Symplectic manifolds

Def Say smooth manifold M is symplectic if have

2-form ω on M, namely a section ω of $\Lambda^2 T^\vee M$

such that ① ω is closed ($d\omega = 0$)

② the tangent space T_pM

is symplectic for all $p \in M$.

Note By the above $\omega^n \neq 0$, giving a non-vanishing section of $\Lambda^{2n} T^\vee M$, namely an orientation.

Def A diffeomorphism $f: M \to N$ is a symplectomorphism if $f^* \omega_N = \omega_M$.

Rem The group of self-symplectomorphisms is generally infinite dimensional, in contrast to situation in algebraic geometry.

Submanifolds

Def Say a submanifold $N \subset M$ is Lagrangian if T_pN is Lagrangian in T_pM for all $p \in N$.

Similarly define (co)isotropic submanifolds.

Rem Such N is Lagrangian iff $\omega|_N = 0$

and $\dim N = n = \frac{1}{2}\dim M$

Cotangent bundles

For any manifold N, dim n, have

cotangent bundle $T^\vee N$, dim $2n$.

Notation: projection $\pi : T^\vee N \to N$

point $(p,\alpha) \in T^\vee N$ for $\alpha \in T_p^\vee N$

Def: tautological 1-form θ on $T^\vee N$ has value

$- d\pi^*(\alpha)$ at point (p, α)

(Recall differential $d\pi$ gives map of bundles

$T T^\vee N \to TN$ with dual $T^\vee N \to T^\vee T^\vee N$)

<u>Prop</u> $\omega = d\theta$ is symplectic form on $T^\vee N$.

<u>Proof</u> First, $d\omega = 0$ using $d^2 = 0$.

Now take local coordinates $x_1 - x_n$ on N

and take further coordinates $y_1 - y_n$ such

that we may write $\alpha = \sum_i y_i dx_i$.

Then $\theta = -\sum_i y_i dx_i$

and $\omega = \sum dx_i \wedge dy_i$

the canonical symplectic form. ✓

<u>Rem</u> For $N = \mathbb{R}^n$, get $T^\vee N \cong \mathbb{R}^{2n}$

Lagrangian examples

① Fibres $T^\vee_p N \subset T^\vee N$

② Zero section $Z \subset T^\vee N$, $Z \cong N$.

③ For 1-form α on N, the graph $\Gamma(\alpha)$

$$\Gamma(\alpha) = \{(p, \alpha(p)) \mid p \in N\} \subset T^\vee N$$

is Lagrangian iff α is closed.

Because $\theta|_\Gamma = -\pi^*(\alpha)$ (pullback of 1-form)

and so $\omega|_\Gamma = d\theta|_\Gamma = -d\pi^*(\alpha) = -\pi^*(d\alpha)$

<u>Example</u> The torus $L = \{x_i^2 + y_i^2 = a_i\}$ in $\mathbb{R}^{2n}$ for $a_i > 0$

is a Lagrangian, and compact.

Symplectomorphisms and Lagrangians

Consider a diffeomorphism $f: M \to N$ of symplectic manifolds. Then $M \times N$ has a symplectic form

$$\omega = \pi_M^*(\omega_M) - \pi_N^*(\omega_N)$$

where ω_M, ω_N are projections to M, N.

Take the graph $\Gamma(f)$ of f

$$\Gamma(f) = \{(p,q) \mid q = f(p)\} \subset M \times N$$

Now $\pi_N = f \circ \pi_M$ on Γ, so on Γ have

$$\omega = \pi_M^*(\omega_M) - \pi_M^* f^*(\omega_N)$$

$$= \pi_M^*(\omega_M - f^*(\omega_N)) = 0 \text{ iff } f \text{ symplectomorphism}$$

Hence, beautifully, $\boxed{f \text{ symplectomorphism iff } \Gamma(f) \text{ Lagrangian.}}$