Lecture 9 • Geography of irreducible/symplectic 4-manifolds

• spin^c structures on 4-manifolds

X: simply-connected 4-manifold, closed, oriented

X is irreducible iff $X \not\cong_{\text{diff}} X_0 \# X_1$ with $X_0, X_1 \not\cong_{\text{top}} S^4$.

Recall $c(X) := 2\chi(X) + 3\sigma(X)$ $\quad \chi_h(X) := \dfrac{\chi(X)+\sigma(X)}{4}$

$t(X) := \begin{cases} 0 & Q_X \text{ is even} \\ 1 & Q_X \text{ is odd} \end{cases}$

Freedman's theorem: $(c(X), \chi_h(X), t(X)) \overset{\text{determine}}{\rightsquigarrow}$ homeomorphism

Question: Which triple (c, χ_h, t) can be realized by smooth, irreducible, simply connected X?

($\Leftrightarrow$ which simply connected 4-manifold admits a smooth structure?)

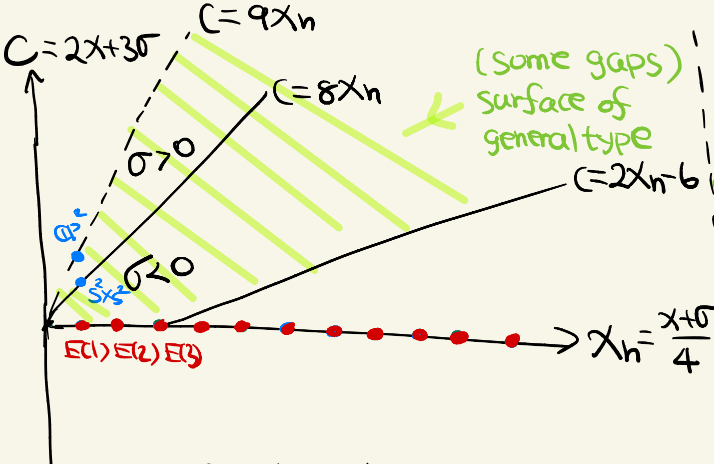
First, $\chi_h(X) = \dfrac{\chi(X)+\sigma(X)}{4} = \dfrac{b^+(X)+1}{2} \in \mathbb{Z} \cup \mathbb{Z}+\frac{1}{2}$

Conjecture: For irreducible X, $\chi_h(X)$ or $\chi_h(\bar{X})$ must be integer.

Fact: $\chi_h(X) \in \mathbb{Z}$ iff X has an almost complex structure.

So far, the only tool to prove X is irreducible is via Seiberg-Witten / Donaldson invariant, which only works when b^+ is $\chi_h(X) \in \mathbb{Z}$.

From now on, let's assume $\chi_h(X) \in \mathbb{Z}$



complex picture

Recall: • rational / ruled X minimal, simply connected $\Rightarrow X \cong_{diff} \mathbb{CP}^2$ or $S^2 \times S^2$

• surface of general type. Then

- $c \le 9\chi_h$ (Bogomolov-Miyaoka-Yau inequality)
- $c \geq 2\chi_h - 6$ $(c > 0)$ (Noether inequality)

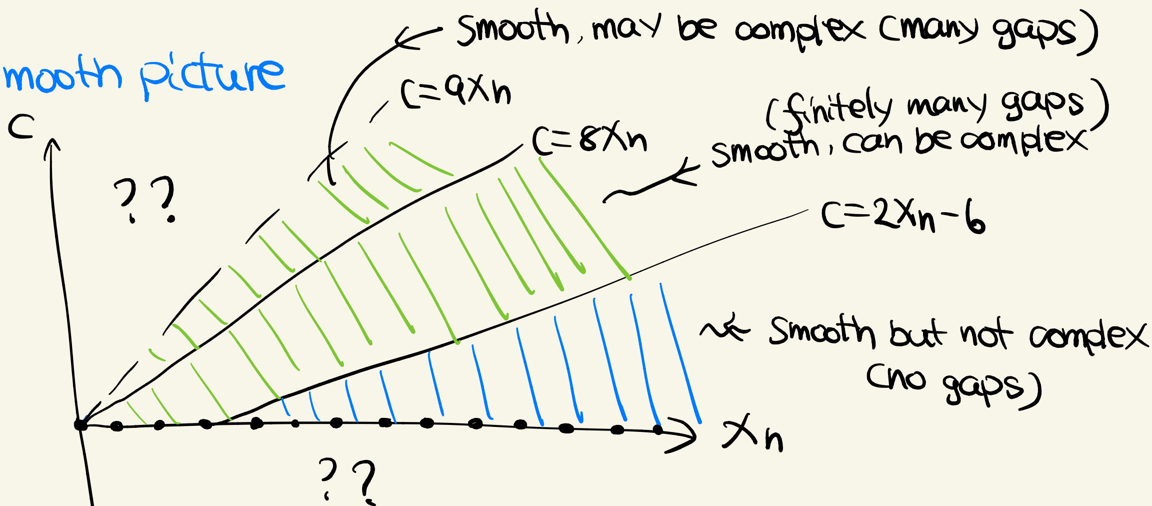
Max Noether (father of Emmy Noether)

• X is elliptic $E(n)$ $\Rightarrow c = 0$

It's believed all (c, χ_h) with $2\chi_h - 6 \le c \le 9\chi_h$ can be realized by complex surface (not necessarily simply connected)

On the BMY line, $\tilde{X} = D^4$ so is never simply connected

Smooth picture



Q1: Does there exists smooth simplex connected with $c \geq 9\chi_h$ or $c < 0$?

Take $t(X) \in \{0, 1\}$ into account.

$t=0 \Rightarrow Q_X = mE_8 \oplus n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow c \equiv 8\chi_h \mod 16$

with this extra constraint, all discussions above still holds.

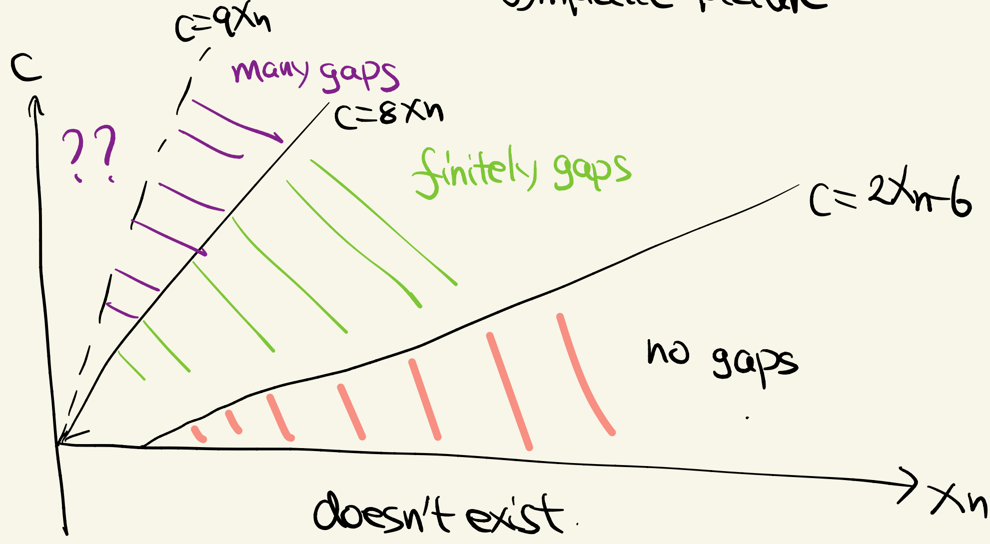
Symplectic geography problem

X is symplectic if $\exists\, \omega \in \Omega^2(X)$ s.t. $d\omega = 0$

$\omega \wedge \omega \cdots \wedge \omega \in \Omega^{\frac{\dim(X)}{2}}(X)$ nowhere vanishing

Algebraic $\Rightarrow$ Symplectic $\Rightarrow$ Smooth.

Symplectic picture



Theorem (Taubes) If X is a minimal symplectic with $\pi_1(X) = 1$. Then $c \geq 0$.

Irreducible $\Rightarrow$ minimal ("$\Leftarrow$" if X is simply connected (Kostchick, Taubes))

The Symplectic BMY-conjecture

Conjecture: Let X be a symply connected, symplectic 4-mfd. Then $c(X) \le 9\chi_h(X)$ ($\Leftrightarrow 3\sigma(X) \le \chi(S)$)

Botnay problem: Which 4-mfds admits an exotic smooth structure?

- $b^+ > 1$, $\pi_1 = 1$, irreducible: all known examples are homeomorphic to a symplectic X s.t. $\exists T^2 \hookrightarrow X$ with
  - $T^2 \cdot T^2 = 0$
  - $[T^2] \neq 0 \in H_2(X; \mathbb{Z})$
  - $\pi_1(X \setminus T^2) = 1$

Theorem (Fintushel-Stern) Any such X admits ∞ smooth str.

- $b^+ = 1$ $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ for $n \geq 2$ has ∞ smooth structures.

  (Park, Akhmedov, Fintushel, Stern, Stipsicz, Szabó ...)

  unknown: $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, $\mathbb{CP}^2$

- $b^+ = 0$ unknown (e.g. S^4)

- Some other unknown examples: T^4, $S^1 \times S^3$.

Horikowa surfaces $H(4n-1)$ $H'(4n-1)$ ← complex mfds not deformation equivalent. unknow whether they are diffeo or not.

- exotic $\mathbb{RP}^4$: not s-cobordant to $\mathbb{RP}^4$ (Cappel-Shaneson)

Next: Seiberg-Witten invariants

Clifford modules

$$\Delta : C^\infty(\mathbb{R}^n, \mathbb{C}^m) \longrightarrow C^\infty(\mathbb{R}^n, \mathbb{C}^m)$$
$$\vec{f} \longmapsto -\left(\sum_{i=1}^n \frac{\partial}{\partial x_i^2}\right)\cdot \vec{f}$$

Self-adjoint, positive definite $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$

$\langle f, \Delta f\rangle \geq 0$

Here $\langle f, g\rangle = \int f \cdot \bar{g}\, d\mathrm{vol}$.

Q: Can we write $\Delta = D^2$ for some self-adjoint operator?

- $n=1$ $\quad D = i \cdot \frac{\partial}{\partial x}$
- $n=2 \quad m=2$ $\quad D = \begin{pmatrix} 0, & \frac{\partial}{\partial \bar{z}} \\ -\frac{\partial}{\partial z}, & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial y}$

In general $D = \sum_{i=1}^n A_i \frac{\partial}{\partial x_i}$ $\quad A_i \in \mathrm{End}(\mathbb{C}^m)$ s.t. $A_i = \bar{A}_i^T$

$$A_i \cdot A_j = \begin{cases} -1 & i=j \\ -A_j \cdot A_i & i \neq j \end{cases}$$

$\boxed{\text{Definition:}}$ Let $\{e_1, e_2, \dots e_n\}$ be orthonormal basis of H.
The Clifford algebra $Cl(H)$ is a real algebra generated by e_i, subject to relation $\begin{cases} e_i e_j + e_j \cdot e_i = 0 \\ e_i^2 = -1 \end{cases}$

Definition: Let H be a real vector space. Consider the tensor algebra $T(H)=\bigoplus_{n\geq 0} \underbrace{H\otimes\cdots\otimes H}_{n \text{ times}}$. We consider the ideal generated by $\{v\otimes v+\|v\|^2 1\}$. The quotient is called the Clifford algebra.

$\boxed{\text{Example}}$: $H=\mathbb{R}$ $\quad Cl(H)=\mathbb{C}$

$H=\mathbb{R}^2$ $\quad Cl(H)=\mathbb{H}$

Definition: A Clifford module of H is a Hermitian complex vector space V equipped with a Clifford multiplication $\gamma: H\to \mathrm{End}(V)$ s.t.

- If $\|e\|=1$, the $\gamma(e)^2=-1$
- If $e_1\perp e_2$, then $\gamma(e_1)\cdot\gamma(e_2)+\gamma(e_2)\cdot\gamma(e_1)=0$
- $\gamma(e)^*=-\gamma(e)$.

(So V is just a representation of $Cl(H)$)

$\boxed{\text{Theorem}}$: • If $n=2k$, then $\exists!$ finite dimensional irreducible Clifford module (S,γ) up to isomorphism $\dim_{\mathbb{C}} S=2^k$

• If $n=2k+1$, there are exactly two Clifford modules (S,γ), $(S,-\gamma)$ up to isomorphism. $\dim_{\mathbb{C}} S=2^k$.

$\boxed{\text{Example}}$: $H=\mathbb{R}^3$ $\quad S=\mathbb{C}^2$ $\quad \gamma(e_i)=B_i$, where

$B_1=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$ $\quad B_2=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}$ $\quad B_3=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$ $\leftsquigarrow$ Pauli matrices

Example $H = \mathbb{R}^4$ $\quad S = \mathbb{C}^4 = S^+ \oplus S^-$

$$\gamma(e_0) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \gamma(e_i) = \begin{pmatrix} 0 & -B_i \\ B_i & 0 \end{pmatrix} \quad i = 1, 2, 3$$

Given a Clifford module S over H, we define the $\boxed{\text{Dirac operator}}$ $\not{D} : C^\infty(H, S) \to C^\infty(H, S)$

by $\displaystyle \not{D}\phi = \sum_{i=1}^{n} \gamma(e_i) \cdot \frac{\partial \phi}{\partial e_i}$

where $\{e_i\}$ is a set of orthonormal basis for H.

Then $\not{D}$ is a self-adjoint operator $\not{D}^2 = \Delta$.

Want to do this on manifolds. Want to define

$\not{D} : C^\infty(M, S) \to C^\infty(M, \tilde{S})$ ← spinor bundle.

Need: • spin$^{\mathbb{C}}$ structure γ

• spin$^{\mathbb{C}}$ connection ∇

X: n-dim smooth, Riemannian manifold

$\boxed{\text{Definition}}$: A spin$^{\mathbb{C}}$ structure on X is a Hermitian bundle $S \to X$ with a bundle map $\rho : T_x X \to \mathrm{End}(S)$ s.t. $\forall x$, $\rho_x : T_x X \to \mathrm{End}(S_x)$ is an irreducible clifford module. (In particular $\forall u, v \in T_x X$ $u \perp v$

$$\rho(u)^2 = -\|u\|^2 \cdot \mathrm{Id} \quad \rho(u)\rho(v) + \rho(v)\rho(u) = 0$$

ρ extends to $Cl(T_*X) \xrightarrow{\rho} \mathrm{End}(S)$

Consider $\rho(e_1 e_2 \cdots e_n) : S \to S$

$$\rho(e_1 \cdots e_n)^2 = \rho(e_1 \cdots e_n e_1 \cdots e_n) = (-1)^{n + \frac{n(n-1)}{2}}$$
$$= (-1)^{\frac{n(n+1)}{2}}$$

Assume $n = 4k$. Then $\rho(\prod e_i)$ squares to (-1)

so $S = S^+ \oplus S^-$ according to ± 1 eigenvalue.

Moreover, $\forall v \in T_x X$ $\quad v \cdot \prod e_i = -\prod e_i \cdot v$

so $\rho(v) \cdot \rho(\prod e_i) = -\rho(\prod e_i) \cdot \rho(v)$

so $\rho(v) : S_x^{\pm} \longrightarrow S_x^{\mp}$

Specialize to dim 4, we get :

A spin$^{\mathbb{C}}$ structure on a Riemannian manifold X consists of

- Two rank-2 Hermitian bundles S^+, S^-
- bundle map $\rho : T_* X \longrightarrow \mathrm{Hom}(S^+, S^-)$

s.t. at each x, we can choose orthonormal basis $\{e_i\} \subset T_x X$ and Hermitian basis of $S^{\pm}$ s.t.

$\rho(e_0) = I \qquad \rho(e_i) = B_i \quad (i = 1, 2, 3)$

(we can extend ρ to $T_* X \to \mathrm{End}(S^+ \oplus S^-)$)

$$\rho(e_0) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \rho(e_i) = \begin{pmatrix} 0 & B_i \\ B_i & 0 \end{pmatrix} \quad i = 1, 2, 3.$$

There is a "principal bundle" definition of $Spin^{\mathbb{C}}$ str.

A little algebra again.

S: irreducible Clifford module of H

Given $g : H \to H$ isometry can we find $\tilde{g} : S \to S$ (isometry)

s.t. the diagram

$$\begin{tikzcd} H \arrow[r, "g"] \arrow[d, "e"'] & H \arrow[d, "e"] \\ \mathrm{End}(S) \arrow[r] & \mathrm{End}(S) \end{tikzcd}$$

$$f \longmapsto \tilde{g} \circ f \circ \tilde{g}^{-1}$$

commutes?

Yes. But $\tilde{g}$ is not unique. (Schur's lemma: $\forall \tilde{g}, \tilde{g}'$ differs by a scalar mutiplication.

So we get a principal S^1-bundle $S^1 \hookrightarrow \{\tilde{g} \text{ that lifts some } g\} \to SO(n)$ -- ①

$Spin(n)$: nontrivial 2-fold cover of $SO(n)$

$Spin(n) \longrightarrow SO(n)$ denote kernel by $\{\pm 1\}$

$$Spin^{\mathbb{C}}(n) = Spin(n) \times S^1 / (z,w) \sim (-z,-w)$$

We have a bundle $S^1 \hookrightarrow Spin^{\mathbb{C}}(n) \to SO(n)$ -- ②

$$[(z,w)] \longmapsto w^2$$

Proposition: ① $\cong$ ②

Corollary: A $spin^{\mathbb{C}}$ structure on X is the same as a lift of $SO(n) \hookrightarrow Fr \to X$ to $Spin^{\mathbb{C}} \hookrightarrow P \to X$.

Example: $n=4$ $\quad SO(4) = SU(2) \times SU(2) / \pm 1$

$$Spin(4) = SU(2) \times SU(2)$$

$$Spin^{\mathbb{C}}(4) = (SU(2) \times SU(2) \times S^1) / \pm 1$$

$$= \{(A, B) \in U(2) \times U(2) \mid \det A = \det B\}$$

So if we consider the associated bundle, we get $S^{\pm}$ with $\det(S^+) = \det(S^-)$

$$\begin{matrix} \det(S^+) \\ \shortparallel \\ \Lambda^2(S^+) \end{matrix}$$