \varphi''(x) = 2\delta_a(x).$$

Thus, if Itô's formula is applicable for $\varphi(B_t)$, we would obtain a formal expression $L_t^a = \int_0^t \delta_a(B_s)\,ds$. Here, δ_a is the Dirac's δ-measure at a. Indeed, it is known that for $^\forall f \in C_b(\mathbb{R})$,

$$\int_{\mathbb{R}} f(a) L_t^a \, da = \int_0^t f(B_s)\, ds \quad \text{a.s.} \tag{7}$$

holds. Formally saying, this can be observed by

$$\begin{aligned} \int_{\mathbb{R}} f(a) L_t^a \, da &= \int_{\mathbb{R}} f(a)\, da \int_0^t \delta_a(B_s)\, ds \\ &= \int_0^t ds \int_{\mathbb{R}} f(a) \delta_{B_s}(a)\, da = \int_0^t f(B_s)\, ds. \end{aligned}$$

[Remark] (ii) Taking $1_{\{a\}}$ instead of δ_a, we have $\int_0^t 1_{\{a\}}(B_s)\,ds = 0$ a.s. (differently from L_t^a), that is, the sojourn time (staying time) of the Brownian motion at a single point a is 0. Indeed, this is seen from

$$E\Big[\int_0^t 1_{\{a\}}(B_s)\,ds\Big] \underset{\text{Fubini's theorem}}{=} \int_0^t E[1_{\{a\}}(B_s)]\,ds$$
$$= \int_0^t P(B_s = a)\,ds = 0.$$

(iii) It is known that L_t^a has a continuous modification in (a,t). $\square$

[Proposition 22.2] L_t^a increases only at t such that $B_t = a$. Namely, $\int_0^t 1_{\{B_s \neq a\}} \, dL_s^a = 0$ a.s. holds, where dL_s^a is the Stieltjes measure determined from L_s^a. $\square$

[Proof] Assume $a = 0$ for simplicity. Then, by (5) (Tanaka's formula) with $a = 0$, we have

$$|B_t| = |B_0| + \int_0^t \mathrm{sgn}(B_s) \, dB_s + L_t^0. \tag{8}$$

Then, applying Itô's formula for $d\varphi(|B_t|)$ with $\varphi(x) = x^2$, since $\varphi'(x) = 2x, \varphi''(x) = 2, d|B_t| = \mathrm{sgn}(B_t) \, dB_t + dL_t^0$ and $\varphi(|B_t|) = B_t^2$, we have

$$\begin{aligned} B_t^2 &= B_0^2 + 2\int_0^t |B_s| \mathrm{sgn}(B_s) \, dB_s + 2\int_0^t |B_s| \, dL_s^0 + \int_0^t \mathrm{sgn}(B_s)^2 \, ds \\ &= B_0^2 + 2\int_0^t B_s \, dB_s + 2\int_0^t |B_s| \, dL_s^0 + t, \end{aligned}$$

from which we obtain $\int_0^t |B_s| \, dL_s^0 = 0$. This implies that $|B_s| = 0$ for dL_s^0-a.e.s, so that we obtain the conclusion (with $a = 0$). $\square$

[Definition 22.1] We call L_t^a the local time of the Brownian motion B_t at a. $\square$

It is known that the local time has the following expression:

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[a-\varepsilon, a+\varepsilon]}(B_s)\, ds, \quad \text{a.s.} \tag{9}$$

Indeed, taking $f(a) = \frac{1}{2\varepsilon} 1_{[b-\varepsilon, b+\varepsilon]}(a)$ in (7), we have

$$\frac{1}{2\varepsilon} \int_{b-\varepsilon}^{b+\varepsilon} L_t^a\, da = \frac{1}{2\varepsilon} \int_0^t 1_{[b-\varepsilon, b+\varepsilon]}(B_s)\, ds.$$

However, by Remark (iii), L_t^a is continuous in a. Therefore, as $\varepsilon \downarrow 0$, LHS converges to L_t^b, and this shows (9) for $a = b$.

As we saw in Remark (ii), the sojourn time of the Brownian motion at the point a is 0. However, the sojourn time on its neighborhood $[a-\varepsilon, a+\varepsilon]$ scaled by $(2\varepsilon)^{-1}$ has the limit L_t^a as $\varepsilon \downarrow 0$. It is called the local time in a sense that we consider the time spent by the Brownian motion in a neighborhood of a under a proper scaling.

22.3 Skorohod's SDE

Reflecting Brownian motion can be constructed also by solving Skorohod's SDE. It is a problem to find a pair (X_t, ψ_t), which satisfies the following conditions (10)–(12), given a starting point $x \in \bar{\mathbb{R}}_+$ and the Brownian motion B_t starting at 0:

$$X_t \text{ is an } \bar{\mathbb{R}}_+\text{-valued continuous process} \tag{10}$$

$$\begin{aligned} &\psi_t \text{ is a continuous increasing process s.t. } \psi_0 = 0, \\ &\quad \text{and it increases only at } t \text{ such that } X_t = 0, \\ &\quad \text{that is, } \int_0^t 1_{\{0\}}(X_s)\, d\psi_s = \psi_t \text{ a.s.} \end{aligned} \tag{11}$$

$$X_t = x + B_t + \psi_t \tag{12}$$

This problem can be discussed for each fixed ω. Indeed, let a sample path $B_t \in W = C([0,\infty), \mathbb{R})$ be given, [Lemma 22.3] (X_t, ψ_t) satisfying (10)–(12) exists uniquely. Indeed, ψ is given by

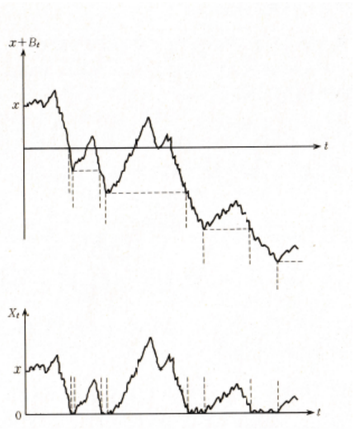
$$\psi_t = \sup_{s\in[0,t]} \{(-x - B_s) \vee 0\}.$$

In particular, when $x = 0$, we have $\psi_t = -\inf_{s\leq t} B_s$. $\square$

[Proof] The picture of the next page suggests that we may take ψ_t as above. To show uniqueness, let us assume that there exist another pair $(\tilde{X}_t, \tilde{\psi}_t)$ satisfying the conditions (10)–(12). Then, since $X_t - \tilde{X}_t = \psi_t - \tilde{\psi}_t$ is of bounded variation in t, we have

$$\begin{aligned} 0 \leq (X_t - \tilde{X}_t)^2 &= 2\int_0^t (X_s - \tilde{X}_s)\, d(\psi_s - \tilde{\psi}_s) \\ &= -2\int_0^t \tilde{X}_s\, d\psi_s - 2\int_0^t X_s\, d\tilde{\psi}_s \leq 0 \end{aligned}$$

Here, we use (11) for the 2nd equality and (10), (11) for the last inequality. Therefore, we obtain $X_t = \tilde{X}_t$, which implies the uniqueness. $\square$



Solution of the Skorohod equation

The next theorem shows that $X_t = B_t - \inf_{s \le t} B_s$, which is a solution of Skorohod's SDE when $x = 0$, is a reflecting Brownian motion (or has the same distribution).

[Theorem 22.4] (Lévy) Two dimensional processes $(|B_t|, L_t^0)$ and $(B_t - I_t, -I_t)$ have a same distribution, where $I_t := \inf_{s \le t} B_s$. $\square$

[Proof] Set $\tilde{B}_t := \int_0^t \mathrm{sgn}(B_s)\, dB_s$, the term of stochastic integral appearing in Tanaka's formula (8) with $a = 0$. Then, $\tilde{B}_t$ is a martingale with quadratic variation

$$\langle \tilde{B} \rangle_t = \int_0^t \mathrm{sgn}(B_s)^2\, ds = t.$$

Therefore, by Lévy's theorem, $\tilde{B}_t$ is a Brownian motion. Moreover, given this $\tilde{B}_t$ first, $(|B_t|, L_t^0)$ is a solution of Skorohod equation with a starting point $x = 0$. Therefore, by the uniqueness of the solution, we have

$$(|B_t|, L_t^0) = (\tilde{B}_t - \inf_{s \le t} \tilde{B}_s, - \inf_{s \le t} \tilde{B}_s).$$

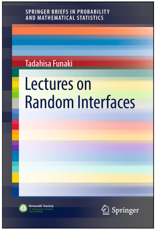
However, by replacing $\tilde{B}_t$ with B_t, the distribution of the RHS does not change so that we obtain the conclusion. $\square$

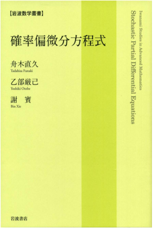
Part III. Applications of Stochastic Analysis

In this part, we discuss stochastic partial differential equations (SPDEs) as an application of stochastic analysis.

Textbooks:

[4] T. Funaki, Lectures on Random Interfaces, SpringerBriefs, 2016.
[7] T. Funaki, Y. Otobe, B. Xie (舟木直久, 乙部厳己, 謝賓):
確率偏微分方程式, 岩波書店, 2019.





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