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NON-INVERTIBLE KNOTS EXIST

H. F. TROTTER

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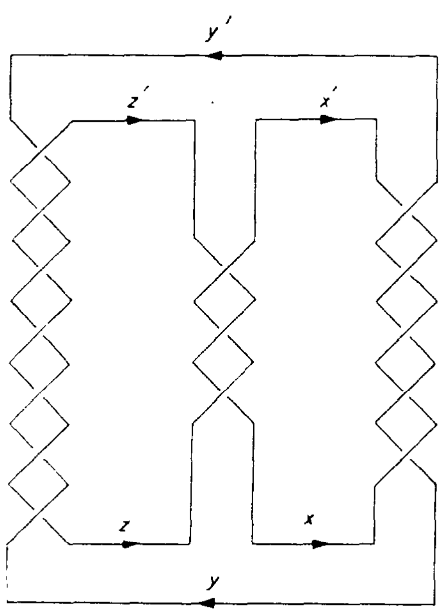
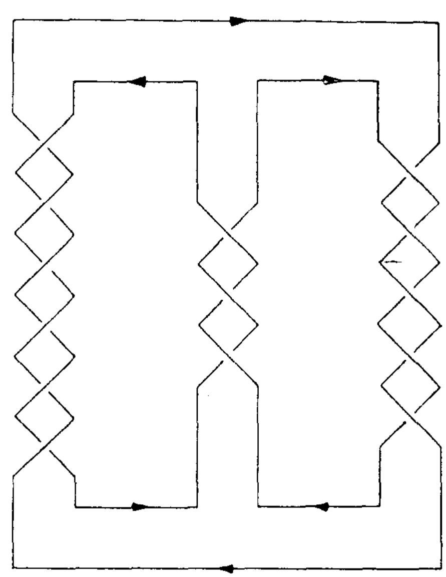
THE TITLE states our main result and gives the expected answer to problem 10 of [3]. We consider two oriented knots K, $K' \subset S^3$ to be *equivalent* if and only if there is a homeomorphism $h : S^3 \to S^3$ which preserves the orientation of S^3 and carries K onto K' so that their orientations match. Two knots which differ only in their orientation are said to be *inverses* of each other, and a knot is *invertible* if it is equivalent to its inverse. The theorem given below describes an infinite family of tame non-invertible knots. The simplest member of the family is shown in Fig. 1(a).



$p = 7,\ q = 3,\ r = 5$

(a)



$p = 6,\ q = -3,\ r = -5$

(b)

FIG. 1

Our examples are all pretzel knots [6, p. 9], i.e., knots with projections in which the crossings lie on three two-stranded braids as in Fig. 1. We shall use $K(p, q, r)$ to denote a pretzel knot in which $|p|$, $|q|$, $|r|$ are the numbers of crossings in the braids, and the signs of p, q, r depend on the directions of twist in the corresponding braids.

THEOREM. *Let p, q, r be odd integers such that $|p|$, $|q|$, and $|r|$ are distinct and greater than* 1. *Then the pretzel knot $K(p, q, r)$ is non-invertible.*

Remark (1). Fox [2, p. 143] defines a knot to be +amphicheiral if it is equivalent to its reflection, and −amphicheiral if it is equivalent to the inverse of its reflection. It seems probable that none of the knots satisfying the hypotheses of our theorem are amphicheiral (in either sense). The determinant of $K(p, q, r)$ is $|pq + qr + rp|$, and if p, q, r are all odd and have the same sign, this number has the form $4n + 3$, and the quadratic form can be used to show that the knot is not amphicheiral [6, p. 30] . Hence if K is, say, $K(3, 5, 7)$ (with some specified orientation) and K^* is its reflection, then no two of K, K^*, K^{-1}, K^{*-1} are equivalent. Since these knots are of genus one, they are prime, and it follows from the unique factorization theorem (see section 7 of [2]) that the product knot $K \# K^*$ is +amphicheiral but not −amphicheiral, while the reverse holds for $K^{-1} \# K^*$. Since many knots (e.g., $K(1, 1, 1)$, the cloverleaf) are known to be invertible and non-amphicheiral, and others (e.g., $K(1, -3, 1)$, the figure-eight knot) are both invertible and amphicheiral, there are examples to illustrate all possible combinations of invertibility and amphicheirality. The interesting question of whether there are *prime* knots which are amphicheiral but non-invertible remains open.

Remark (2). The hypotheses of the theorem, which are sufficient conditions for the non-invertibility of a pretzel knot, are very nearly necessary. For example, if $p = r$ then $K(p, q, r)$ can obviously be inverted by a 180° rotation about a central vertical axis. The equivalence class of $K(p, q, r)$ is unchanged by a cyclic permutation of the crossing-numbers, and hence $K(p, q, r)$ is invertible unless p, q, r are all distinct. If one of the numbers is even, as in Fig. 1(b), a 180° rotation about a horizontal axis gives an inversion. (If more than one of the numbers is even, the diagram represents a link with more than one component.) Finally, it follows from the work of Bankwitz and Schumann [1, pp. 265, 279] (and is not hard to show directly) that $K(p, q, r)$ is invertible if any of p, q, r has the value ± 1. Thus in view of the theorem given in this paper, the question of invertibility is settled for all pretzel knots except those of type $K(p, q, -q)$ with $|p| \neq |q|$ and $|p|, |q| \geq 3$.

Remark (3). Fox and Harrold [4] have shown that the existence of tame non-invertible knots implies the existence of certain types of (wild) non-invertible arcs.

Proof. We first reduce the problem to a question in group theory, following a suggestion of Fox [3, p. 169]. Given an oriented knot K, certain classes of elements of its group $G = \pi_1(S^3 - K)$ may be defined as follows (cf. [2, p. 123]). An element which for any neighborhood N of K can be represented by a path $\gamma\beta\gamma^{-1}$, where γ runs from the base-point to a point in $N - K$ and β is a loop in $N - K$ such that $\beta \sim K$ in N and $\beta \sim 0$ in $S^3 - K$, is a *longitude* of K. An element which has linking number +1 with K (here we assume given an orientation for S^3 as well as for K), and for any neighborhood N can be represented by $\gamma\beta\gamma^{-1}$ with β a loop in $N-K$ such that $\beta \sim 0$ in N, is a *meridian* of K. (We have deviated here from the usual definitions, which do not mention orientations, and under which the inverse of a meridian or longitude is again a meridian or longitude.) If two knots are equivalent, the homeomorphism showing them to be so induces an isomorphism of their groups which carries meridians into meridians and longitudes into longitudes. In particular, the group of an invertible knot admits an automorphism taking the class of meridians into the class of their inverses, and the class of longitudes into the class of their inverses. We shall call such an automorphism an *inversion*.

Let p, q, r satisfy the hypotheses of the theorem, and define k, l, m by $p = 2k+1$, $q = 2l+1$, $r = 2m+1$. We calculate a presentation of G, the group of $K(p,q,r)$, by the Wirtinger method [5, ch. 6]. Let x, y, z, x', y', z' be generators corresponding to meridians around the arcs as labelled in Fig. 1(a). Each braid yields relations expressing the generators at its top in terms of those at its bottom. Two such relations are

$$y' = (xy^{-1})^m x (xy^{-1})^{-m}$$
$$y' = (yz^{-1})^{k+1} z (yz^{-1})^{-k-1},$$

and the others are similar. On eliminating the generators x', y', z', we are left with generators x, y, z and relations

$$(1)\qquad \begin{aligned} (xy^{-1})^m x (xy^{-1})^{-m} &= (yz^{-1})^{k+1} z (yz^{-1})^{-k-1} \\ (yz^{-1})^k y (yz^{-1})^{-k} &= (zx^{-1})^{l+1} x (zx^{-1})^{-l-1} \\ (zx^{-1})^l z (zx^{-1})^{-l} &= (xy^{-1})^{m+1} y (xy^{-1})^{-m-1}. \end{aligned}$$

The generators x, y, z are meridians. A longitude commuting with x is

$$(2)\qquad w = (xy^{-1})^{-m}(yz^{-1})^{k+1}(zx^{-1})^{-l}(xy^{-1})^{m+1}(yz^{-1})^{-k}(zx^{-1})^{l+1},$$

as may be read from the diagram. (As a partial check, it can be verified algebraically that the relation $xw = wx$ is a consequence of the relations (1).)

If G has any inversions then it has one which carries w onto w^{-1}, for every longitude is conjugate to w, and following any inversion by a suitable inner automorphism will give an inversion with the desired property. Accordingly we assume

3) There exists an inversion α of G with $\alpha(w) = w^{-1}$,

and the proof will be complete when we derive a contradiction from (3).

The group G is rather intractable, and we simplify the problem by means of a homomorphism used by Reidemeister [6, p. 65] in classifying pretzel knots. Let H be the subgroup of G generated by the squares of meridians. H is a normal subgroup, and is carried onto itself by any inversion. Hence α induces an automorphism of G/H.

A presentation for G/H is obtained by adjoining the relations

$$(4)\qquad x^2 = y^2 = z^2 = 1$$

to (1). Since (4) implies $x = x^{-1}$, $y = y^{-1}$, $z = z^{-1}$, the first relation in (1) may be replaced by

$$(xy)^m x(yx)^m = (yz)^{k+1} z(zy)^{k+1},$$

and multiplying each side on the right by y gives $(xy)^r = (yz)^p$. Treating the other relations similarly, we find that G/H has a presentation with generators x, y, z and relations (4) and

(5) $$(xy)^r = (yz)^p = (zx)^q.$$

Let U be the subgroup of G/H generated by $(xy)^r$. Let F be the commutator subgroup of G/H; it is the unique subgroup of index 2, and is generated by xy, yz, and zx. U is the center of F and hence a characteristic subgroup of G/H. (U is obviously contained in the center of F. As will appear from the geometric interpretation of $(G/H)/U$ given below, F/U has trivial center, which implies that U is precisely the center of F.)

Now $W = (G/H)/U$ has a presentation with generators x, y, z and relations

(6) $$x^2 = y^2 = z^2 = (xy)^r = (yz)^p = (zx)^q = 1.$$

The supposed inversion α induces an automorphsim of W which takes the image of w onto its inverse.

This image is

(7) $$\begin{aligned} v &= (xy)^{-m}(yz)^{k+1}(zx)^{-l}(xy)^{m+1}(yz)^{-k}(zx)^{l+1} \\ &= ((xy)^{-m}(yz)^{-k}(zx)^{-l})^2. \end{aligned}$$

We show the impossibility of (3) and complete the proof of the theorem by showing that

(8) There is no automorphism of W which takes v into v^{-1}.

Note that at this point there is no loss of generality in assuming $p, q, r > 0$. The relations (6) are unaffected by any change of sign of the exponent. Changing the sign of, say, p means that $-k-1$ replaces k in (7), but since $(yz)^k = (yz)^{-k-1}$ in W this also has no effect.

The group W has a well-known geometric interpretation [8, p. 55; 7]. Let ξ, η, ζ be three lines in the hyperbolic plane which form a triangle with angles π/p, π/q, π/r opposite the sides ξ, η, ζ respectively. The reflections with these three lines as axes generate a properly discontinuous group of isometries, which has the original triangle as a fundamental domain. Identifying x, y, z with the reflections in ξ, η, ζ gives an isomorphism of W with this group of isometries. The product xy becomes a rotation of angle $2\pi/r$ about the intersection of ξ and η, in the direction to carry ξ onto its reflection in η. (We interpret xy as giving the result of following the action of x with that of y.) Thus $(xy)^{-m}$ is a rotation through an angle of $\pi-\pi/r$, in the direction as shown in Fig. 2, where $(yz)^{-k}$, $(zx)^{-l}$ are similarly indicated.

The product $(xy)^{-m}(yz)^{-k}(zx)^{-l}$ is now easily seen to give a translation along ξ (to the left, in the figure as shown) through a distance equal to the perimeter of the triangle. The element v is thus a translation through twice the distance, and is non-trivial.

The following lemma is perhaps a classical result, but I have found no statement of in the literature.

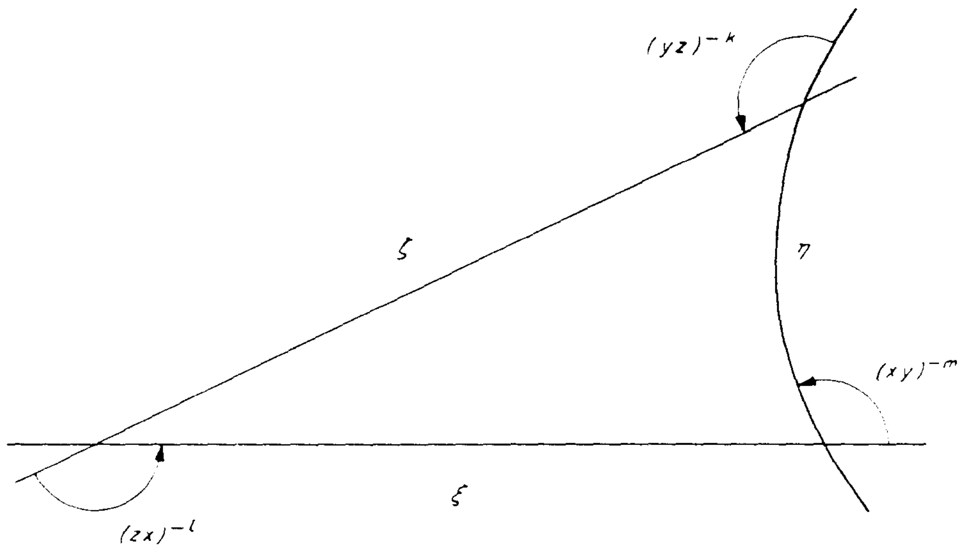


FIG. 2

LEMMA. *The group W has no outer automorphisms.*

Proof. The reflections in W can be characterized as the only elements of order 2. (Since p, q, r are odd, there are no rotations of period 2.) Suppose that x', y', z' are the images of x, y, z under some given automorphism, and let ξ', η', ζ' be the corresponding axes. Now $(x'y')^r = 1$, and therefore ξ' and η' meet at an angle which is some integral multiple of π/r. Applying the same argument to $y'z'$ and $z'x'$ shows that ξ', η', ζ' form a triangle with angle-sum $\geq \pi/p + \pi/q + \pi/r$, and hence with area less than or equal to the area of a fundamental triangle. Consequently they bound a fundamental triangle, and there is some t in W which carries it onto the triangle bounded by ξ, η, ζ. Since p, q, r are distinct, t must in fact carry ξ' onto ξ, η' onto η, and ζ' onto ζ. Hence $x' = txt^{-1}$, $y' = tyt^{-1}$, $z' = tzt^{-1}$, the given automorphism is an inner automorphism, and the lemma is proved.

The lemma implies that if (8) is false, there is some g in W such that $v^{-1} = gvg^{-1}$. Such a g would have to be an isometry taking the axis ξ onto itself with reversal of direction, and would therefore have to be either a rotation with angle π and center on ξ or a reflection in an axis perpendicular to ξ. Since p, q, r are all odd there are no such elements in W, and the proof is complete.

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Princeton University,
Princeton, N.J., U.S.A.