Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 7. Rauzy–Veech induction in details. Continued fractions and cutting sequences as a particular case of the Rauzy–Veech induction

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Rauzy–Veech induction

Fundamental domain in the space of zippered rectangles

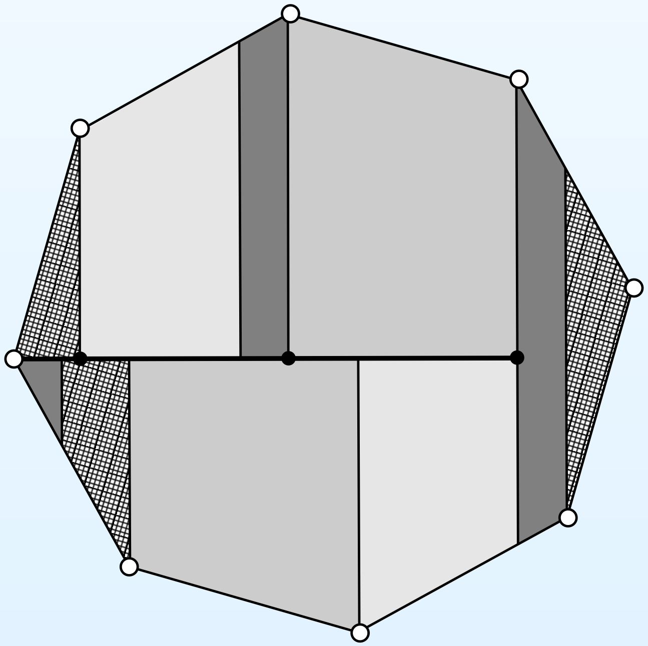
Assume that the vertical foliation on a translation surface S in some stratum $\mathcal{H}(d_1, \ldots, d_n)$ is uniquely ergodic. Launch a horizontal ray $(0x)$ in the positive direction from the singularity P_1. There is a discrete set of points b on the ray such that the first return map of the vertical flow to the subinterval $[0, b[$ is an interval exchange transformation with the minimal possible number of subintervals $m = 2g + n - 1$ under exchange. Choose among such b the smallest one satisfying $|0, b| \geq 1$. Denote by X the resulting horizontal interval of length $|0, b|$. By construction it has his left endpoint at P_1.

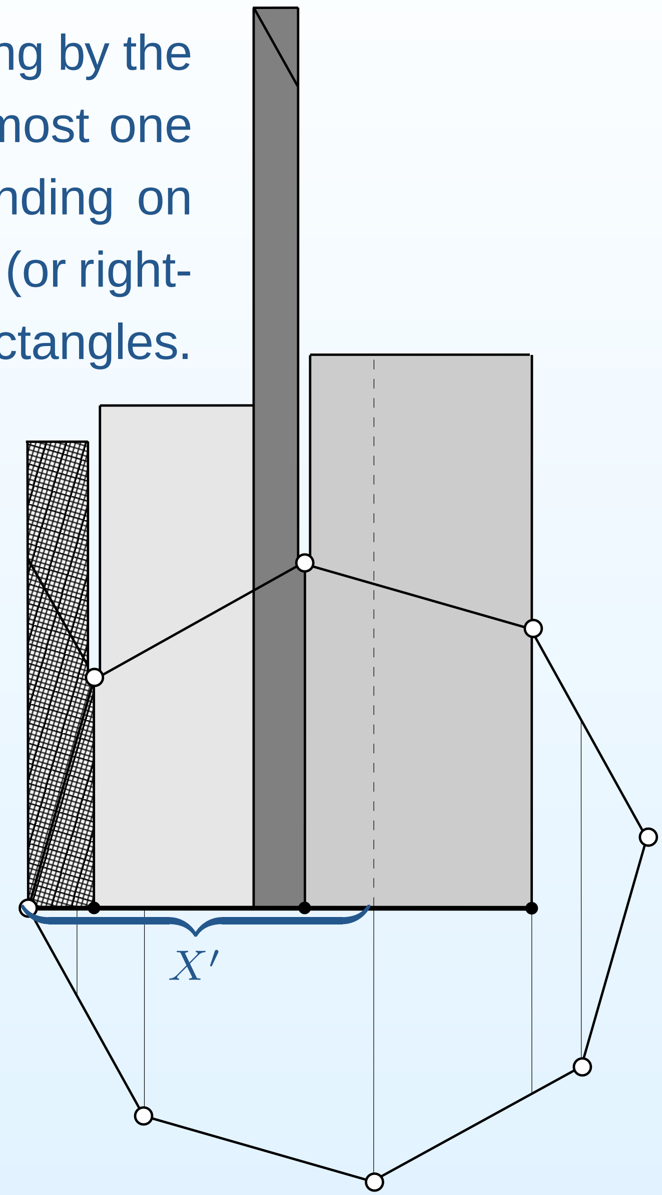
Unwrap the flat surface into m “zippered rectangles” with the base X. Denote by π the permutation such that the rectangles are glued to the bottom of the interval in the order $\pi^{-1}(1), \ldots, \pi^{-1}(m)$.

Reciprocally, take a collection of m rectangles over a base X which top horizontal sides are glued to the base X in the order $\pi^{-1}(1), \ldots, \pi^{-1}(m)$. “Zip” the adjacent rectangles up to the height $a_1, \ldots, a_{m-1}$. If the lengths of the remaining “unzipped” parts of the vertical sides pairwise match, we can glue a closed translation surface from the “zippered rectangles”.

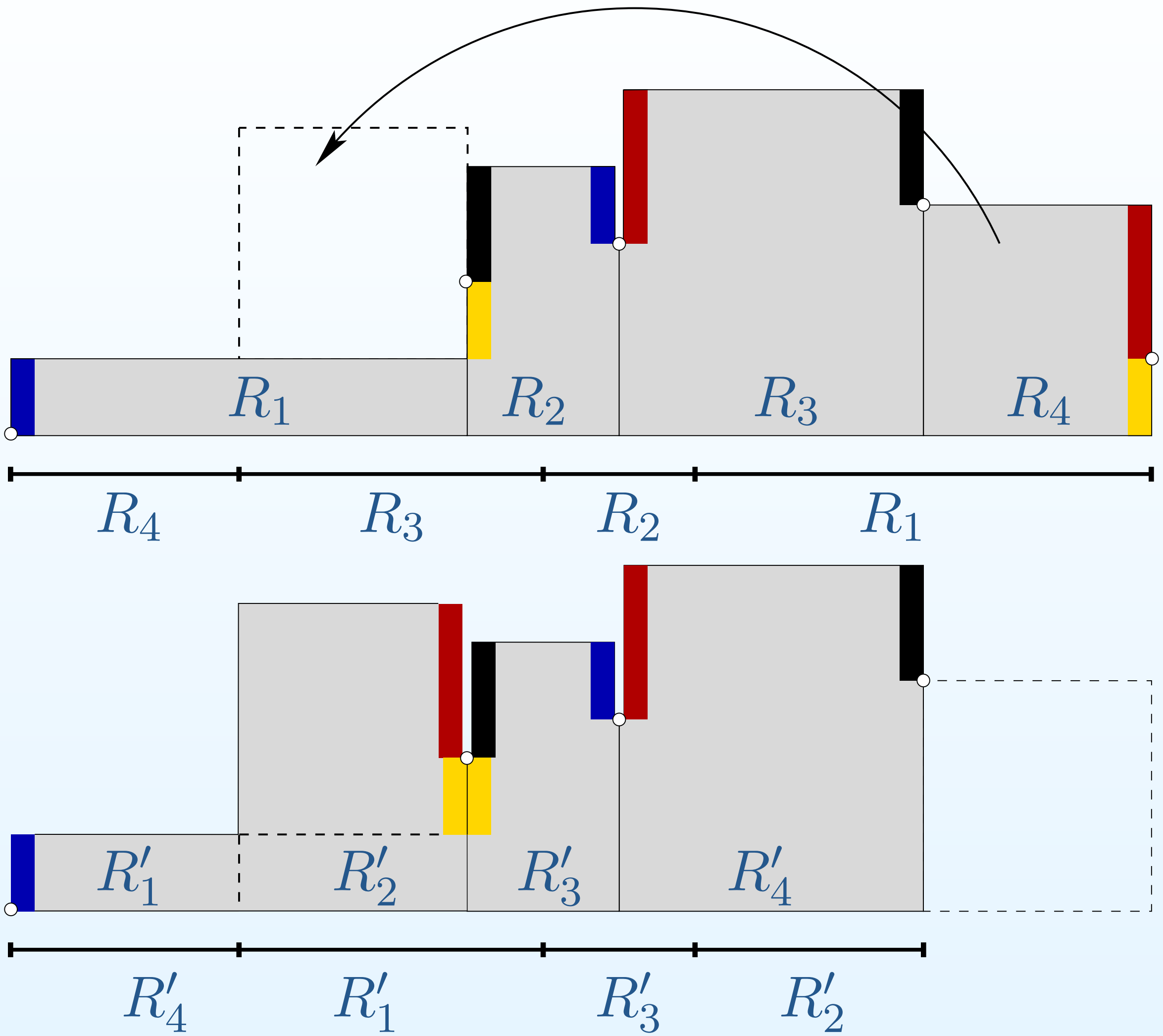
Rauzy–Veech induction geometrically

We shorten the base of the zippered rectangles building by the length of the smallest of the two intervals: the rightmost one and the one going to the rightmost position. Depending on which interval is shorter, the entire rightmost rectangle (or rightmost part of it) gets placed atop of one of the other rectangles.



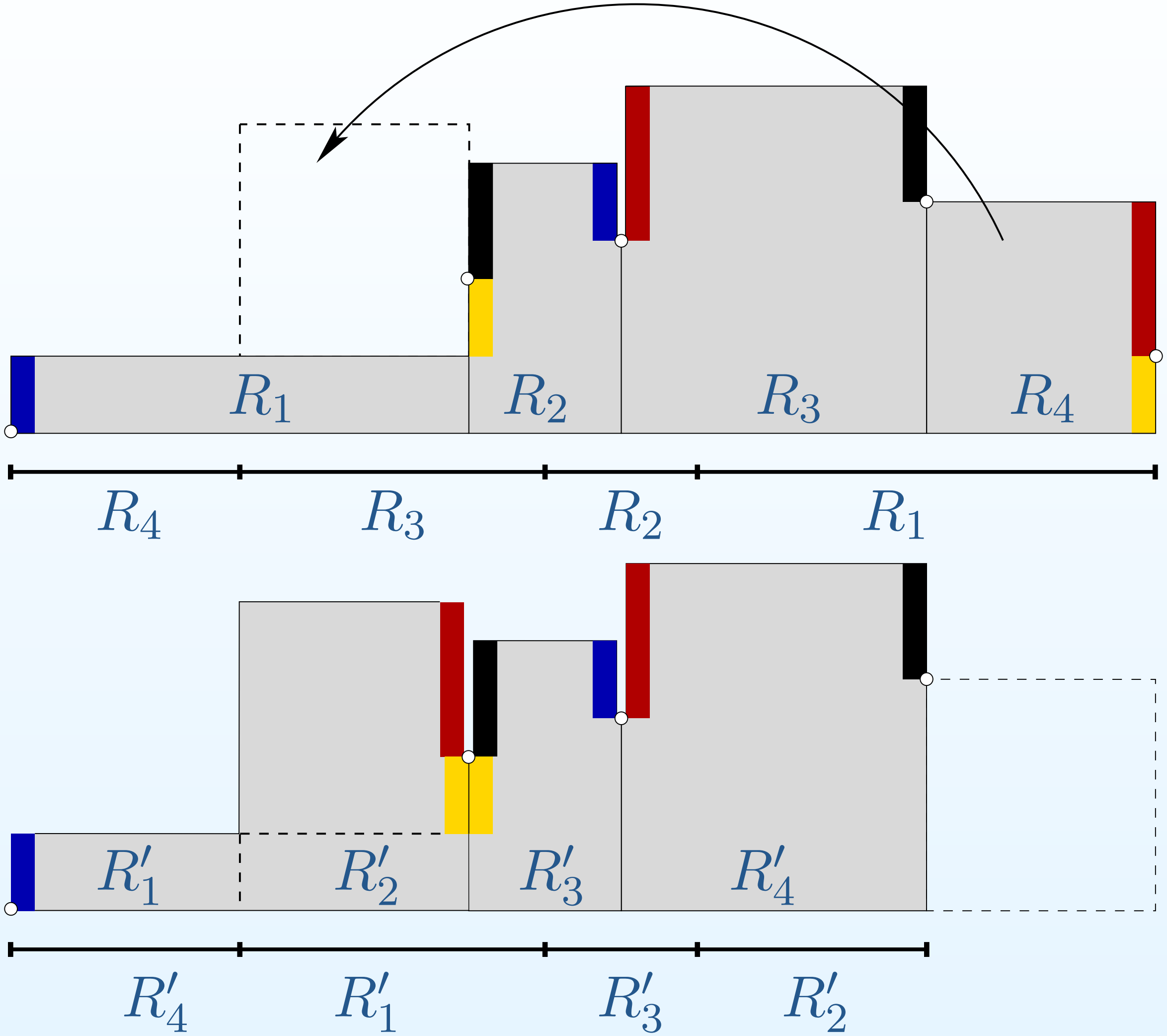


Rauzy move I. Winner on the bottom: $\lambda_m < \lambda_{\pi^{-1}(m)}$



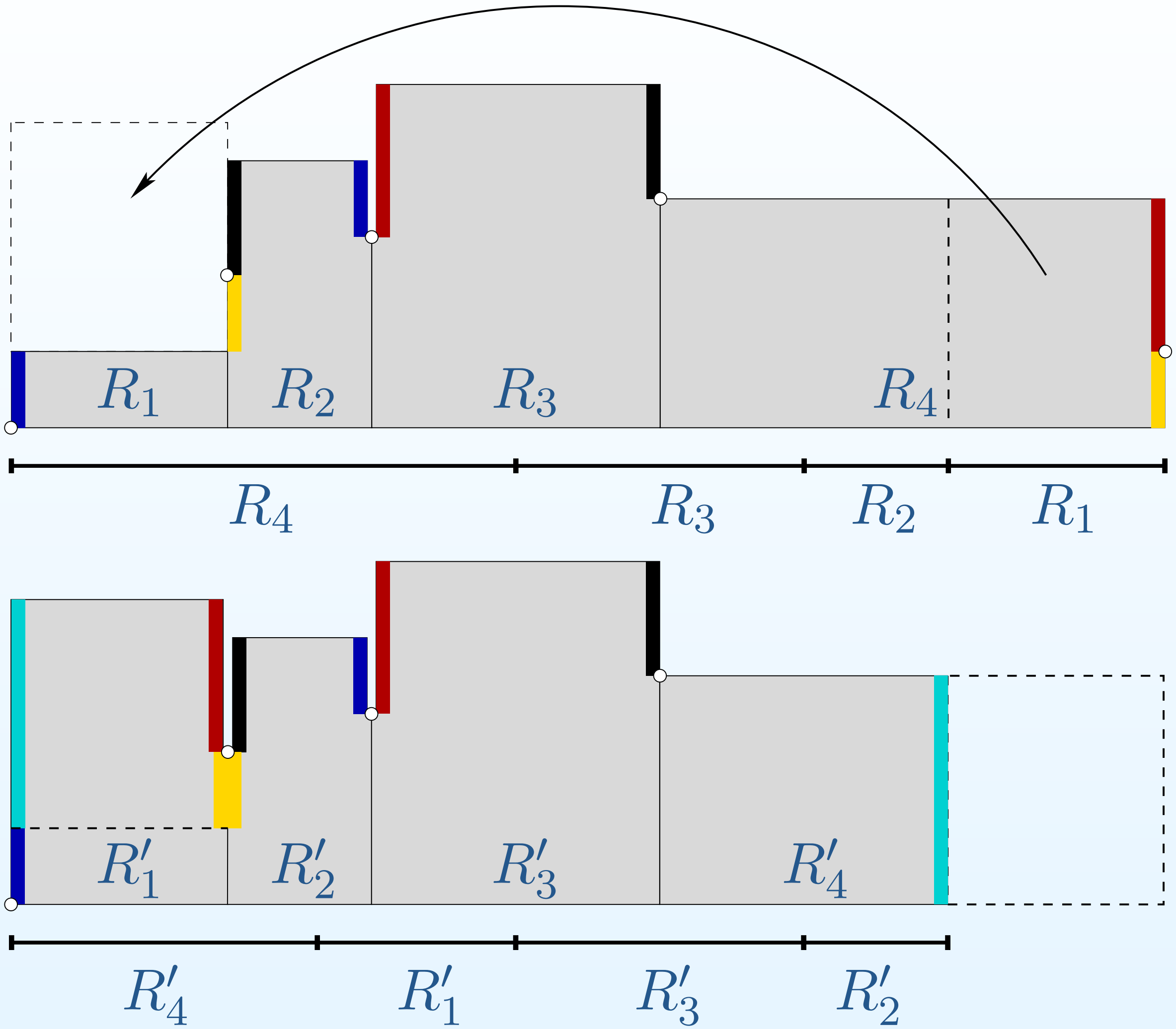
Type I modification: the rightmost rectangle R_4 on top of X is narrower than the rectangle $R_1 = R_{\pi^{-1}(4)}$ glued to the rightmost position at the bottom of X.

Rauzy move I. Winner on the bottom: $\lambda_m < \lambda_{\pi^{-1}(m)}$



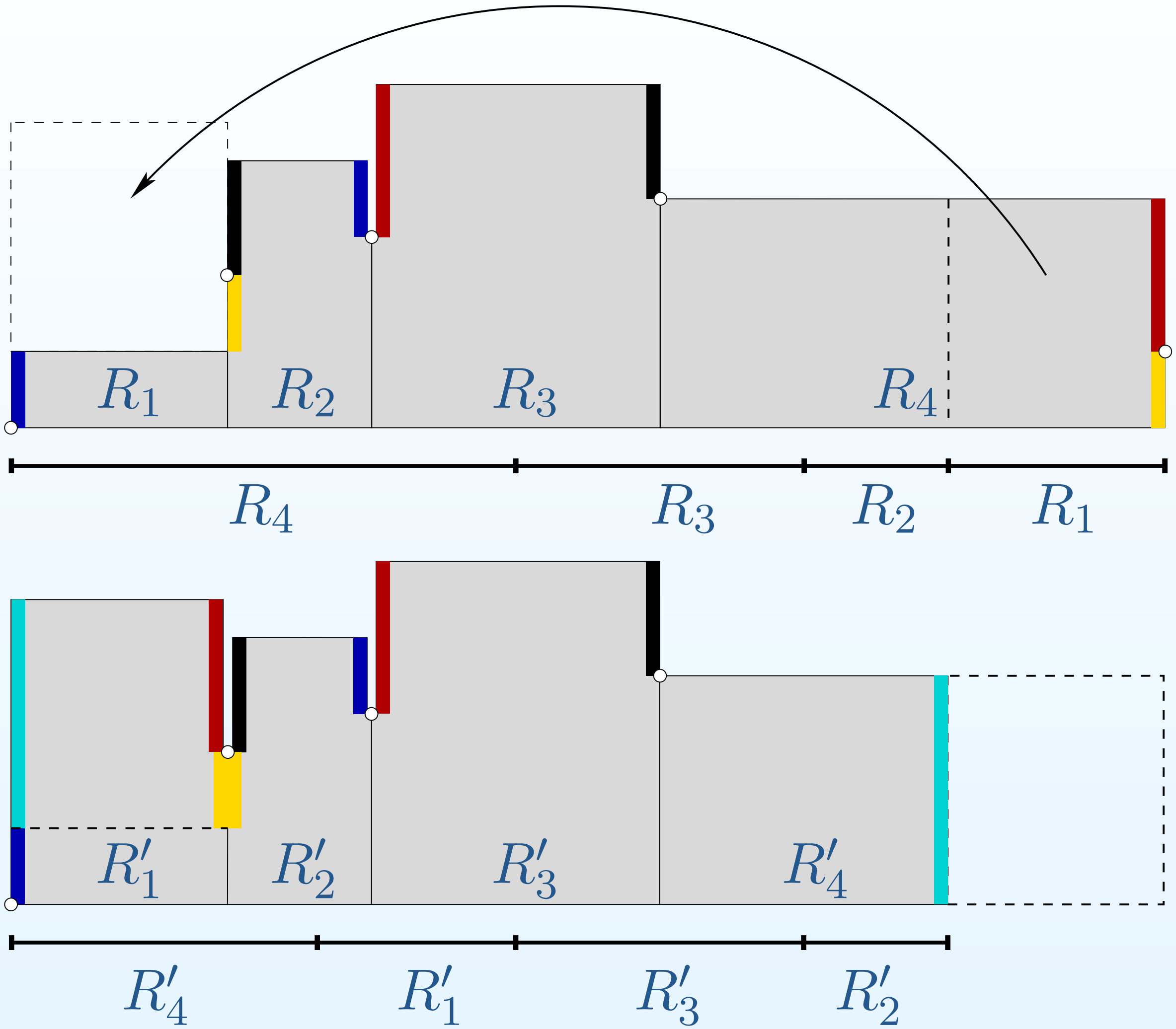
$$\begin{pmatrix} 1 & 2 \to & 3 \to & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = \pi'$$

Rauzy move I. Winner on the top: $\lambda_m > \lambda_{\pi^{-1}(m)}$



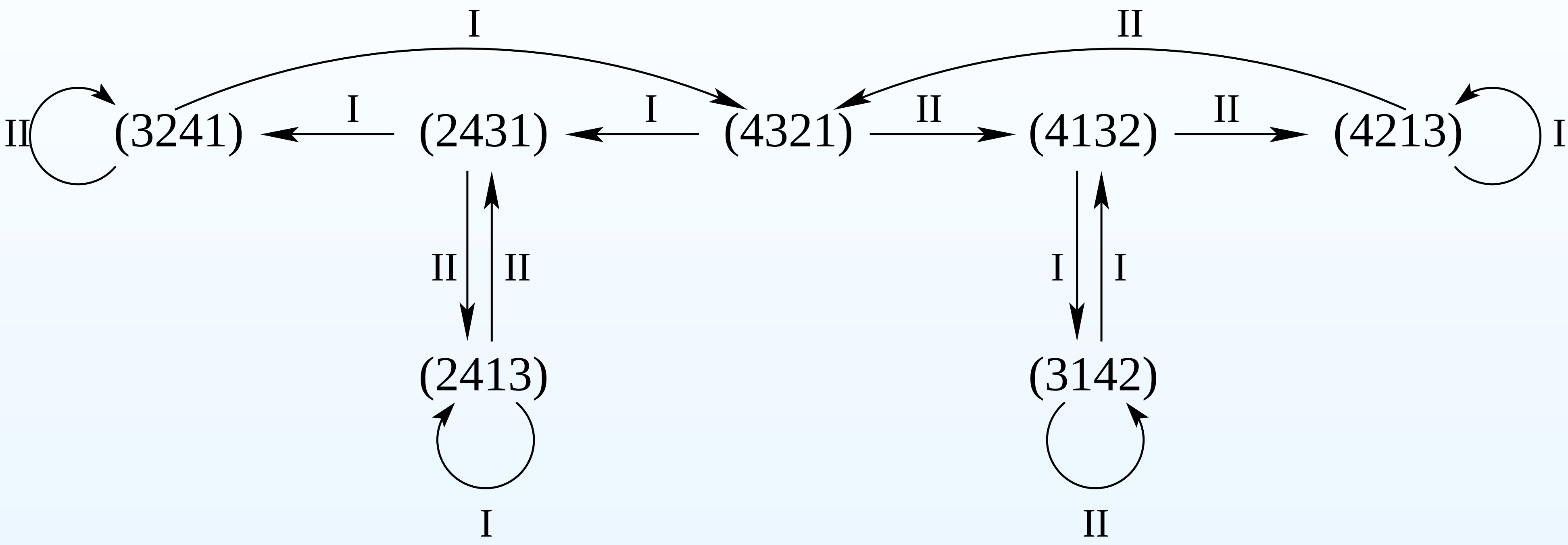
Type II modification: the rightmost rectangle R_4 on top of X is wider than the rectangle $R_1 = R_{\pi^{-1}(4)}$ glued to the rightmost position at the bottom of X.

Rauzy move I. Winner on the top: $\lambda_m > \lambda_{\pi^{-1}(m)}$



$$\begin{pmatrix} 1 & 2 & & 3 & & 4 \\ 4 & 3 & \to & 2 & \to & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = \pi'$$

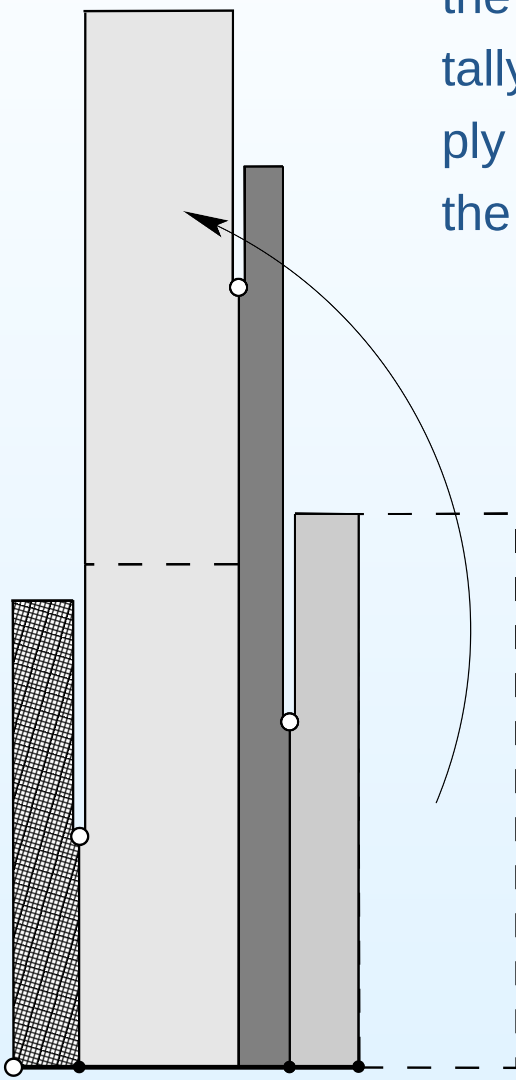
Rauzy class of the stratum $\mathcal{H}(2)$



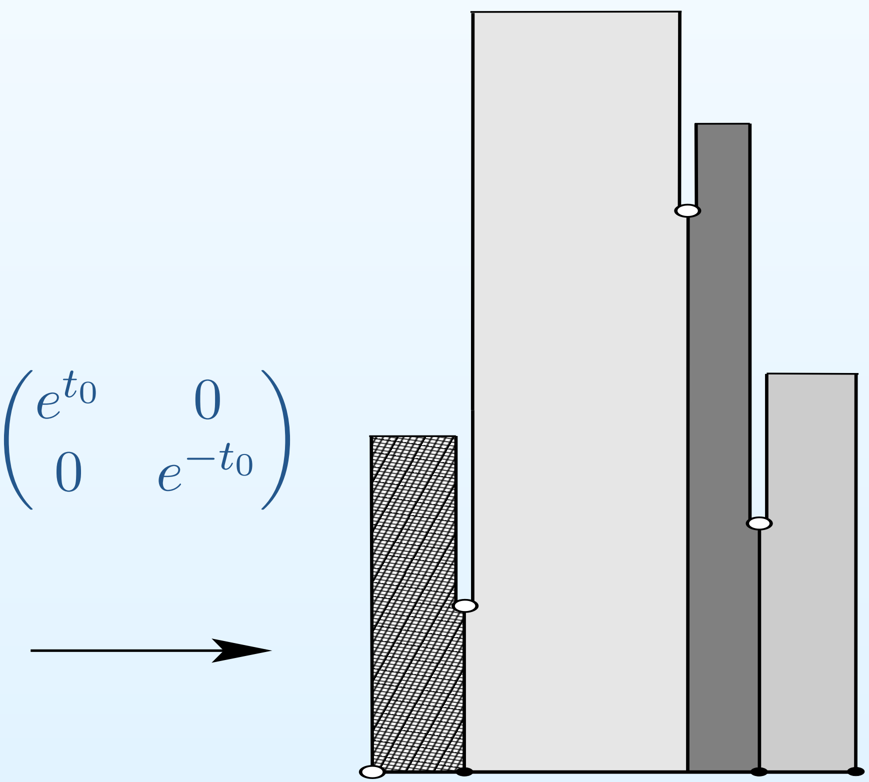
Rauzy class of the stratum $\mathcal{H}(2)$ contains all nondegenerate irreducible permutations of 4 elements. There are 7 such permutations.

Rauzy–Veech induction geometrically

We can apply the modification as above and then expand the resulting tall and narrow zippered rectangles horizontally and contract them vertically by a factor e^{t_0} or first apply expansion-contraction and them modify the building: the two operations commute.



$$\begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{pmatrix}$$



First return map on the space of zippered rectangles

A *Rauzy class* $\mathfrak{R}(\pi)$ of a nondegenerate permutation $\pi \in S_m$ is the orbit of π under Rauzy moves of types I and II. For any $\pi' \in \mathfrak{R}(\pi)$ one, actually, has $\mathfrak{R}(\pi') = \mathfrak{R}(\pi)$.

For every $\pi \in \mathfrak{R}$ consider all *admissible* zippered rectangles having the following properties: total area 1; the first return map to the base X is an interval exchange with the permutation π; "zipping" the free vertical parts of the rectangles one gets a closed translation surface. Consider a fundamental domain in the constructed space by selecting zippered rectangles with the base of length $|X| \geq 1$, but such that after the first Rauzy move the shortened base X' becomes already strictly shorter than 1. The resulting *space of zippered rectangles* if a finite cover over the associated stratum $\mathcal{H}(d_1, \ldots, d_n)$ of Abelian differentials of area one.

Consider a zippered rectangle $Z = (\lambda, \pi, H, a)$ which belongs to the "wall" Υ of the fundamental domain defined by the condition $1 = |X| = \lambda_1 + \cdots + \lambda_n$. Apply the Teichmüller flow to the translation surface defined by Z. The trajectory of the flow returns back to the "wall" Υ for the first time after a time $-\log(1 - \min(\lambda_m, \lambda_{\pi^{-1}(m)}))$. We get the first return map $\mathcal{S} : \Upsilon \to \Upsilon$.

From zippered rectangles to interval exchanges

Zippered rectangles pattern naturally defines an interval exchange transformation — the first return map of the vertical flow to the base X of zippered rectangles.

The map $\mathcal{S}$ of the subspace Υ of zippered rectangles projects to our map $\mathcal{T}$ on the space of interval exchange transformations:

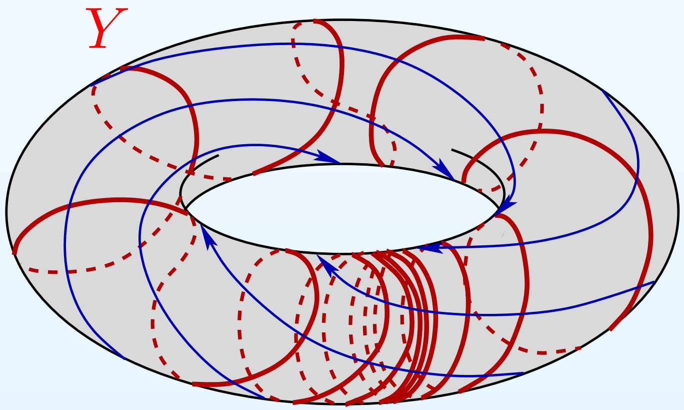
$$\begin{array}{ccc}
\text{Section } \Upsilon \text{ in the space of zippered rectangles} & \xrightarrow{\mathcal{S}} & \text{Section } \Upsilon \text{ in the space of zippered rectangles} \\
\downarrow & & \downarrow \\
\text{Space of interval exchange transformations} & \xrightarrow{\mathcal{T}} & \text{Space of interval exchange transformations}
\end{array}$$

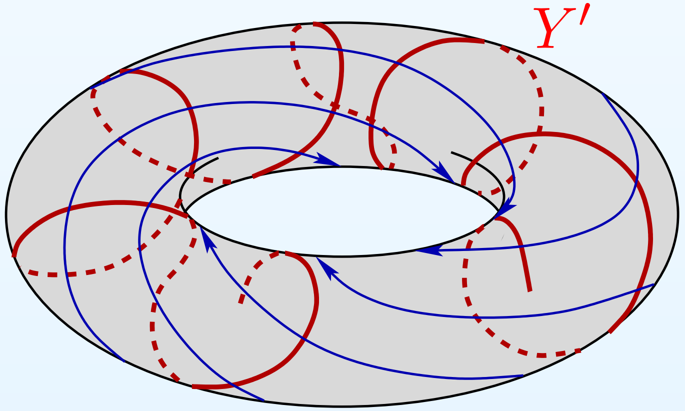
The map $\mathcal{S} : \Upsilon \to \Upsilon$ induced by the first return of the Teichmüller flow to the section Υ in the space of zippered rectangles is a suspension of the map $\mathcal{T}$ on the space of interval exchanges. In his fundamental Annals'82 paper, W. Veech proved that both maps are ergodic with respect to the natural Lebesgue measures, and deduced from this ergodicity of the Teichmüller flow.

Choice of a section

Consider a directional flow on a torus. (This time we use the torus as an example of an abstract multidimensional manifold with finite measure.)

Consider two different sections to this flow. Taking as a section the curve Y represented on the left picture we get a section of infinite measure though the measure of the torus is finite and the flow is very nice. Taking as a section a finite piece $Y' \subset Y$ as on the right picture we get a section of finite measure.





In both cases the first return map of the ergodic flow to the section is ergodic, but the mean return time to the left subsection is zero.

Additive and multiplicative Euclidean algorithms

Additive Euclidean algorithm ("Farey algorithm")

Having two intervals X_1 and X_2 of total length $x + (1 - x) = 1$ chop off the shorter one from the longer one **once**. Rescale proportionally the resulting intervals normalizing the total length to 1. We get a map $\mathcal{F} : [0; 1] \to [0; 1]$ with an invariant measure $d\mu_{\mathcal{F}}$.

$$\mathcal{F}(x) = \begin{cases} \frac{x}{1-x} & \text{for } 0 < x \le \frac{1}{2} \\ \frac{2x-1}{x} & \text{for } \frac{1}{2} < x < 1 \end{cases}, \qquad d\mu_{\mathcal{F}} = \begin{cases} \frac{1}{x} & \text{for } 0 < x \le \frac{1}{2} \\ \frac{1}{1-x} & \text{for } \frac{1}{2} < x < 1 \end{cases}$$

Multiplicative Euclidean algorithm

Having two intervals X_1 and X_2 chop off the shorter one from the longer one **as many times as possible**. For the resulting intervals normalize the length of the longer one to 1. We get a map $\mathcal{G} : [0; 1] \to [0; 1]$ with an invariant **probability** measure $d\mu_{\mathcal{G}}$:

$$\mathcal{G}(x) = \left\{ \frac{1}{x} \right\}, \qquad d\mu_{\mathcal{G}} = \frac{1}{\log 2} \cdot \frac{dx}{(1+x)}.$$

Smaller section

The natural section Υ chosen by W. Veech is too large: the corresponding invariant measure (induced from the measure on the space of flat surfaces) is infinite.

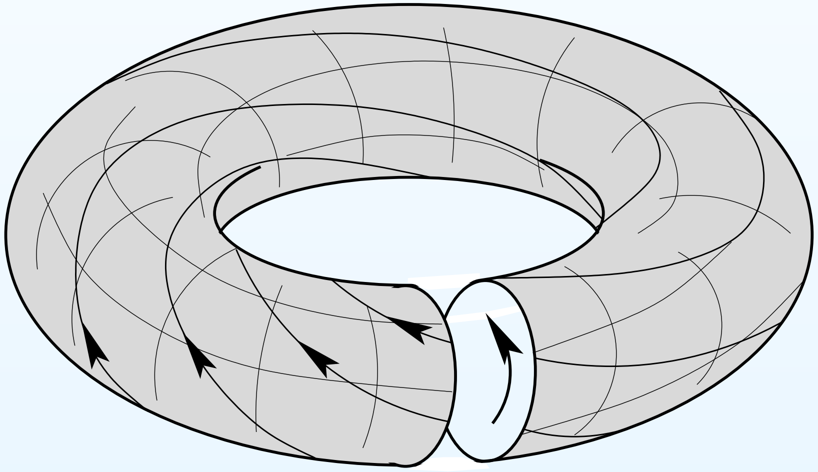
Choosing an appropriate subset $\Upsilon' \subset \Upsilon$ one can get finite invariant measure. Moreover, the subset Υ' can be chosen in such way that the corresponding first return map $\mathcal{S}' : \Upsilon' \to \Upsilon'$ of the Teichmüller geodesic flow is a suspension of some natural map $\mathcal{G}$ on the space of interval exchange transformations (analogous to the multiplicative Euclidean algorithm).

The Teichmüller geodesic flow is ergodic which implies ergodicity of the maps $\mathcal{S}'$ and $\mathcal{G}$. To apply Oseledets theorem one should, actually, consider the induced cocycle B over this new map $\mathcal{G}$ instead of the cocycle A over the map $\mathcal{T}$ described above.

Example of renormalization: Euclidean algorithm

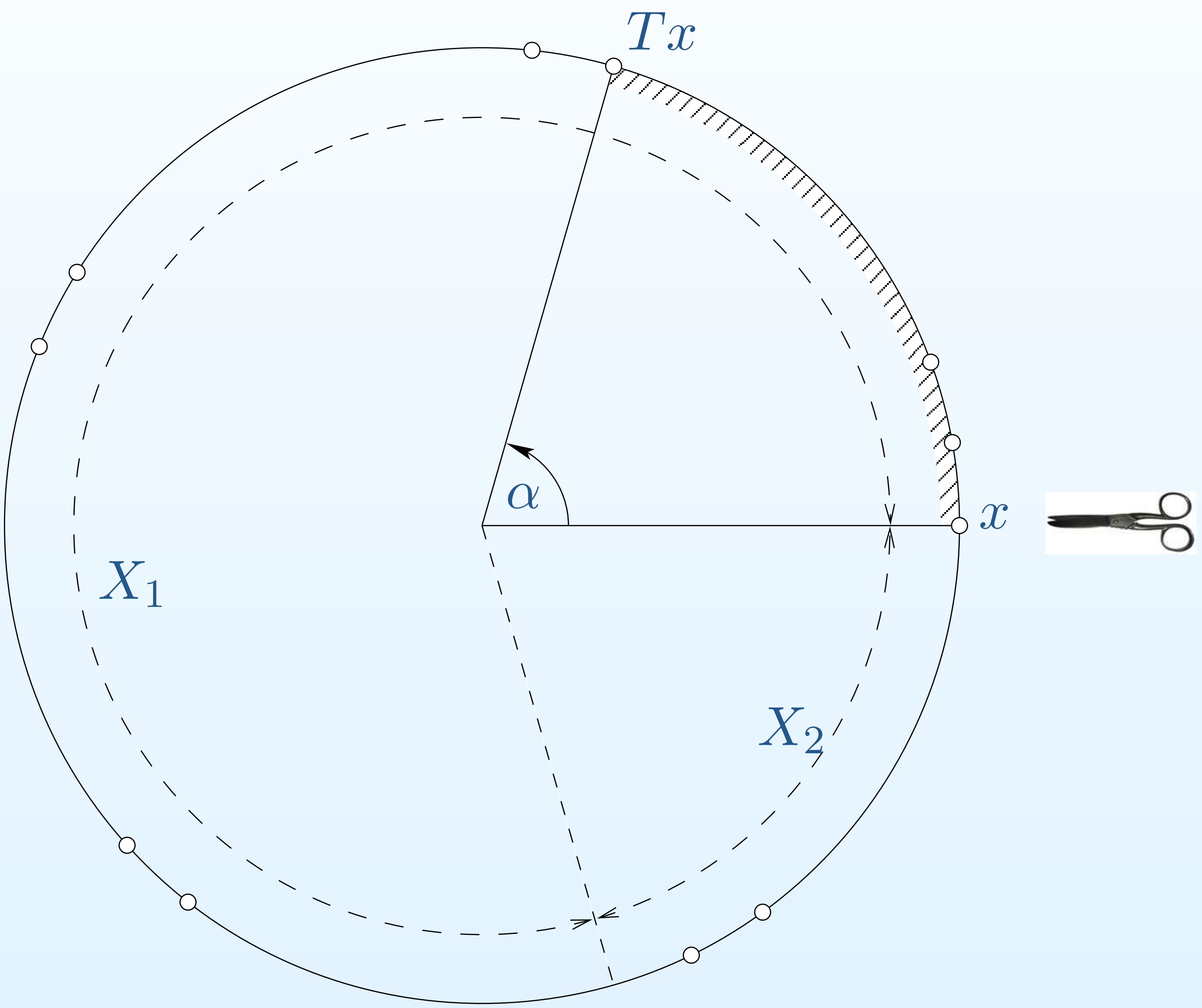
First return to a meridian

Consider a rotation of a circle $T : S^1 \to S^1$ obtained as the first return to a meridian of a directional flow on a torus.



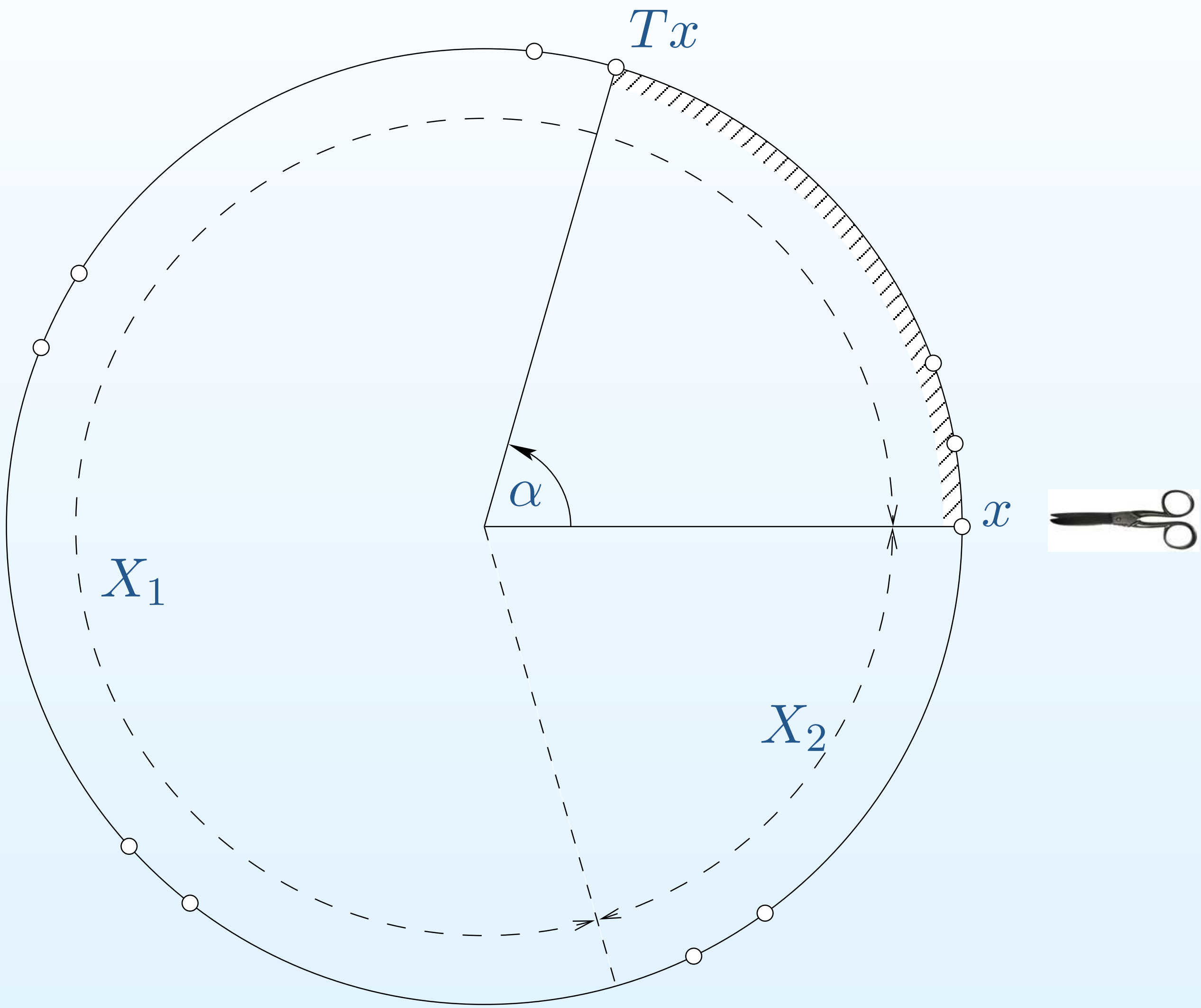
Rotation as an interval exchange transformation

A circle cut at a point becomes an interval. A rotation of the circle becomes an interval exchange transformation of subintervals X_1, X_2.



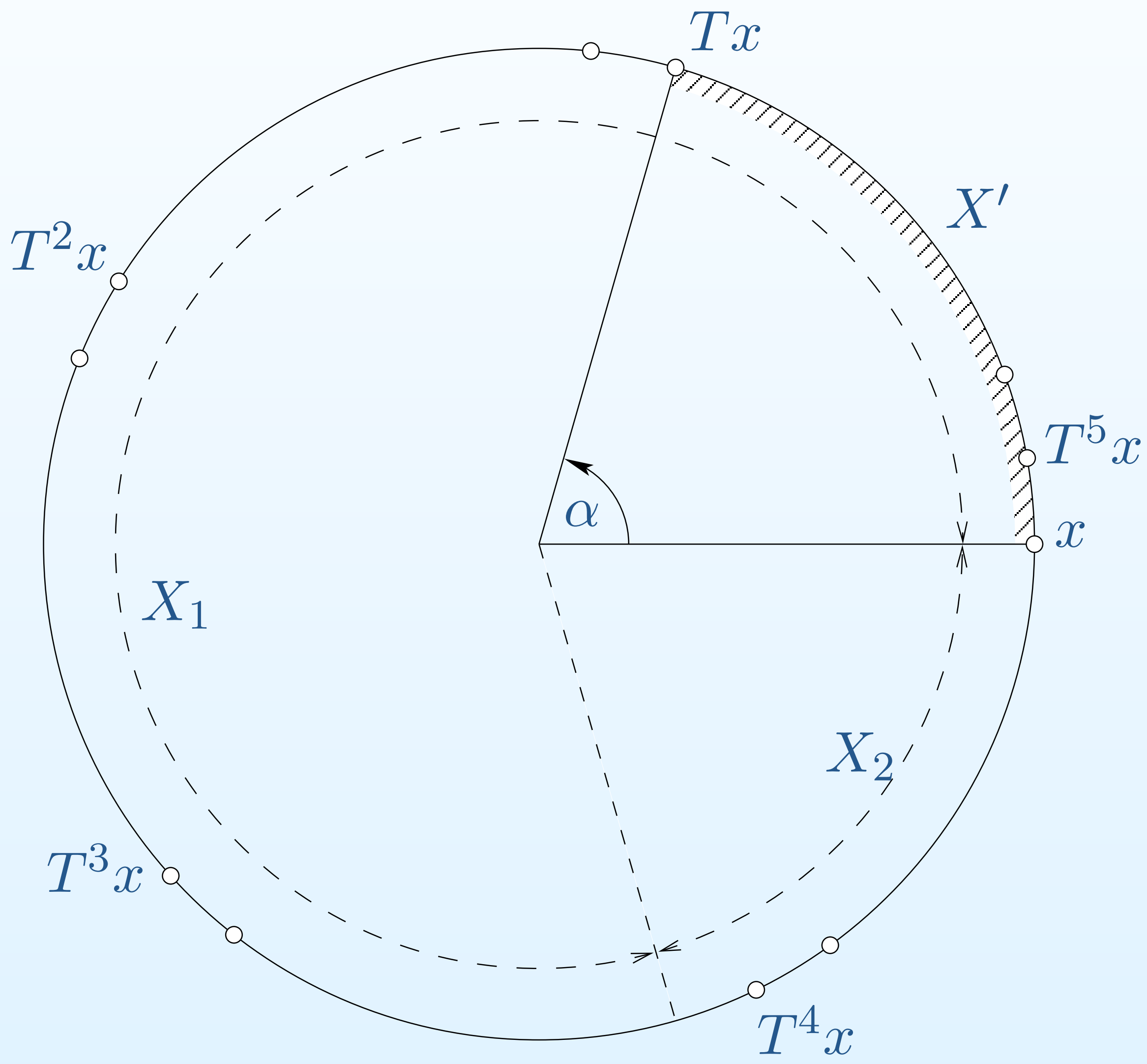
Rotation as an interval exchange transformation

Normalizing the length of the circle by 1 we get $|X_2| := \lambda = \alpha/(2\pi)$ and $|X_1| = 1 - \lambda$.



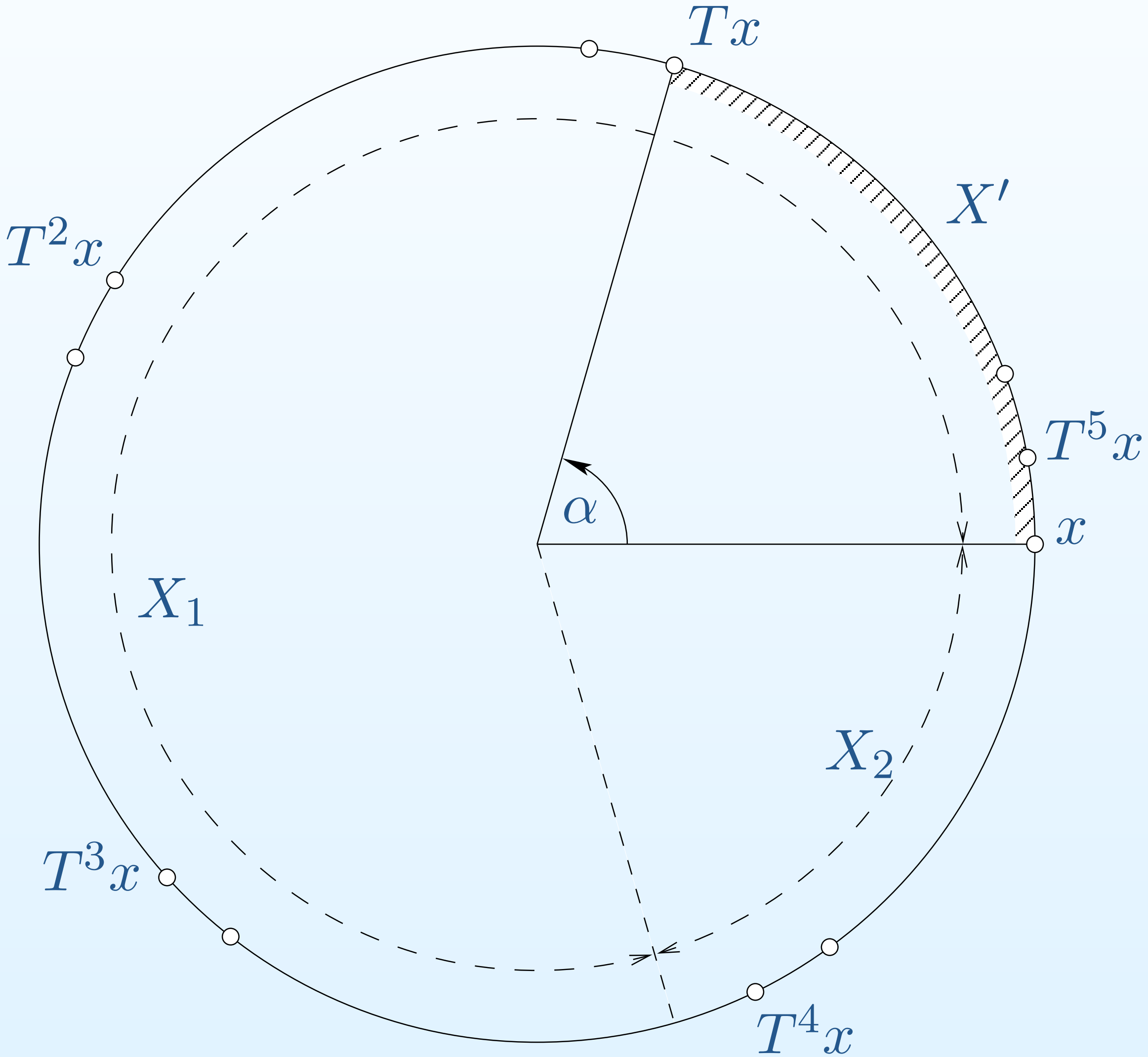
Induction to a subinterval

Suppose now that we can see only the subinterval $X' = [x, Tx[$. We would not see Tx, T^2x, T^3x, T^4x anymore.



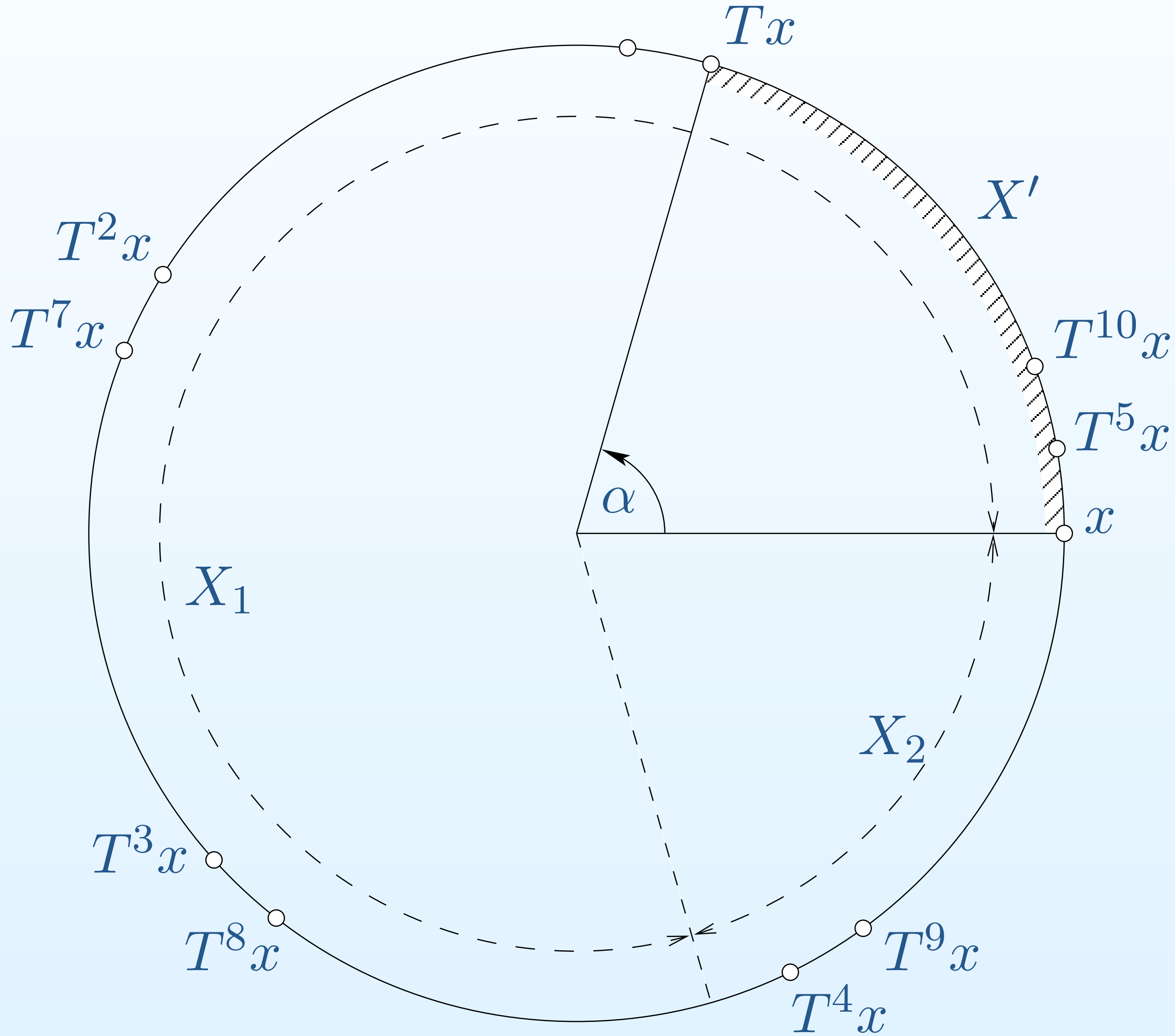
Induction to a subinterval

The first point of the trajectory $x, Tx, \ldots$ which visits X' is T^5x. This is the image $T'(x)$ of the induced map $T' : X' \to X'$.



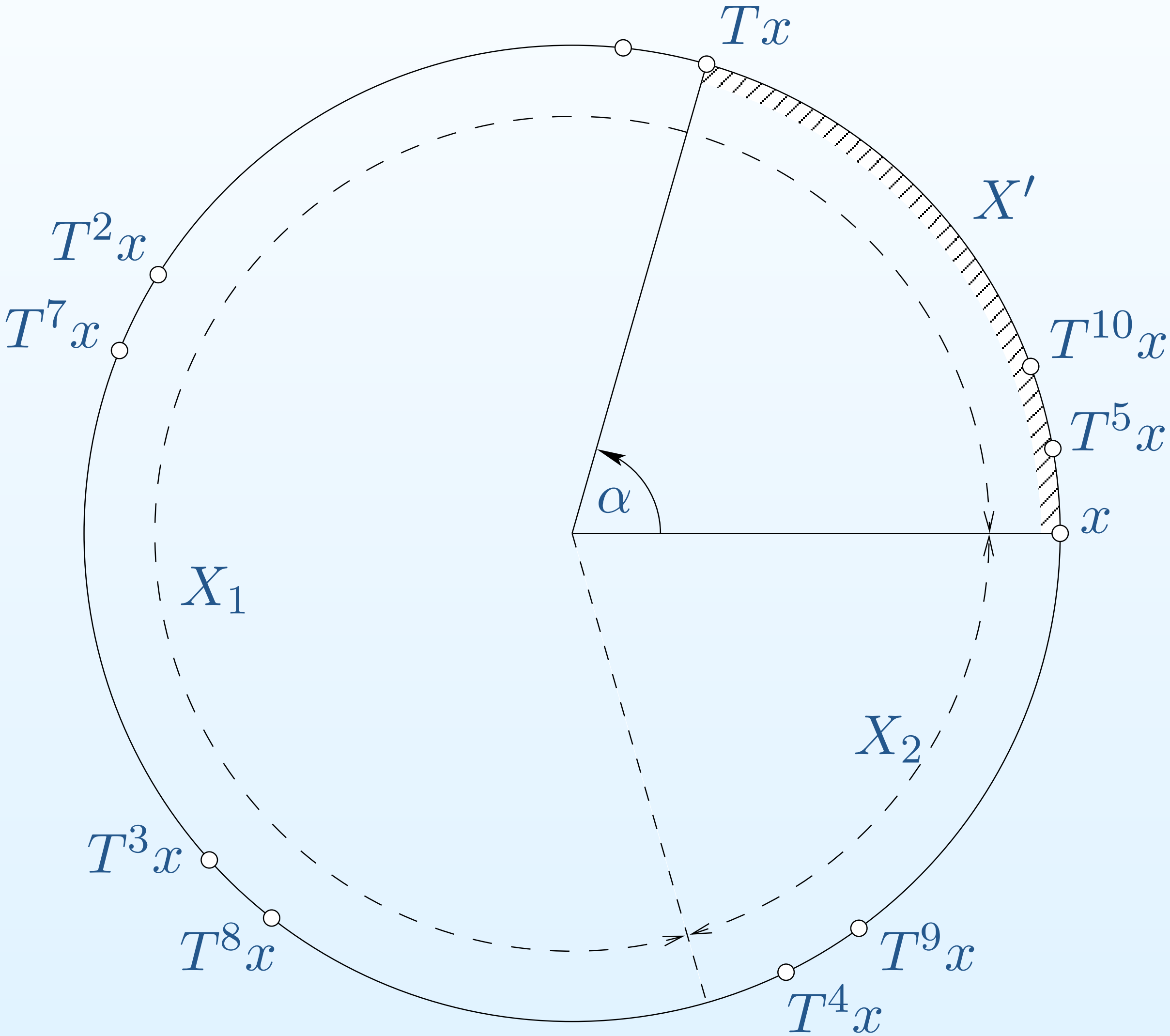
Induced map

Invisible trajectory $T^6x, T^7x, \ldots$ will produce the next visible point $T^{10}x = T'(T'x)$.



Induced map

The distances between x and $T'x = T^5x$ and between $T'x = T^5x$ and $T'(T'x) = T^{10}x$ are the same. They are equal to $(1 - \{1/\lambda\}) \cdot \lambda$.



Euclidean Algorithm

$T' : X' \to X'$ is again an interval exchange of two intervals $X'_1 \sqcup X'_2$ of lengths $|X'_1| = \{1/\lambda\} \cdot \lambda$ and $|X'_2| = (1 - \{1/\lambda\}) \cdot \lambda$.

Identifying the endpoints x and Tx of the interval X' we get a circle; the map T' becomes a rotation of this circle. Having started with a rotation by angle $\alpha = +2\pi \cdot \lambda$ we get a rotation T' by angle $\alpha' = -2\pi \cdot \left\{\frac{1}{\lambda}\right\}$.

The resulting *renormalization map* $G : [0,1] \to [0,1], \lambda \mapsto \left\{\frac{1}{\lambda}\right\}$ corresponds to the Euclidean algorithm It can be considered as a map from the space of interval exchange transformations of two intervals to intself.

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Euclidean Algorithm

$T' : X' \to X'$ is again an interval exchange of two intervals $X_1' \sqcup X_2'$ of lengths $|X_1'| = \{1/\lambda\} \cdot \lambda$ and $|X_2'| = (1 - \{1/\lambda\}) \cdot \lambda$.

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Continued fractions

The map G is ergodic with respect to the finite invariant measure

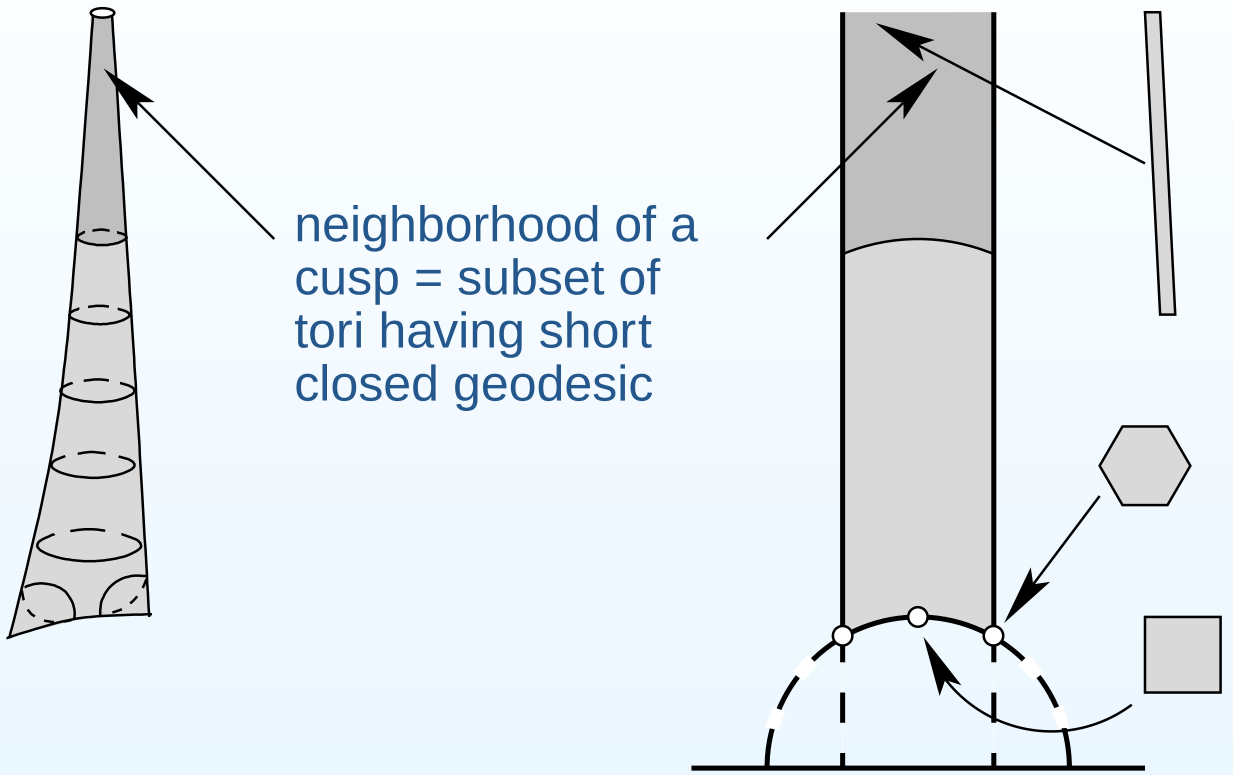
$$d\mu = \frac{1}{\log 2} \cdot \frac{d\lambda}{(\lambda + 1)}.$$

Integer numbers $n_1 = \left[\frac{1}{\lambda}\right], \ldots$ provide a decomposition of λ into a continued fraction:

$$\lambda = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ldots}}}}$$

Geodesic flow on the modular curve versus Euclidean algorithm

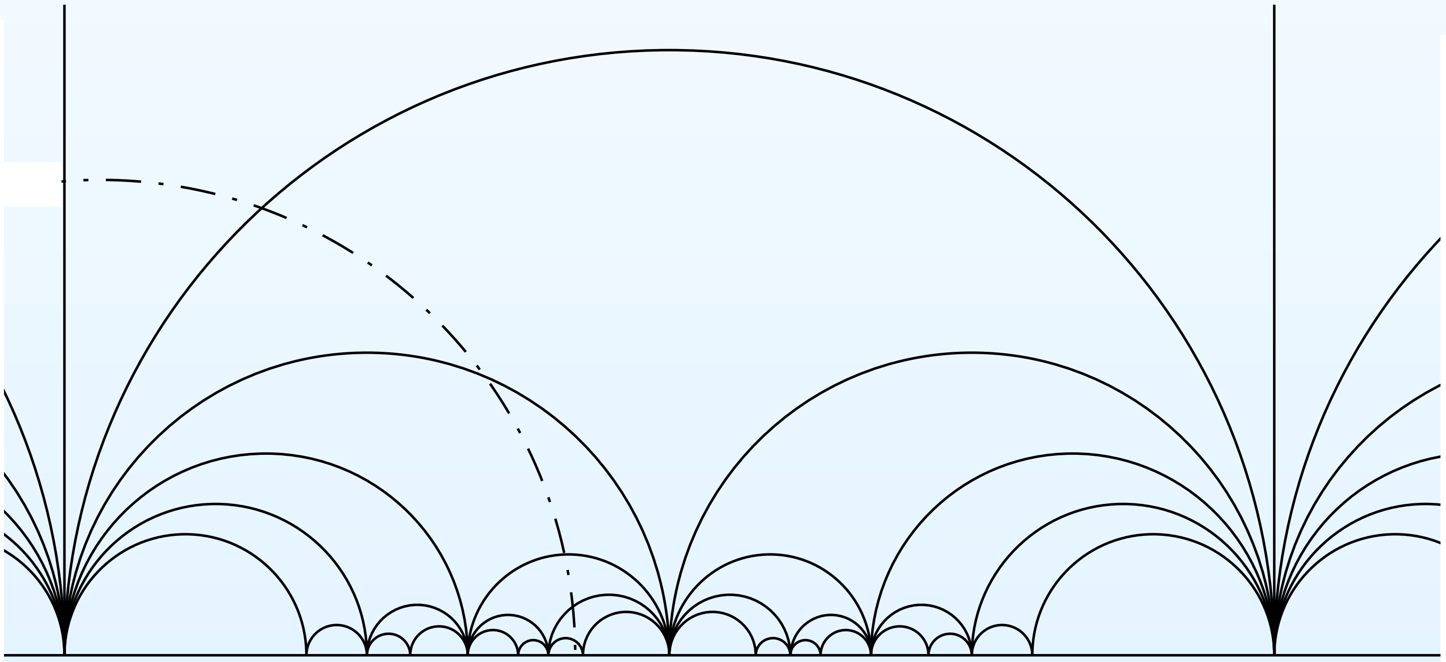
Action of the diagonal subgroup



Consider the action of the diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on the space of flat tori. The orbits of this action project to geodesics in the standard hyperbolic metric on the modular surface.

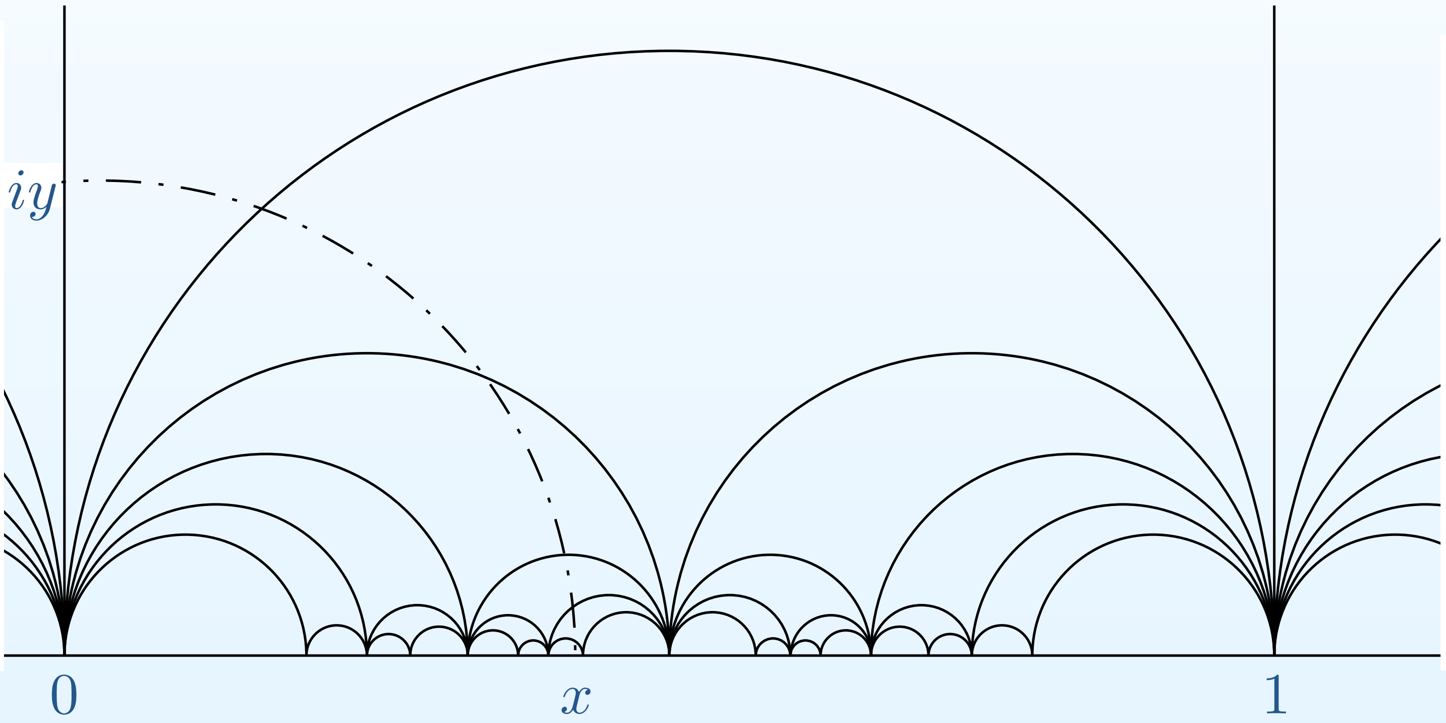
Teichmüller geodesic flow

Consider the standard tiling of the universal cover $\mathbb{H}^2$ of the modular curve by ideal triangles. The fundamental domain of the tiling is a triple cover over the modular curve. Geodesics on the modular curve unfold to geodesics on the hyperbolic plane $\mathbb{H}^2$.



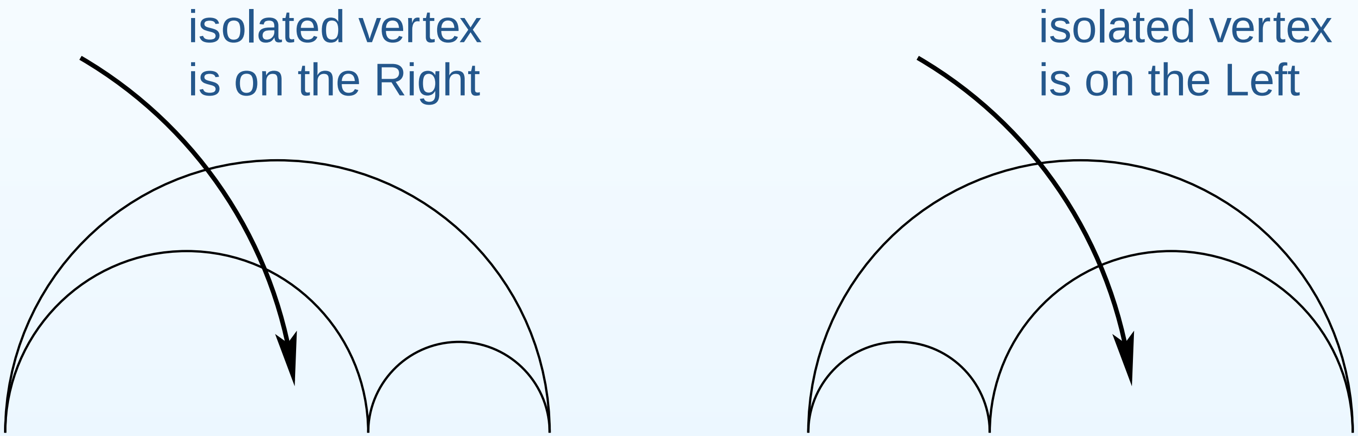
Geometric coding of a continued fraction

Consider a real number $0 < x < 1$. Consider a geodesic segment γ joining some point iy of the vertical axes with the point x at the absolute. Let us trace the way in which γ crosses the fundamental domains of the tiling.

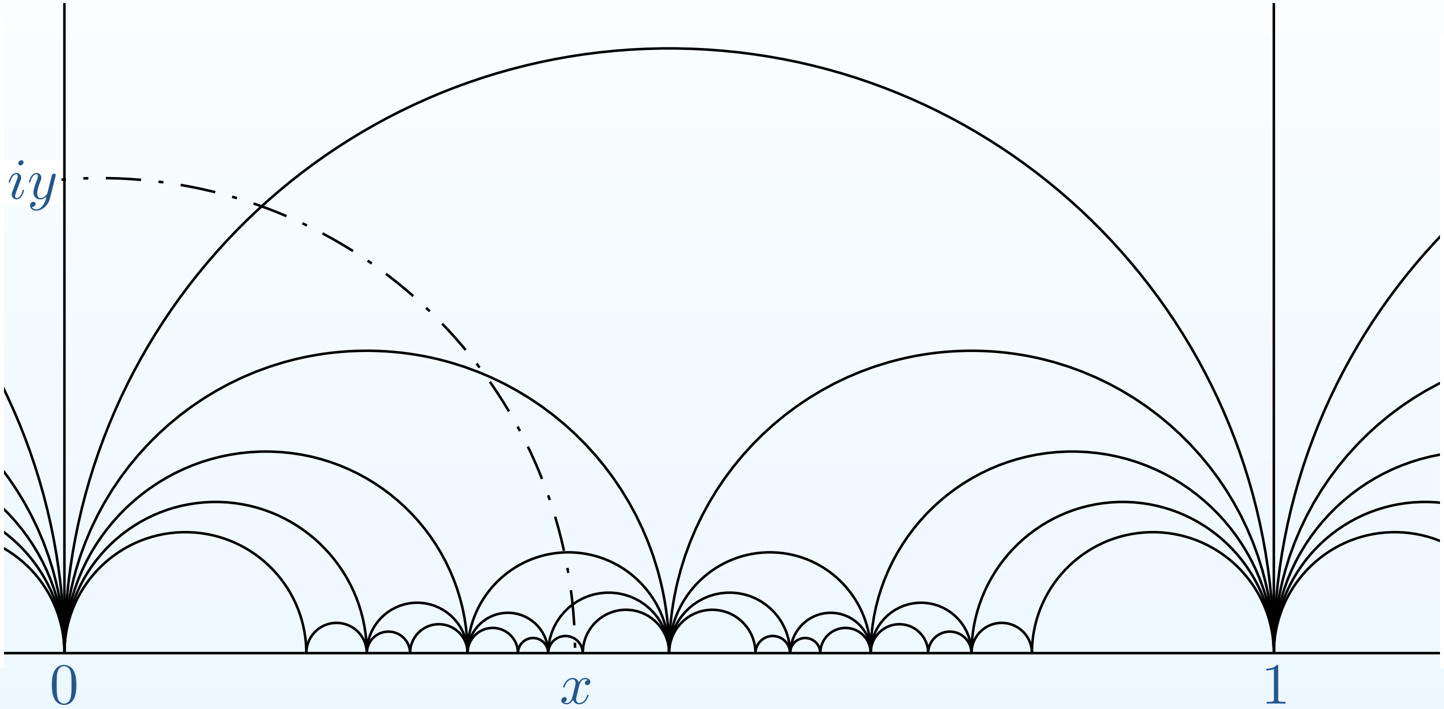


Geometric coding of a continued fraction

Every time γ crosses a triangle of the tiling, we encode the crossing by one of the letters "L" or "R" using the following rule:



Geometric coding of a continued fraction

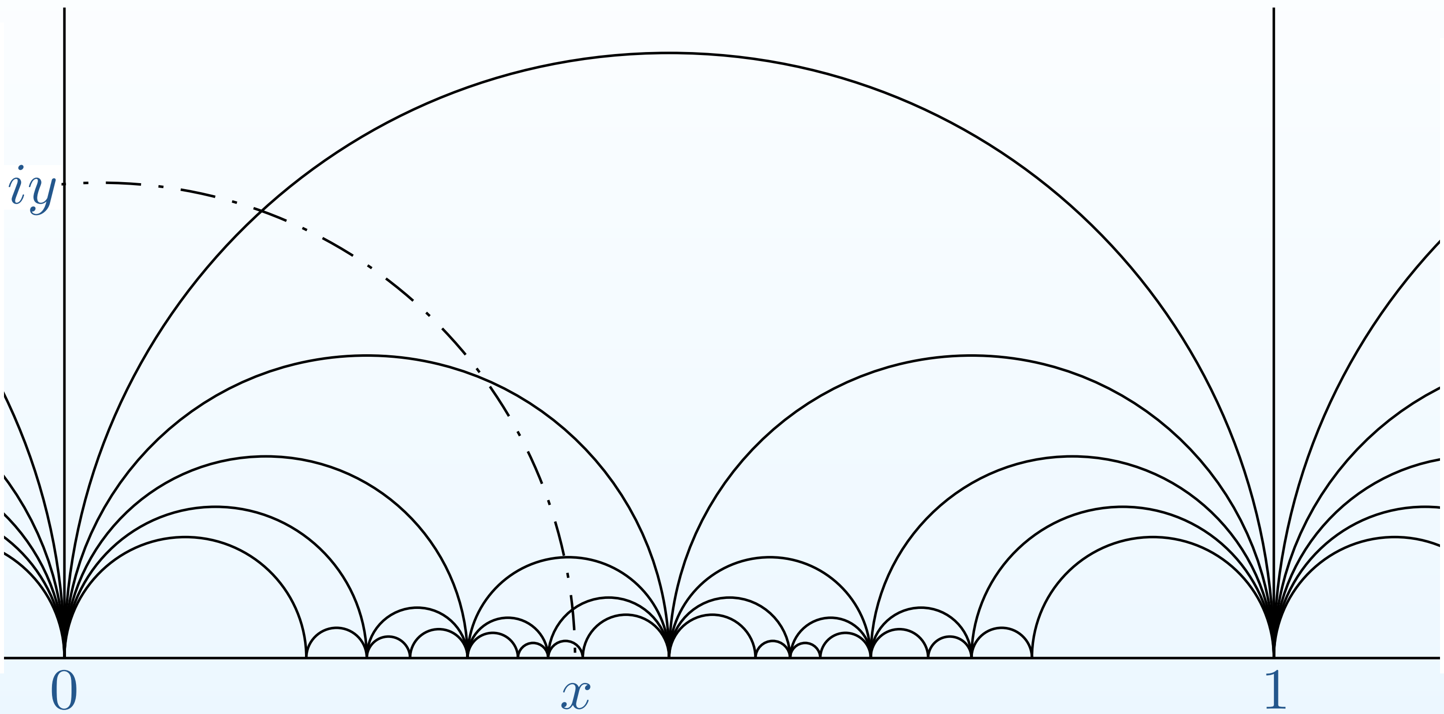


Example Following a geodesic γ landing at $x = (\sqrt{85} - 5)/10 \approx 0.421954$ we get the following cutting sequence

$$R, R, L, L, R, L, L, R, R, L, \ldots$$

which we encode by $R^2 L^2 R^1 L^2 R^2 L^1 \ldots$.

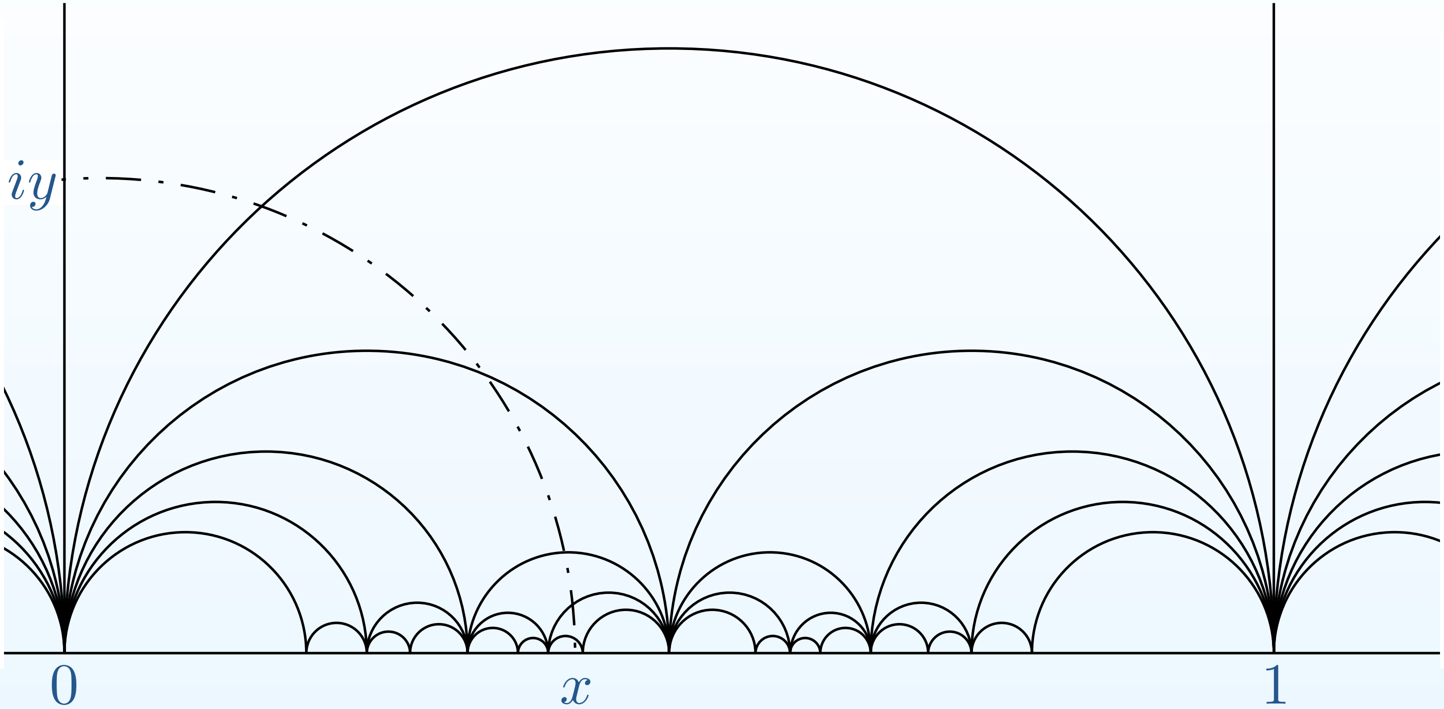
Geometric coding of a continued fraction



Theorem (C. Series) *Consider an irrational $x \in (0,1)$. Let γ be a geodesic segment launched from some iy and landing at x; let $R^{n_1}L^{n_2}R^{n_3}L^{n_4}\ldots$ be the induced cutting sequence. Then,*

$$x = [0; n_1, n_2, n_3, n_4, \ldots].$$

Geometric coding of a continued fraction



Here $[0; n_1, n_2, n_3, n_4, \ldots] = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\cdots}}}}$