2021-10-20 Kähler geometry

郑恺, 同济大学 will give a mini-course on cscK metrics next week:
http://ymsc.tsinghua.edu.cn/en/content/show/247-344.html

Join Zoom Meeting
https://us02web.zoom.us/j/81649775126?pwd=UzYyYXNhSGh4UVFOamRuNXl0VHJmdz09

Meeting ID: 816 4977 5126
Passcode: Kahler

Def A Kähler metric g is an extremal Kähler metric if

$$g^{i\bar{j}} \frac{\partial S}{\partial \bar{z}^j} \frac{\partial}{\partial z^i} \quad \left(= \underrightarrow{\operatorname{grad}' S} \right)$$

is a holomorphic vector field.

Fact (Calabi) Ω Kähler class

$$\Phi : \Omega_+ \to \mathbb{R}, \qquad \Phi(g) = \int_M \operatorname{Scal}(g)^2 \, \omega_g^n$$

Ω_+: Kähler forms

Φ is called the Calabi energy.

g is extremal $\Longleftrightarrow$ g is a critical point of Φ.

Example

$$\{\text{KE}\} \subset \{\text{cscK}\} \subset \{\text{extremal}\}.$$

Matsushima Lichnerowicz Calabi

Th (Calabi) 1980's $\mathfrak{h}(M) = \sum \mathfrak{h}_\lambda(M)$

$$\mathfrak{h}_\lambda = \{ X \in \mathfrak{h}(M) \mid [\mathrm{grad}' s, X] = \lambda X \}$$

$$\boxed{\mathfrak{h}_0 = \mathfrak{k} \otimes \mathbb{C}} \quad \text{reductive.}$$

Theorem (F)

$$f : \mathfrak{h}(M) \longrightarrow \mathbb{C} \qquad \text{scal curv.}$$
$$\cup$$
$$X \longmapsto -\int_M u_X \, S \, \omega^m$$

$\left(\operatorname{grad}_g u_X = X, \quad \int_M u_X \, \omega^m = 0 \right)$

is independent of $\omega \in [\omega_0]$.

In particular if $f \neq 0$ then $\nexists$ cscK metric.

Remark Fano case $\qquad m = \dim_{\mathbb{C}} M$

$$\mathrm{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial F \quad , \quad R_{i\bar j} - g_{i\bar j} = F_{i\bar j}$$

$$\therefore \; S - m = \Delta F$$

$$\mathrm{Fut}(X) = \int_M XF\,\omega^m = \int_M u_X^{\,i} F_{i}\,\omega^m$$

$$= -\int_M u_X \Delta F\,\omega^m = -\int_M u_X (S-m)\,\omega^m$$

$$= -\int_M u_X S\,\omega^m = f(X).$$

<u>proof</u> of last statement of Theorem

If $\exists$ cscK metric so $S = \text{const}$.

$$f(X) = -\int u_X S\, \omega^n = -\underset{\text{const}}{S} \int_M u_X\, \omega^n$$

$$= 0.$$

☺

<u>Lemma</u> If $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + t\varphi_{i\bar{j}}$ $\quad t \in [0, \varepsilon)$

and X is holo vector field then u and $\tilde{u}$

satisfying $\operatorname{grad}'_g u = X$, $\operatorname{grad}'_{\tilde{g}} \tilde{u} = X$

are related by

$$\tilde{u} = u + t u^i \varphi_i \,.$$

$$g^{i\bar{j}} \frac{\partial u}{\partial \bar{z}^j} \frac{\partial \varphi}{\partial z^i}$$

☺ $\quad \tilde{g}^{i\bar{j}} (u + t u^k \varphi_k)_{\bar{j}} = \tilde{g}^{i\bar{j}} (u_{\bar{j}} + t u^k \varphi_{k\bar{j}})$

$$= \tilde{g}^{i\bar{j}} \Big(u^k \underbrace{\left(g_{k\bar{j}} + t \varphi_{k\bar{j}} \right)}_{\tilde{g}_{k\bar{j}}} \Big)$$

$$= u^i = X^i$$

$$\frac{d}{dt}\int (u + t u^k \varphi_k)\, \omega_t^m \Big|_{t=0}$$ ← for convenience

exercise.

$$= \int u^k \varphi_k\, \omega^m + \int u\, \Delta\varphi\, \omega^m$$

$$= -\int u\,\Delta\varphi \cdot \omega^n + \int u\,\Delta\varphi\, \omega^n = 0.$$

So if $\int_M u\,\omega^n = 0$ then $\int_M \tilde{u}\,\tilde{\omega}^m = 0$

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<u>Proof</u> of Thm (first part). just for convenience.

$$\frac{d}{dt}\int (u + t u^k \varphi_k)\, S_t\, \omega_t^m \Big|_{t=0}$$

$$= \int u^k \varphi_k \cdot S\, \omega^n - \int u(\Delta^2\varphi + R^{i\bar{j}}\varphi_{i\bar{j}})\,\omega^m$$

$$+ \int u S\,\Delta\varphi\, \omega^m \qquad g + t\,\nabla\bar{\nabla}\varphi$$

☺ $$S_t = -g_t^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det g_t$$

$$\frac{dS_t}{dt} = -\Delta^2\varphi - R^{i\bar{j}}\varphi_{i\bar{j}}$$

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$$= -\int u\, \nabla^k S \cdot \nabla_k \varphi\, \omega^n - \int u\, \Delta^2 \varphi \cdot \omega^n$$

$$+ \int \left(\nabla_{\bar j} u\, R^{i\bar j} + u\, \underbrace{R^{i\bar j}{}_{,\bar j}}_{S^{,i}} \right) \varphi_i\, \omega^n$$

$$= \int \left(\nabla_k u \cdot \cancel{\nabla^k} S + u\, \cancel{\Delta S} \right) \varphi\, \omega^n - \int \Delta^2 u \cdot \varphi\, \omega^n$$

$$- \int \left(u_{,i\bar j} R^{i\bar j} + \nabla_{\bar j} u \cdot R^{i\bar j}{}_{,i} + \cancel{u_{,i} S^{,i}} \right.$$

$$\left. + \cancel{u S^{,i}{}_{,i}} \right) \varphi\, \omega^n$$

$$= -\int \left(\Delta^2 u + u_{,i\bar j} R^{i\bar j} + \nabla_{\bar j} u \cdot S^{\bar j} \right) \varphi\, \omega^n$$

$$= -\int \mathcal{D} u \cdot \varphi\, \omega^n = 0$$

$$\left(\because\ \underline{\text{grad}'\, u \quad \text{holo}} \right)$$

$$\left(\begin{aligned} &\underline{\nabla_{\bar j} \nabla^i u = 0} \iff \nabla_{\bar j} \nabla_{\bar i} u = 0 \\ &\iff \mathcal{D} u = \nabla^{\bar j} \nabla^{\bar i}\ \nabla_{\bar j} \nabla_{\bar i} u. \end{aligned} \right)$$

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Rem $\underline{\underline{\text{Fut}}} = f : \mathfrak{f}(M) \to \mathbb{C}$

Futaki invariant.

Now we turn to K-stability.

Application of an idea in GIT.

What is geometric invariant theory?

- To construct a good moduli space of algebraic geometric objects.
  (such as holo vector bundles. or varieties.)
- You often have to take a quotient by a complex Lie group action.
- If you take a quotient naively, you do not get a good space.
  e.g. can not be Hausdorff.
  can not be compactified.
- If you discard "unstable" orbits you can get a good moduli space.

due to Mumford. ("L'Enseignement"

__Book.__ Mumford – Fogarty – Kirwan.

Stability in algebraic sense.

(later symplectic sense
in terms of moment map
__Donaldson – Kronheimer__)