L. La Fuente

The formal moment map geometry of the space of symplectic connections

arXiv:2106.13608.

Day 1:

References

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[3] B.V. Fedosov, A simple geometrical construction of deformation quantization, Journal of Differential Geometry 40, 213-238 (1994).

[4] T. Foth, A. Uribe, The manifold of compatible almost complex structures and geometric quantization, Comm. Math. Phys. 274 (2), 357–379 (2007).

[5] A. Futaki, L. La Fuente-Gravy, Kähler geometry and deformation quantization with moment maps, ICCM proceedings 2018, 31–66 (2020).

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+ Book: Quantization, Geometry and Non-Commutative structure in Mathematical Physics, in Mathematical Physics studies, Editors Cardona, Morales, Ocampo, Paycha, Reyes Lega,

Chap 2: Deformation quantization and group actions by S. Gutt.

I) Introduction and definitions

A) Goals of the lectures

- Teach Fedosov construction of $*$-product
- Present the paper arXiv: 2106.13608 :
  - Formal analogue of Foth-Uribe's paper
  - Deform the Cahen-Gutt moment map picture on the space of symplectic connections
  - Applications: - Study of automorphisms of $*$-products
    - Hamiltonian diffeomorphism

Main Tool: On (M,ω) a closed symplectic mfd

$\longrightarrow$ Deformation quantization: Fedosov construct:

∇ symplectic connection $\xrightarrow{\text{Fedosov}}$ $*_\nabla$ a star product.

$\longleftarrow$

Any "natural" $*$-product determines a symplectic connection.

B) Symplectic connections

(M, ω) symplectic manifold
 └ closed non-degenerate 2-form

<u>Def 1:</u> A symplectic connection ∇ on (M, ω) is a linear connection of TM st:

- $\nabla \omega = 0$ $\quad (\omega$ is ∇-parallel$)$
- $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ (∇ has no torsion)

<u>Facts:</u> ① There always exists a symplectic connection on (M, ω).

<u>Proof:</u> Take ∇^0 any linear connection which is torsion-free.

Define a tensor N on M by

$$(\nabla^0_X \omega)(Y, Z) =: \omega(N(X, Y), Z) \quad , \; X, Y, Z \in TM$$

anti-sym

$$\cdot \; \underset{XYZ}{\oplus} \omega(N(X,Y),Z) = 0$$

Then, define $\nabla_X Y := \nabla^0_X Y + \frac{1}{3} N(X,Y) + \frac{1}{3} N(Y,X)$.

Check ∇ is a symplectic connection.

$\square$

② Given ∇ a symplectic connection

Consider $A \in \Gamma\, \Lambda^1 M \otimes \mathrm{End}(TM)$

$$\rightsquigarrow \tilde{\nabla}_X Y := \nabla_X Y + A(X) Y \qquad X, Y \in \Gamma TM$$

$\tilde{\nabla}$ is symplectic connection iff $\underline{A}(\cdot, \cdot, \cdot) := \omega(\cdot, A(\cdot)\cdot)$ is totally symmetric

iff $\underline{A} \in \Gamma(\underbrace{S^3 T^* M}_{\text{symmetric 3-tensors on } M})$

③ In <u>Darboux coordinates</u>: $(\underset{\text{open in } M}{U}, \{x^i\})$ st $\omega_{|U} = \underset{\text{constant}}{\omega_{ij}} dx^i \wedge dx^j$

∇ symplectic connection: $(\nabla_{|U}) = d + \Gamma^k_{ij} dx^i$

$$\Rightarrow \omega_{\ell k} \Gamma^k_{ij} \text{ is symmetric in } i, j, \ell$$

<u>Def2:</u> · We denote by $\mathcal{E}(M,\omega)$ the space of symplectic connections on (M,ω)

· Given a symplectic connection ∇ on (M,ω):

$$\mathcal{E}(M,\omega) = \nabla + \Gamma(S^3T^*M)$$

is an affine space.

The curvature tensor R of a symplectic connection ∇ is

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

<u>Properties</u>: $X,Y,Z,T \in TM$

· $R(X,Y) \in \mathrm{End}(TM,\omega)$

· $\boxed{\underset{XYZ}{\overset{\curvearrowright}{\oplus}}}\, R(X,Y)Z = 0$ (1st Bianchi)

cyclic sum

· $\omega(T, R(X,Y)Z)$ is anti-symmetric in X,Y
symmetric in T,Z

· The Ricci tensor: $Ric(X,Y) := Tr(Z \mapsto R(Z,X)Y)$

Ric is symmetric in X,Y.

<u>Remark:</u> In Riemannian geom: Scalar curvature $g^{ij}Ric_{ij}$

In sympl geom: No g
ω is anti-symmetric
No scalar curvature!

· $\underset{X,Y,Z}{\overset{\curvearrowright}{\oplus}}(\nabla_X R)(Y,Z) = 0$

. Consider now $\varphi \in \mathrm{Symp}(M,\omega)$ ie $\left| \begin{array}{l} \varphi : M \to M \text{ is a diffeo of } M \\ \varphi^* \omega = \omega \end{array} \right.$

$\to$ Natural action on $\mathcal{E}(M,\omega)$ by

$$(\varphi.\underbrace{\nabla}_{\in \mathcal{E}(M,\omega)})_X Y := \varphi_* (\nabla_{\varphi_*^{-1} X} \varphi_*^{-1} Y) \quad , \quad X, Y \in \Gamma TM$$

$\Rightarrow \varphi . \nabla \in \mathcal{E}(M,\omega)$.

. The infinitesimal action: For X being a symplectic v.f. $(\mathcal{L}_X \omega = 0)$

$$\underbrace{(\mathcal{L}_X \nabla)_Y}_{\in \Gamma(\Lambda^1 M \otimes \mathrm{End}(TM))} Z = [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z]$$

$$= \nabla^2_{(Y,Z)} X - R(X,Y) Z$$

$$\frac{d}{dt}\Big|_0 (\phi^X_{-t} . \nabla)_Y Z$$

for ϕ^X_t being the flow of X.

Def 3: The symplectic structure on $\mathcal{E}(M,\omega)$

We consider (M,ω) closed sympl mfd, ∇ a symplectic connection

. The space $\mathcal{E}(M,\omega) = \nabla + \Gamma(S^3T^*M)$ is a Fréchet manifold modelled on $\Gamma(S^3T^*M)$

. For $\underline{A}, \underline{B} \in T_\nabla \mathcal{E}(M,\omega) \simeq \Gamma(S^3T^*M)$, we set

independent of ∇

$$\Omega^{\mathcal{E}}_{\nabla}(\underline{A},\underline{B}) := \int_M \Lambda^{i_1 j_1}\Lambda^{i_2 j_2}\Lambda^{i_3 j_3} \underline{A}_{i_1 i_2 i_3} \underline{B}_{j_1 j_2 j_3} \frac{\omega^{\wedge n}}{n!} \quad (\dim M = 2n)$$

where. $\Lambda^{k\ell}\omega_{\ell p} = \delta^k_p$.

. we use Einstein summation convention on repeated indices.

. The space $(\mathcal{E}(M,\omega), \Omega^{\mathcal{E}})$ is a symplectic manifold.

$\hookrightarrow$. $\Omega^{\mathcal{E}}$ is closed, it is actually a constant symplectic form

. $\Omega^{\mathcal{E}}$ is non-degenerate.

. $\mathcal{G}$ acts symplectically on $\mathcal{E}(M,\omega)$.

$\mathrm{Ham}(M,\omega)$

Goal of the lecture:

- Obtain a formal deformation of $\Omega^{\mathcal{E}}$ that is $\mathrm{Ham}(M,\omega)$-invariant.

- Obtain a formal moment map for this action.

C) Star products: definitions

Deformation quantization, Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer.

- Poisson algebra of a symplectic manifold (M, ω).

Def 4: . A Hamiltonian v.f X is a symplectic v.f ($\mathcal{L}_X \omega = 0$) for which

$\exists\, H \in C^\infty(M)$ st $\iota(X_H)\omega = dH$

. The Poisson bracket attached to the symplectic manifold (M, ω) is the map:

$$\{\cdot , \cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

defined by $\{F, G\} = -\,\omega(X_F, X_G)$

It satisfies: . $\{F, G\} = -\{G, F\}$.

. $\{F, GH\} = \{F, G\}H + \{F, H\}.G$ (Leibniz)

. $\underset{FGH}{\circlearrowleft} \{F, \{G, H\}\} = 0$ (Jacobi)

. The triple $(C^\infty(M), \underset{\uparrow}{\cdot}, \{\cdot , \cdot\})$ is called the Poisson algebra of (M, ω)

pointwise product

Def 5: A formal star product $*$ on (M, ω), or a formal deformation quantization of (M, ω), is a product

formal power series with coefficients in $C^\infty(M)$

$$* : C^\infty(M)[[\nu]] \times C^\infty(M)[[\nu]] \longrightarrow C^\infty(M)[[\nu]]$$

$$\Big(F = \sum_{r \geq 0} \nu^r F_r ,\, G\Big) \longmapsto F * G = \sum_{r \geq 0} \nu^r C_r(F, G)$$

$F_r \in C^\infty(M)$

ν a formal parameter

st: ① $*$ is associative.

② The C_r's are bidifferential ν-linear operator.

③ $C_0(F, G) = F.G$

$C_1(F, G) - C_1(G, F) = \{F, G\}$ $\quad \forall F, G \in C^\infty(M)$

④ $F * 1 = F = 1 * F$

Example: (Moyal $\star$-product).

$(\mathbb{R}^{2n}, \omega = \underbrace{\omega_{ij}}_{\text{constant}} dx^i \wedge dx^j)$ linear symplectic vector space.

$F, G \in C^\infty(\mathbb{R}^{2n})[[\nu]]$

$$(F \star_{Moyal} G)(x) := \left[\exp\left(\frac{\nu}{2} \Lambda^{ij} \partial_{y^i} \partial_{z^j}\right) F(y) \cdot G(z)\right]\Big|_{y=z=x} \qquad (\Lambda^{ij}\omega_{jl} = \delta^i_l)$$

$$= F(x) \cdot G(x) + \frac{\nu}{2}\{F, G\} + \sum_{r \geq 2} \left(\frac{\nu}{2}\right)^r \frac{1}{r!} \Lambda^{i_1 j_1} \cdots \Lambda^{i_r j_r} \frac{\partial^r F}{\partial x^{i_1} \cdots \partial x^{i_r}}(x) \cdot \frac{\partial^r G}{\partial x^{j_1} \cdots \partial x^{j_r}}.$$

- Moyal $\star$-product is associative

Proof: It relies on the following trick: $F, G, H \in C^\infty(M)$

$\partial_{t^l} = \frac{\partial}{\partial t^l}$

$$(1) \quad \partial_{t^l}\left[F \star_{Moyal} G(t)\right] = \left[(\partial_{x^l} + \partial_{y^l}) \exp\left(\frac{\nu}{2} \Lambda^{ij} \partial_{x^i} \partial_{y^j}\right) F(x) \cdot G(y)\right]\Big|_{x=y=t}$$

Then,

$$(H \star (F \star G))(x) = \left[\exp\left(\frac{\nu}{2} \Lambda^{ij} \partial_{u^i} \underset{\sim}{\partial_{v^j}}\right) H(u) \cdot \underline{(F \star G)(v)}\right]\Big|_{u=v=x}$$

$$\overset{(1)}{=} \left[\exp\left(\frac{\nu}{2} \Lambda^{ij} \partial_{u^i}(\partial_{x^j} + \partial_{y^j})\right) H(u) \exp\left(\frac{\nu}{2} \Lambda^{rs} \partial_{x^r} \partial_{y^s}\right) F(x) \cdot G(y)\right]\Big|_{u=x=y}$$

$$= \left[\exp\left(\frac{\nu}{2} \Lambda^{ij} (\partial_{u^i}\partial_{x^j} + \partial_{u^i}\partial_{y^j} + \partial_{x^i}\partial_{y^j})\right)(H(u) \cdot F(x) \cdot G(y))\right]\Big|_{u=y=x}$$

$$= ((H \star F) \star G)(x)$$

$\square$

Overview: . Fedosov construction

$\nabla \in \mathcal{E}(M, \omega) \longrightarrow *_\nabla$ star product

. After we will study a bundle of $*$-product algebra

$$\mathcal{V} = \mathcal{E}(M, \omega) \times \left(C^\infty(M)[[\nu]], *_\nabla\right)$$

depending on the base point

$$\begin{array}{c} \mathcal{V} \\ \downarrow \\ \mathcal{E}(M, \omega) \end{array}$$

- Present formal connections
- curvature of formal connection
- formal moment map picture on $\mathcal{E}(M, \omega)$.

II) Fedosov star products:

On a symplectic manifold (M, ω)

Fedosov construction: Choose $\begin{cases} \nabla \in \mathcal{E}(M,\omega) \\ \Omega \in \Omega^2(M)[[\nu]] \\ \text{closed} \end{cases}$ $\rightsquigarrow$ $*_{\nabla,\Omega}$ a star product

A) Formal Weyl algebra bundle

Take $x \in M$ and $\{e_1, \dots, e_n\}$ a basis of T_xM

$\{y^1, \dots, y^{2n}\}$ the dual basis of T_x^*M.

Def 6: The Weyl algebra W_x is the space of formal polynomial functions on T_xM of the form:

$$a(y,\nu) = \sum_{\substack{2k+r \geq 0 \\ k \geq 0,\, r \geq 0}} \nu^k a_{k,i_1 \dots i_r} y^{i_1} \cdots y^{i_r}$$

for $a_{k,i_1\dots i_r} \in \mathbb{R}$, symmetric in $i_1, \dots, i_r$, $2k+r$ is the total degree $\begin{pmatrix} \nu \text{ has degree } 2 \\ y \cdot \text{ - - - } 1 \end{pmatrix}$ W-

endowed with the Moyal product

$$(a \circ b)(y,\nu) := \left(\exp\left(\frac{\nu}{2} \Lambda^{ij} \partial_{z^i} \partial_{u^j}\right) a(z,\nu) \cdot b(u,\nu) \right)\Big|_{z=u=y}$$

$a, b \in W_x$.

There is a filtration wrt the total degree:

$$W_x \supset W_x^1 \supset W_x^2 \supset \cdots$$

and the $\circ$-product preserves this filtration.

Def 7: The formal Weyl algebra bundle is the bundle of algebra over M

$$\begin{array}{l} \mathcal{W} = \coprod_{x \in M} W_x \simeq \prod_{p \geq 0} \underbrace{S^p(T^*M)}_{\text{symmetric tensors on } M}[[\nu]] \\ \big\downarrow \\ M \end{array}$$