Random hyperbolic and flat surfaces (Riemann surfaces seen without glasses) Lecture 2. Magic Wand Theorem

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November 6, 2025

Closed horocycles (answer to a question asked after the previous lecture)
- Two facts from hyperbolic geometry
- Pushing closed horocycles down
- Long closed horocycles
- Riemann Hypothesis through averaging over long horocycles

Translation surfaces as holomorphic 1-forms

Period coordinates and Masur–Veech volume element

Dynamics in the moduli spaces

Magic Wand Theorem

Excercise

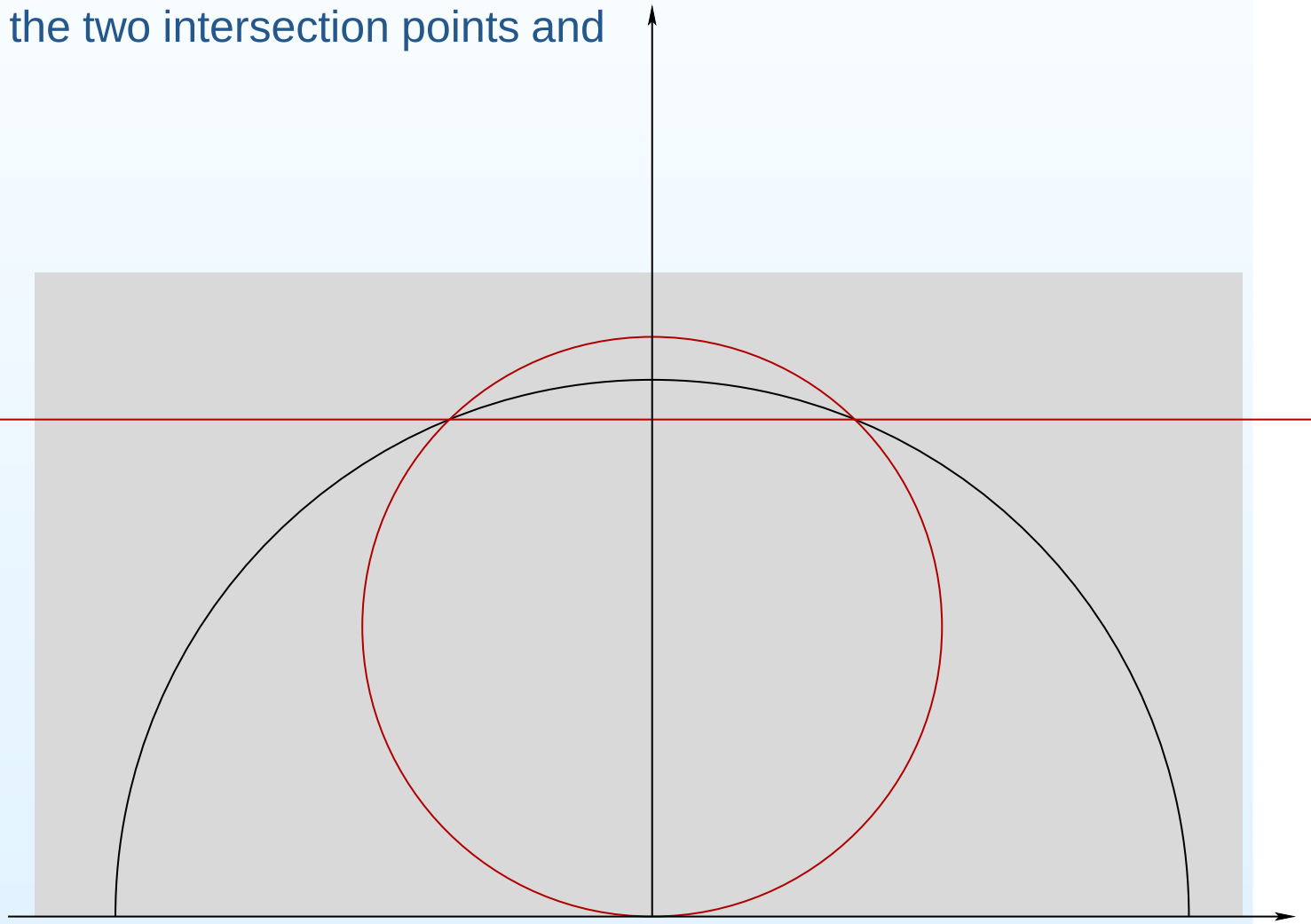
Disintegration of the Masur–Veech volume element in $\mathcal{H}(0)$

Closed horocycles (answer to a question asked after the previous lecture)

Two facts from hyperbolic geometry

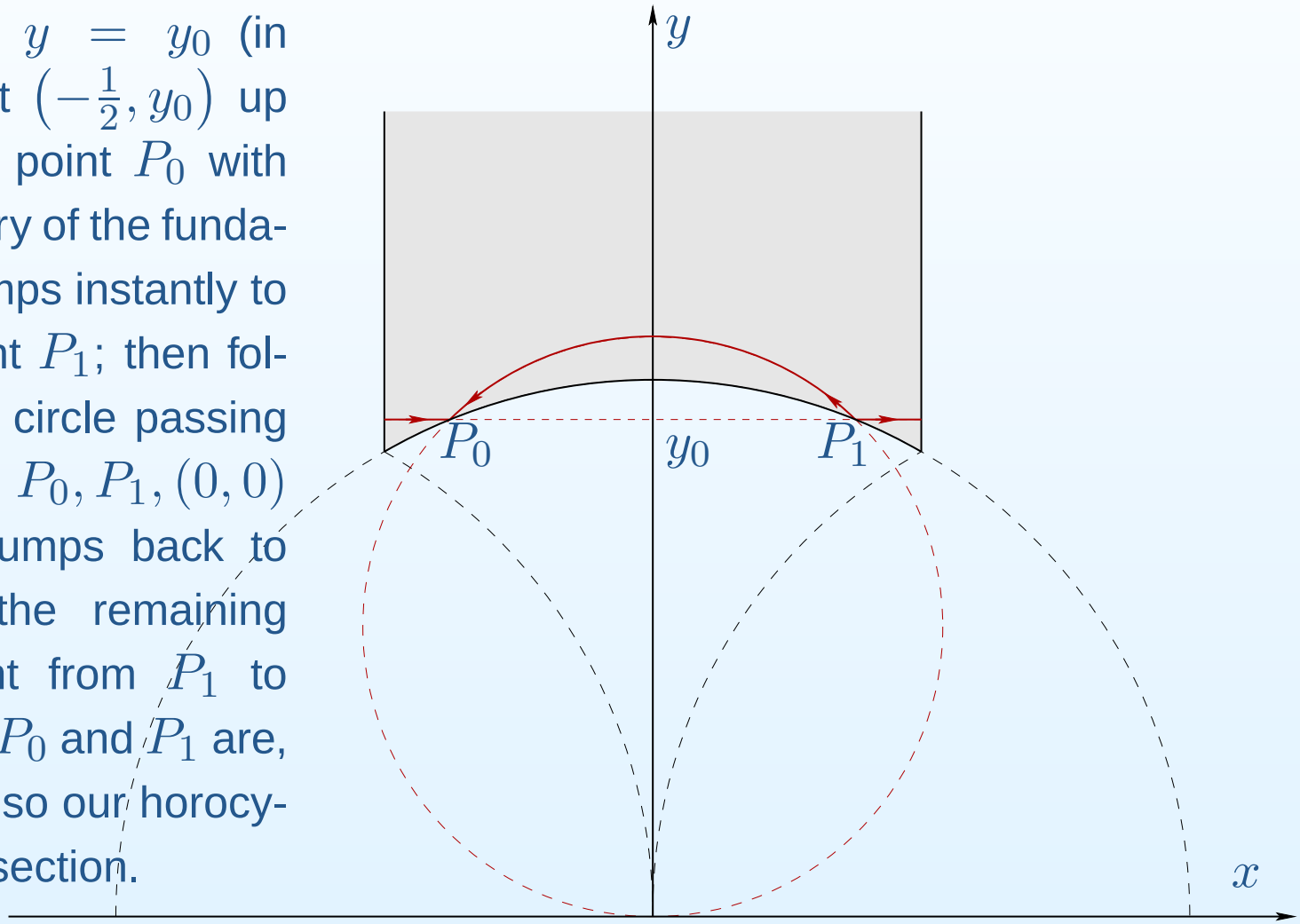
Inversion with respect to the unit circle centered at the origin transforms a horizontal line intersecting it into a circle passing through the two intersection points and through the origin.

Geometrically, horocycles are lines of curvature $|\varkappa| = 1$. In the Poincaré half-plane model they are represented by horizontal lines and by circles tangent to the real axes.



Pushing closed horocycles down

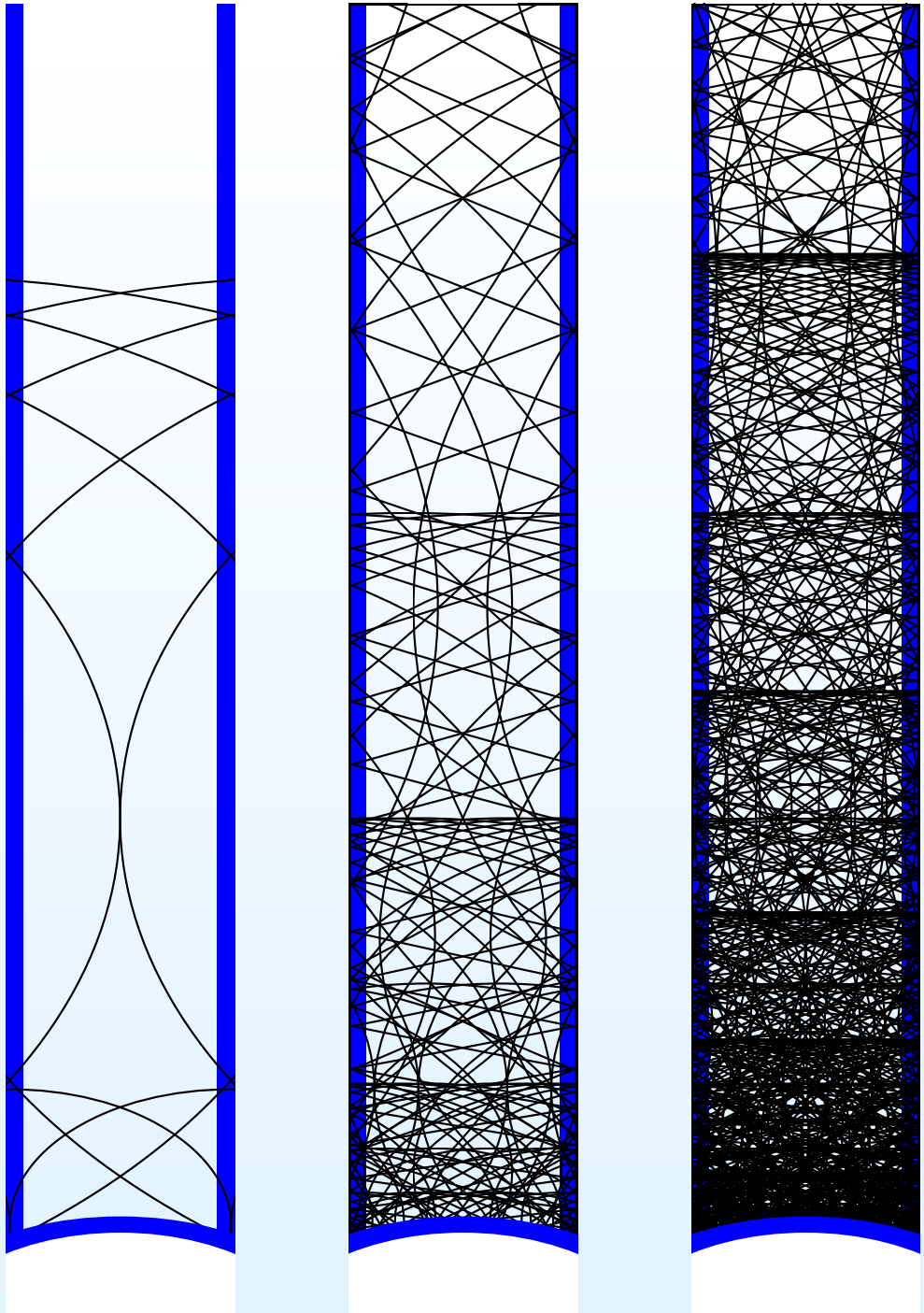
A horocycle at the level y_0 within bounds $\frac{\sqrt{3}}{2} < y_0 < 1$ follows the horizontal line $y = y_0$ (in red) from the point $\left(-\frac{1}{2}, y_0\right)$ up to the intersection point P_0 with the bottom boundary of the fundamental domain, jumps instantly to the symmetric point P_1; then follows an arc of the circle passing through the points $P_0, P_1, (0,0)$ from P_1 to P_0, jumps back to P_1 and follows the remaining horizontal segment from P_1 to $\left(\frac{1}{2}, y_0\right)$. Note that P_0 and P_1 are, actually, identified, so our horocycle has a self-intersection.



Long closed horocycles

Horizontal lines $y = const$ always represent closed horocycles. However, starting from $y = 1$ the horocycles start to self-intersect, and when y becomes small, they more and more densely wrap the modular surface.

The picture (taken from the paper of M. Cardella arXiv:1012:2754) represents horocycles $y = \frac{1}{8}, \frac{1}{100}, \frac{1}{400}$. The modular surface is truncated at $y > 10$.



Riemann Hypothesis reformulation (D. Zagier'81)

The Riemann Hypothesis is equivalent to the following condition:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x,y)\,dx \sim \frac{3}{\pi}\int_{\mathcal{M}} f\,d\mu + O\left(y^{\frac{3}{4}-\varepsilon}\right), \quad \text{as } y \to 0\,,$$

where $\mathcal{M}$ is the modular surface, $d\mu$ is the hyperbolic measure, and f is a sufficiently smooth function on $\mathcal{M}$ satisfying certain rapid decay conditions at the cusp.

The error term $o\left(y^{\frac{1}{2}}\right)$ is known.

Warning. One has to be very careful with the smoothness and with rapid decay conditions: see the paper of A. Verjovsky related to this issue.

Closed horocycles (answer to a question asked after the previous lecture)
Translation surfaces as holomorphic 1-forms
● From flat to complex structure
● From complex to flat structure
● Hyperbolic metric associated to the flat one
Period coordinates and Masur–Veech volume element
Dynamics in the moduli spaces
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Excercise
Disintegration of the Masur–Veech volume element in $\mathcal{H}(0)$

Translation surfaces as holomorphic 1-forms

Holomorphic 1-form associated to a flat structure

Consider the natural coordinate z in the complex plane, where lives the polygon. In this coordinate the parallel translations which we use to identify the sides of the polygon are represented as $z' = z + const$.

Since this correspondence is holomorphic, our flat surface S with punctured conical points inherits the complex structure. This complex structure extends to the punctured points.

Consider now a holomorphic 1-form dz in the complex plane. The coordinate z is not globally defined on the surface S. However, since the changes of local coordinates are defined as $z' = z + const$, we see that $dz = dz'$. Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on S which in local coordinates has the form $\omega = dz$.

The form ω has zeroes exactly at those points of S where the flat structure has conical singularities.

Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of zero a holomorphic 1-form can be represented as $w^d\,dw$, where d is the **degree** of zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \cdots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \ldots, d_n)$ enumerated by unordered partitions $d_1 + \cdots + d_n = 2g - 2$.
- Any holomorphic 1-forms corresponding to a fixed stratum $\mathcal{H}(d_1, \ldots, d_n)$ has exactly n zeroes $P_1, \ldots, P_n$ of degrees $d_1, \ldots, d_n$.

Hyperbolic metric associated to the flat one

We have seen that a translation surface S is nothing but a pair (X, ω), where X is a Riemann surface, and ω is a holomorphic 1-form on it. The space of all pairs (X, ω) forms a vector bundle $\mathcal{H}_g$ over the moduli space $\mathcal{M}_g$ of complex curves (which is also the moduli space of hyperbolic metrics). This vector bundle is called the *Hodge bundle*. The space of holomorphic 1-forms (often called *Abelian differentials*) on a given Riemann surface X is isomorphic to $\mathbb{C}^g$, so the Hodge bundle is g-dimensional.

The projection $\mathcal{H}_g \ni (X, \omega) \mapsto X \in \mathcal{M}_g$ assigns to a flat metric defined by (X, ω) the unique hyperbolic metric in its conformal class. This hyperbolic metric is exactly the one which is associated to the underlying complex structure X by the uniformization theorem.

Warning. Strata $\mathcal{H}(m_1, \dots, m_n)$ *do not* have a structure of a fiber bundle over $\mathcal{M}_g$. We will see that $\dim \mathcal{H}(m_1, \dots, m_n) = 2g + n - 1$. Thus, for example, the minimal stratum $\mathcal{H}(2g-2)$ has dimension $2g$, while $\dim \mathcal{M}_g = 3g - 3$. Indeed, for $g > 2$ a generic Riemann surface X does not have a single holomorphic 1-form with all zeroes collapsed to a single point.

Closed horocycles (answer to a question asked after the previous lecture)

Translation surfaces as holomorphic 1-forms

Period coordinates and Masur–Veech volume element

- Period coordinates
- Masur–Veech volume
- Volume element
- Concise geometro-analytic dictionary

Dynamics in the moduli spaces

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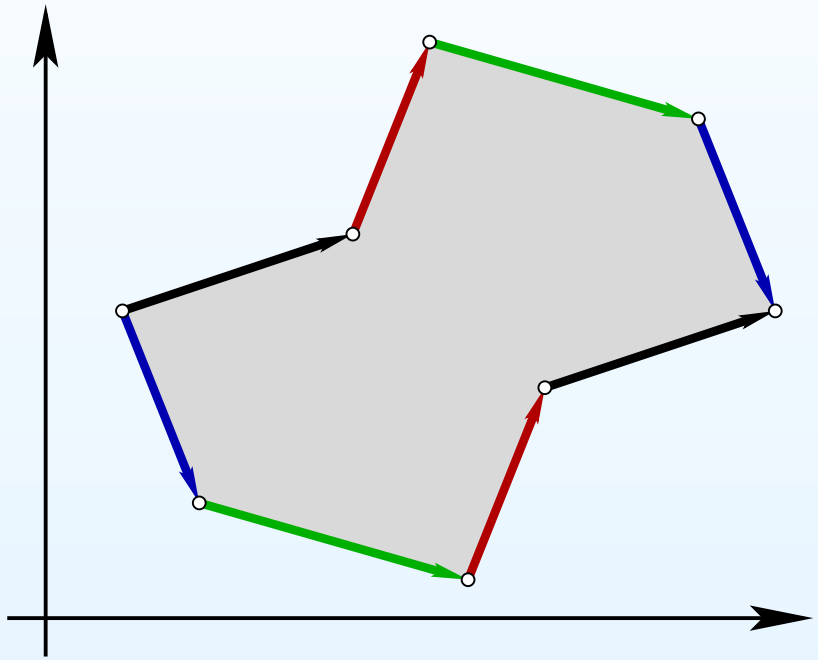
Disintegration of the Masur–Veech volume element in $\mathcal{H}(0)$

Period coordinates and Masur–Veech volume element

Period coordinates and Masur–Veech measure

Vectors defining the sides of the polygonal pattern serve as coordinates in the space of flat surfaces endowed with the distinguished vertical direction. The Lebesgue measure in these coordinates is called the *Masur–Veech measure*.



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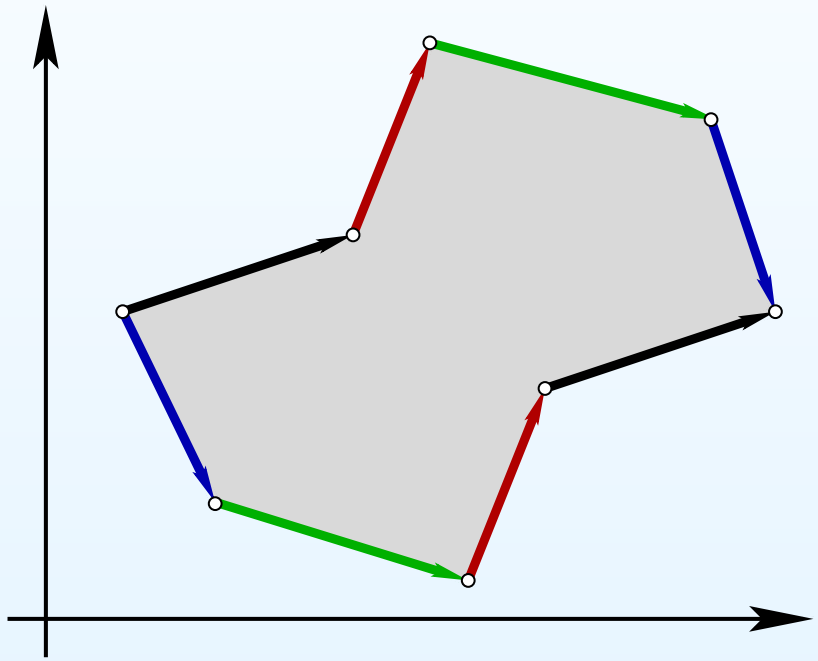

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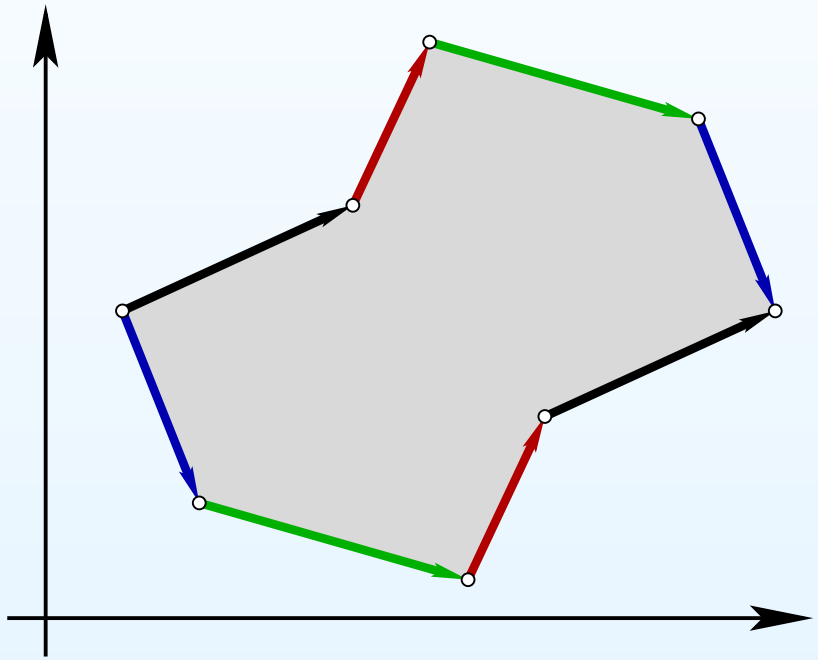

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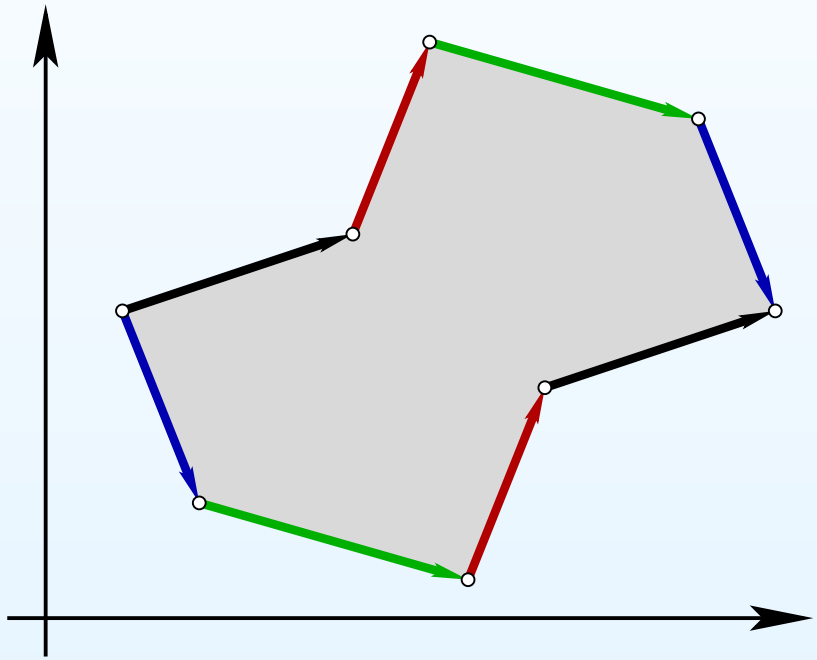
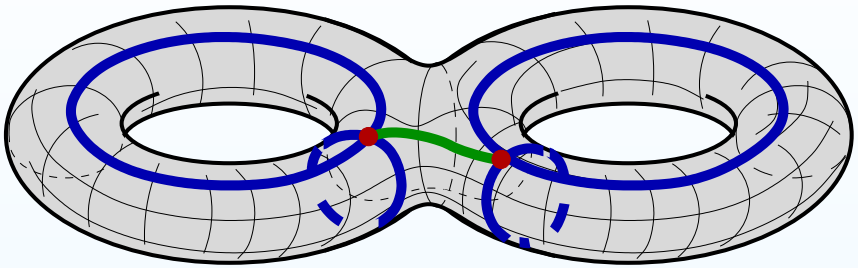
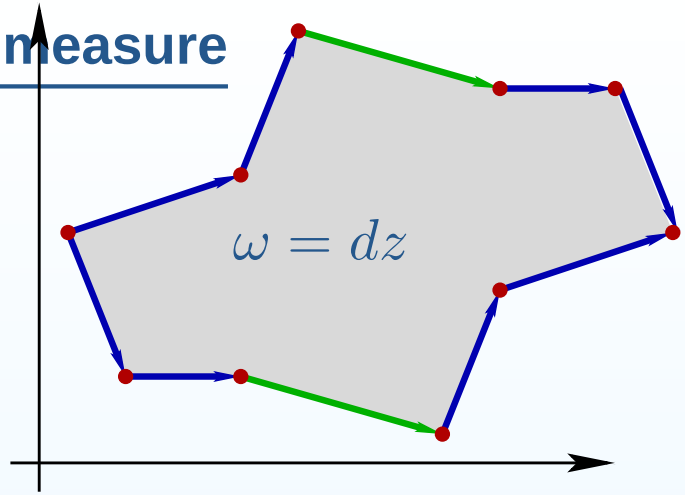


Considered as complex numbers, they represent integrals of the holomorphic form $\omega = dz$ along paths joining zeroes of the form ω. (In polygonal representation the zeroes of ω are represented by vertices of the polygon.)

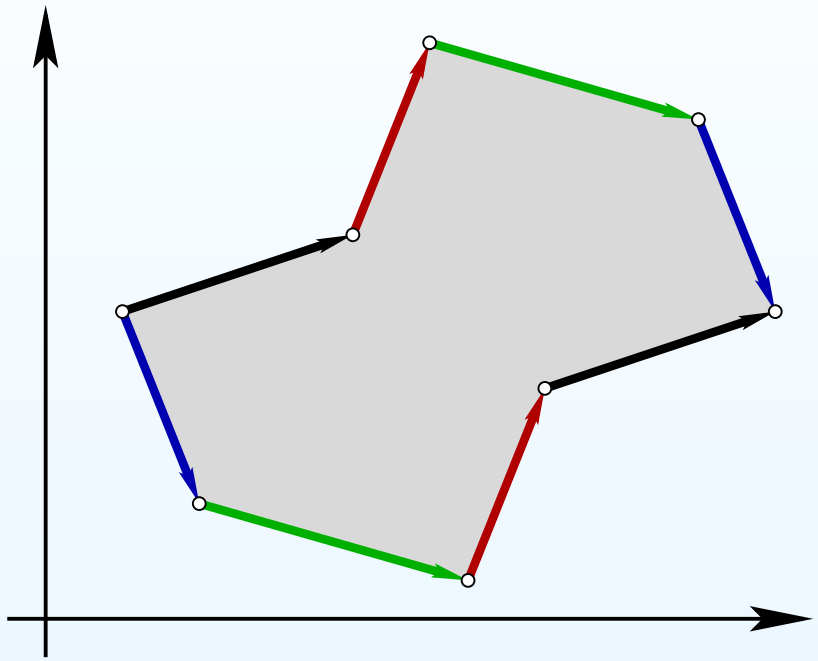
Period coordinates and Masur–Veech measure





Identifying corresponding sides V_j of a polygon by parallel translations we get a Riemann surface X and a holomorphic 1-form ω on it, where $\omega = dz$ in coordinate z on the polygon. The sides V_j become lines on S with endpoints in the collection of points $Y = \{P_1, \dots, P_k\} \subset X$ coming from vertices of the polygon. Since $d\omega = 0$ it defines a relative homology class $[\omega] \in H^1(X, Y; \mathbb{C})$: the value of $[\omega]$ on a cycle c is given by $\int_\gamma \omega$, where $[\gamma] = c$ is any collection of paths representing c. It is easy to check that vectors V_j generate $H_1(X, Y; \mathbb{C})$. Considered as complex numbers, they represent integrals $\mathbb{C} \ni V_j = \int_{V_j} dz$ of ω over the corresponding relative cycles. Thus, the collection of vectors uniquely determines $[\omega] \in H^1(X, Y; \mathbb{C})$. Reciprocally, any cohomology class in $H^1(X, Y; \mathbb{C})$ sufficiently close to $[\omega]$ defines a collection of deformed integrals over paths V_j, and, hence a deformed polygon.

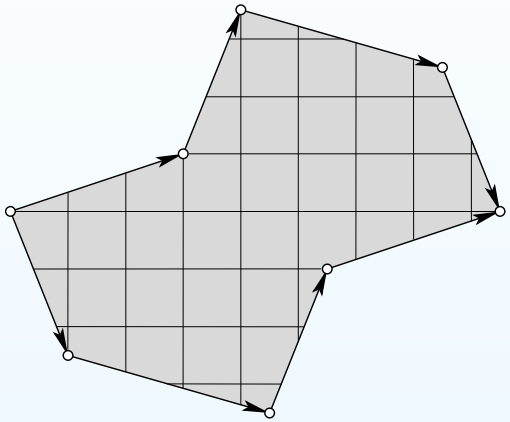
Period coordinates and Masur–Veech measure



In other words, the moduli space $\mathcal{H}(m_1, \ldots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \ldots, m_n$, where $\sum m_i = 2g - 2$, is modeled on the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

Flat area of the surface as a positive homogeneous function

We have a natural action of $\mathbb{R}^+$ on any stratum $\mathcal{H}(m_1, \ldots, m_n)$: we can rescale a flat surface by any positive factor r. The flat area gets rescaled by r^2.



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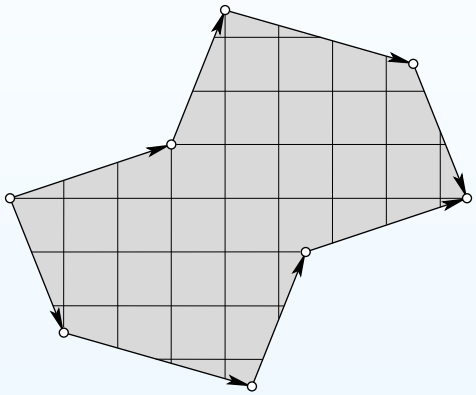

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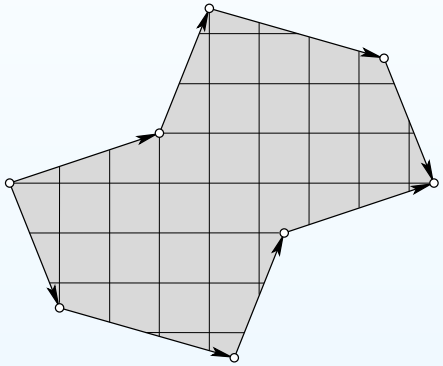

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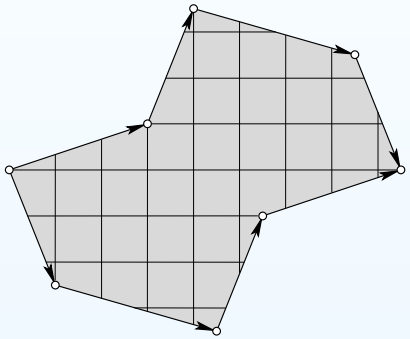

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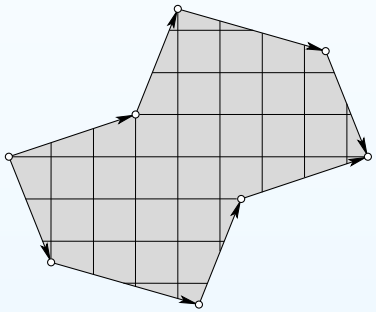

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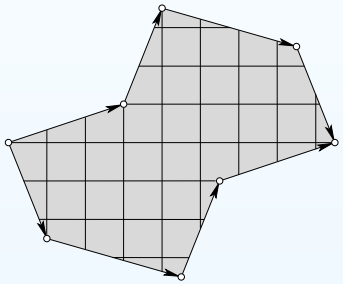

Flat surfaces of area 1 form a real hypersurface $\mathcal{H}_1 = \mathcal{H}_1(m_1, \dots, m_n)$ defined in period coordinates by equation

$$1 = \text{area}(S) = \frac{i}{2} \int_C \omega \wedge \bar{\omega} = \sum_{i=1}^{g} (A_i \bar{B}_i - \bar{A}_i B_i) \, .$$

Any flat surface S can be uniquely represented as $S = (C, r \cdot \omega)$, where $r > 0$ and $(C, \omega) \in \mathcal{H}_1(m_1, \dots, m_n)$. In these "polar coordinates" the volume element disintegrates as $d\nu = r^{2d-1} dr \, d\nu_1$ where $d\nu_1$ is the induced volume element on $\mathcal{H}_1$ and $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n) = 2g + n - 1$.

Excercise. Prove that $\nu_1(\mathcal{H}_{area=1}) = 2d \cdot \nu(\mathcal{H}_{area \leq 1})$.

Period coordinates and Masur–Veech volume element

The moduli space $\mathcal{H}(m_1, \ldots, m_n)$ of pairs (C, ω), where C is a complex curve and ω is a holomorphic 1-form on C having zeroes of prescribed multiplicities $m_1, \ldots, m_n$, where $\sum m_i = 2g - 2$, is modelled on the vector space $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$. The latter vector space contains a natural lattice $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$, providing a canonical choice of the volume element $d\nu$ in these *period coordinates*.

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The area function defined on every stratum $\mathcal{H}(m_1, \dots, m_n)$

$$\text{area}(C, \omega) = \frac{i}{2} \int_C \omega \wedge \bar{\omega} = \frac{i}{2} \sum_{i=1}^{g} (A_i \bar{B}_i - \bar{A}_i B_i) \, .$$

allows to define an analog of a "unit ball" $\mathcal{H}_{\leq 1}$ in any stratum as a subset of those (C, ω) in $\mathcal{H}(m_1, \dots, m_n)$, where $\text{area}(C, \omega) \leq 1$. (Note that in period coordinates the "unit ball" is rather the interior of a "unit hyperboloid".)

Definition.

$$\text{Vol}\, \mathcal{H}(m_1, \dots, m_n) := 2d \cdot \int_{\mathcal{H}_{\leq 1}} d\nu \, ,$$

where $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n)$.

Masur–Veech volume

Summary. Every stratum of Abelian differentials admits

- A local structure of a vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$;
- An integer lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z})$ which allows to normalize the associated Lebesgue measure;
- A positive homogeneous function which allows to define an analog of a unit sphere (or rather of a unit hyperboloid).

Theorem (H. Masur; W. Veech, 1982). *The total volume of any stratum $\mathcal{H}_1(m_1, \dots, m_n)$ or $\mathcal{Q}_1(m_1, \dots, m_n)$ of Abelian differentials or of meromorphic quadratic differentials with at most simple poles is finite.*

Volume element

Note that the vector space $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$ contains a natural integer lattice $H^1(S, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus \sqrt{-1}\,\mathbb{Z})$. Consider a linear volume element $d\nu$ normalized in such a way that the volume of the fundamental domain in this lattice equals one. Consider now the real hypersurface $\mathcal{H}_1(d_1, \dots, d_n) \subset \mathcal{H}(d_1, \dots, d_n)$ defined by the equation $area(S) = 1$. The volume element $d\nu$ can be naturally restricted to the hypersurface defining the volume element $d\nu_1$ on $\mathcal{H}_1(d_1, \dots, d_n)$.

Theorem (H. Masur; W. A. Veech) *The total volume* $\mathrm{Vol}(\mathcal{H}_1(d_1, \dots, d_n))$ *of every stratum is finite.*

The Masur–Veech volumes of the first several low-dimensional strata were computed by M. Kontsevich and A. Zorich about 2000. The first efficient algorithm for evaluation of the Masur–Veech volume was found by A. Eskin and A. Okounkov. In particular, they proved that the Masur–Veech volume of any stratum always has the form $(p/q)\pi^{2g}$ where p/q is a rational number. By 2003 A. Eskin computed these rational numbers up for all strata to genus 10. By now we have much better knowledge of Masur–Veech volumes; we will discuss them in more details later in these lectures.

Concise geometro-analytic dictionary

flat structure (including a choice of the vertical direction)	complex structure and a choice of a holomorphic 1-form ω
conical point with a cone angle $2\pi(d+1)$	zero of degree d of the holomorphic 1-form ω (in local coordinates $\omega = w^d\, dw$)
side $\vec{v}_j$ of a polygon	relative period $\int_{P_j}^{P_{j+1}} \omega = \int_{\vec{v}_j} dz$ of the 1-form ω
family of flat surfaces sharing the same cone angles $2\pi(d_1+1), \ldots, 2\pi(d_n+1)$	stratum $\mathcal{H}(d_1, \ldots, d_n)$ in the moduli space of holomorphic 1-forms
local coordinates in the family: vectors $\vec{v}_i$ defining the polygon	local coordinates in $\mathcal{H}(d_1, \ldots, d_n)$: relative periods of ω in $H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C})$

Closed horocycles (answer to a question asked after the previous lecture)

Translation surfaces as holomorphic 1-forms

Period coordinates and Masur–Veech volume element

Dynamics in the moduli spaces

- Group action
- Masur—Veech Theorem

Magic Wand Theorem

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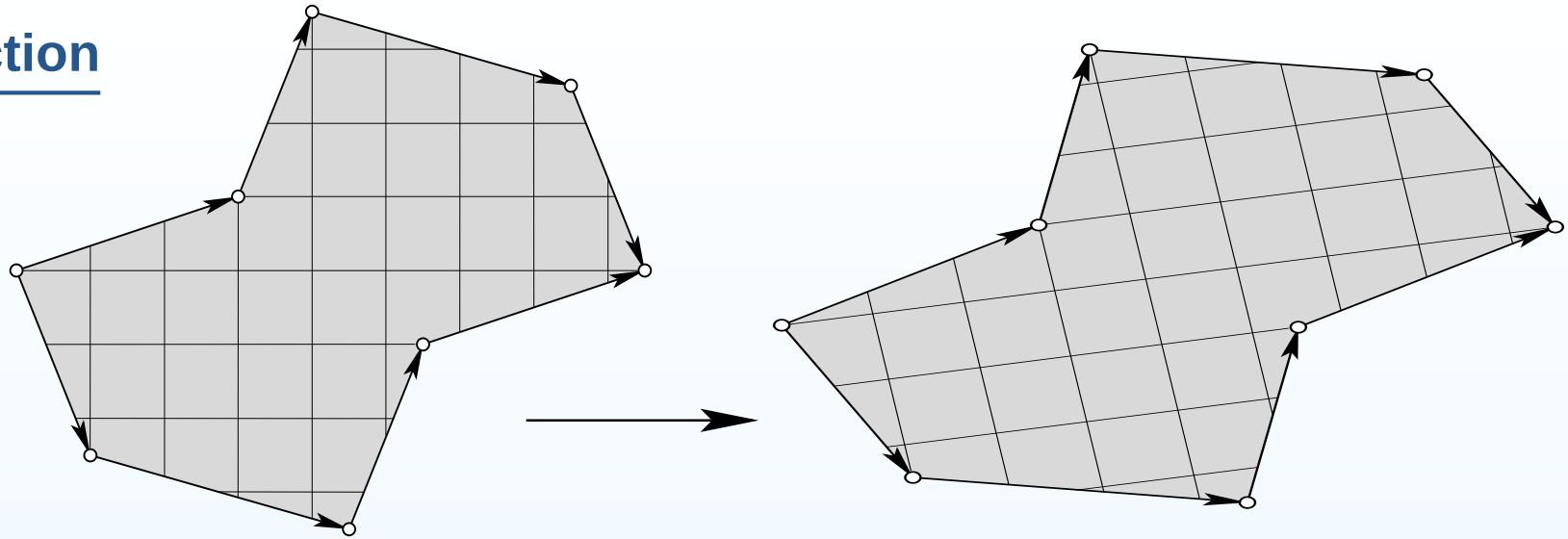
Dynamics in the moduli spaces

Group action

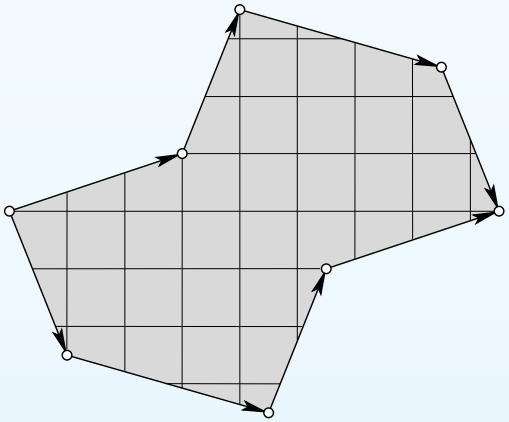


The subgroup $SL(2,\mathbb{R})$ of area preserving linear transformations acts on the "unit hyperboloid" $\mathcal{H}_1(d_1,\ldots,d_n)$. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset SL(2,\mathbb{R})$ induces a natural flow on the stratum, which is called the *Teichmüller geodesic flow*.

Key Theorem (H. Masur; W. A. Veech) *The action of the groups $SL(2,\mathbb{R})$ and $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ preserves the measure $d\nu_1$. Both actions are ergodic with respect to this measure on each connected component of every stratum $\mathcal{H}_1(d_1,\ldots,d_n)$.*

Masur—Veech Theorem

Theorem of Masur and Veech claims that taking an arbitrary octagon as below we can contract it horizontally and expand vertically by the same factor e^t to get arbitrary close to, say, regular octagon.



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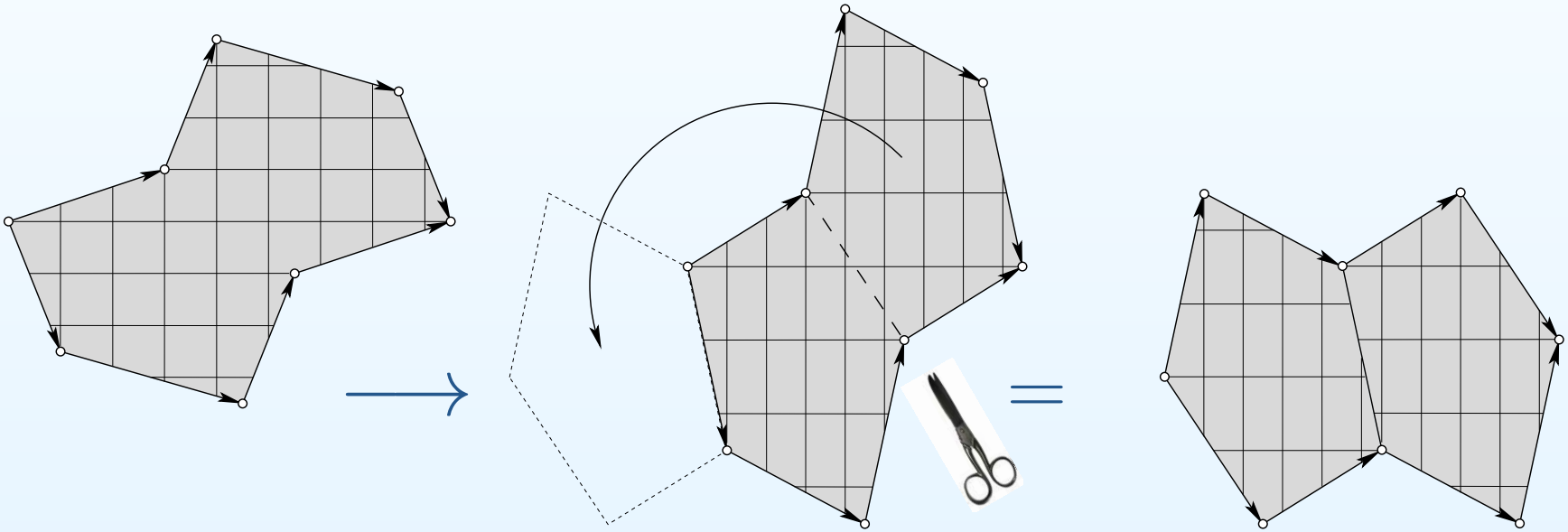
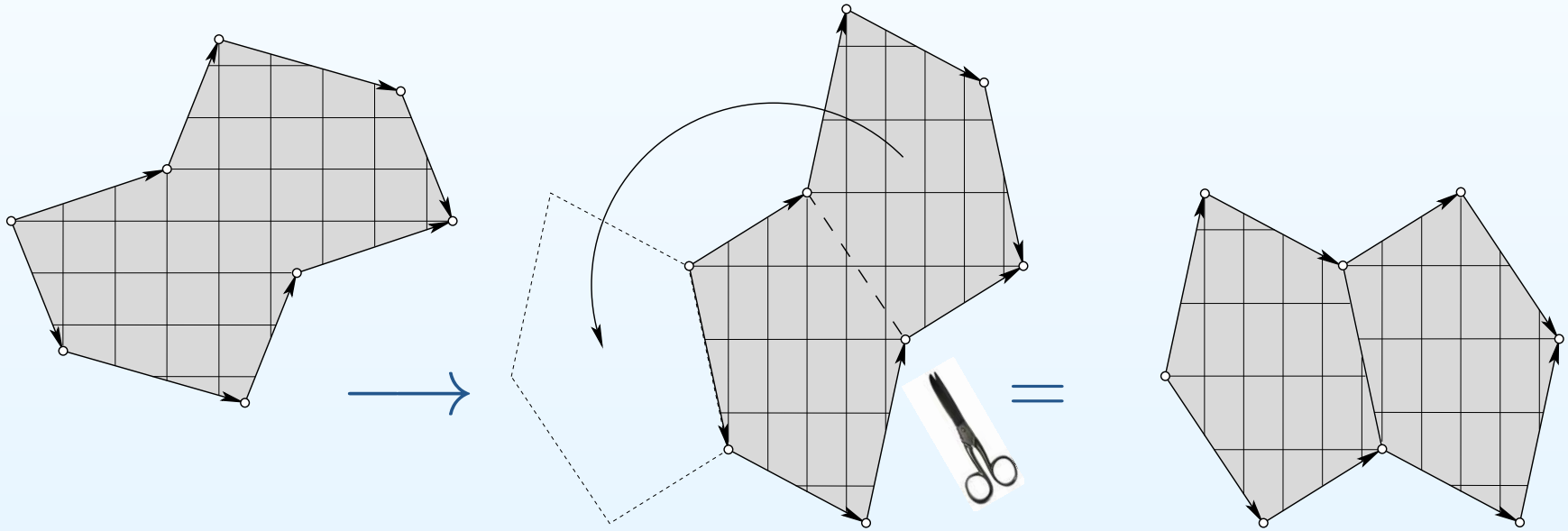
There is no paradox since we are allowed to cut-and-paste!



Masur—Veech Theorem

Theorem of Masur and Veech claims that taking an arbitrary octagon as below we can contract it horizontally and expand vertically by the same factor e^t to get arbitrary close to, say, regular octagon.



The first modification of the polygon changes the flat structure while the second one just changes the way in which we unwrap the flat surface

Closed horocycles (answer to a question asked after the previous lecture)

Translation surfaces as holomorphic 1-forms

Period coordinates and Masur–Veech volume element

Dynamics in the moduli spaces

Magic Wand Theorem

- Invariant measures and orbit closures
- Fields Medal
- Breakthrough Prize
- Why the Magic Wand Theorem is astonishing
- Geometric counterpart of Ratner Theorem

Excercise

Disintegration of the Masur–Veech volume element in $\mathcal{H}(0)$

Magic Wand Theorem

Invariant measures and orbit closures

Magic Wand Theorem (A. Eskin–M. Mirzakhani–A. Mohammadi'14–18). *The closure of any $\mathrm{SL}(2,\mathbb{R})$-orbit is a suborbifold. In period coordinates any $\mathrm{GL}(2,\mathbb{R})$-orbit closure is represented by a complexification of an $\mathbb{R}$-linear subspace.*

Any ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant measure is supported on a suborbifold. In period coordinates this suborbifold is represented by an affine subspace, and the invariant measure is just a usual affine measure on this affine subspace.

Theorem (S. Filip'16) *Any $\mathrm{GL}(2,\mathbb{R})$-invariant orbifold is, actually, an algebraic variety characterized by special arithmetic conditions.*

Further developements (M. Mirzakhani–A. Wright'18). *Hundreds of examples of triangles with small rational angles leading to orbit closures which are as big as a priori possible.*

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Fields Medal

At the International Congress of Mathematics in 2014 Maryam Mirzakhani has received a Fields Medal for *“for her exceptional contributions to dynamics and geometry of Riemann surfaces and their moduli spaces”* becoming the first woman to receive the Fields Medal.

Breakthrough Prize

Alex Eskin got 2020 Breakthrough Prize in Mathematics *“for revolutionary discoveries in the dynamics and geometry of moduli spaces of Abelian differentials, including the proof of the “Magic Wand Theorem” with Maryam Mirzakhani.”*

Why the Magic Wand Theorem is astonishing

For most of dynamical systems (including very nice and gentle ones) certain individual trajectories are disastrously complicated. In particular, after many iterations they might fill wired fractal sets.

For example, the map $f : x \mapsto \{2x\}$ homogeneously winding the circle $S^1 = \mathbb{R}/\mathbb{Z}$ twice around itself has orbits with orbit closures of (basically) any Hausdorff dimension between 0 and 1. The same map has infinite orbits avoiding certain arcs of the circle, etc. Even such elementary maps have certain (rare) orbits with a very bizarre behavior.

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Bernoulli shift. In the binary representation of a real number $x \in [0; 1[$

$$x = \frac{n_1}{2} + \cdots + \frac{n_k}{2^k} + \cdots ,$$

all the binary digits n_k are zeroes or ones. The map f acts on a sequence $(n_1, n_2, \ldots, n_k, \ldots)$ by erasing the first digit. This coding shows that we have, basically, a complete freedom in constructing orbits of f with peculiar behavior.

Geometric counterpart of Ratner Theorem

Consider one of the nicest possible dynamical systems: the geodesic flow on a closed compact Riemann surface of negative curvature. Its orbits live in the three-dimensional unit tangent bundle to the hyperbolic surface.

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Folklore Theorem (H. Furstenberg versus B. Weiss). *For any Riemann surface C of constant negative curvature and any real number d, such that $1 \leq d \leq 3$, there is a trajectory of the geodesic flow on the unit tangent bundle to C such that its closure has Hausdorff dimension d.*

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Situation with "geodesics" of higher dimensions is completely different.

Theorem (N. Shah). *In a compact manifold of constant negative curvature, the closure of a totally geodesic, complete (immersed) submanifold of dimension at least 2 is a totally geodesic immersed submanifold.*

The moduli space is **not** a homogeneous space, so a priori there were no reasons to hope for a rigidity theorem like the Magic Wand Theorem of Eskin, Mirzakhani, and Mohammadi! (Very recently A.Brown, A. Eskin, S. Filip and F. Rodriguez-Herz extended rigidity results to even broader situation.)

Closed horocycles (answer to a question asked after the previous lecture)

Translation surfaces as holomorphic 1-forms

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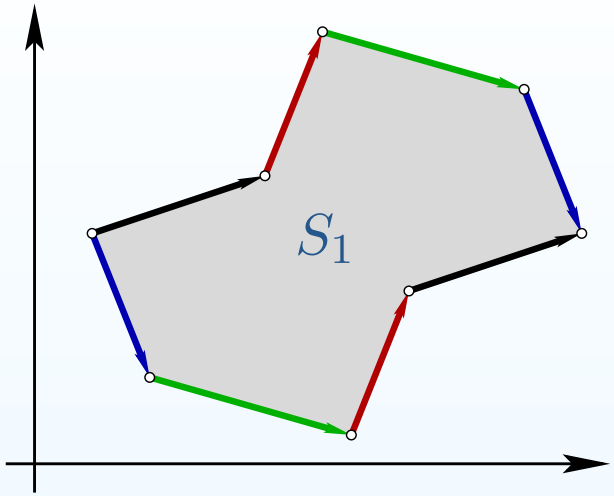
Excercise

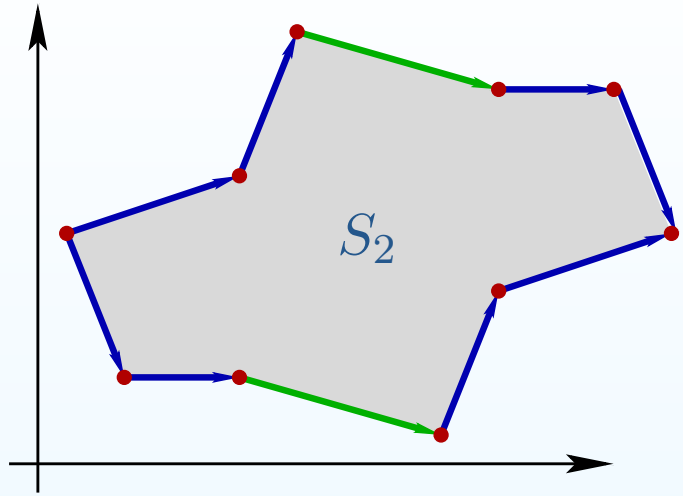
- Hyperelliptic involution and Weierstrass points
- Some references for spectral geometry of flat surfaces

Disintegration of the Masur–Veech volume element in $\mathcal{H}(0)$

Excercise

Hyperelliptic involution and Weierstrass points





• Verify that the surface S_1, obtained by identifying pairs of corresponding sides of the first polygon (respectively S_2 — of the second polygon) by parallel translations, have genus 2. To which strata belong S_1 and S_2?

• It is known that every Riemann surface of genus 2 is *hyperelliptic*, i.e. it admits a holomorphic involution τ such that the quotient over the involution is $\mathbb{C}\mathrm{P}^1$. Describe the hyperelliptic involutions for the surfaces S_1 and S_2.

• Fixed points of a hyperelliptic involution are called *Weierstrass points*. It follows from the Riemann–Hurwitz formula (which is a nice and very simple fact) that there are $2g + 2$ Weierstrass points. Find all Weierstrass points for the surfaces S_1 and S_2.

Some references for spectral geometry of flat surfaces

- Luc Hillairet, https://www.idpoisson.fr/hillairet/recherche/
- Christopher Judge, https://cjudge.pages.iu.edu/research/index.html
- Alexei Kokotov, https://www.concordia.ca/faculty/alexey-kokotov.html
- Alex Eskin, Maxim Kontsevich, Anton Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, Publications de l'IHES, 120:1 (2014), 207–333; arXiv:1112.5872 https://webusers.imj-prg.fr/∼anton.zorich/subjects.html

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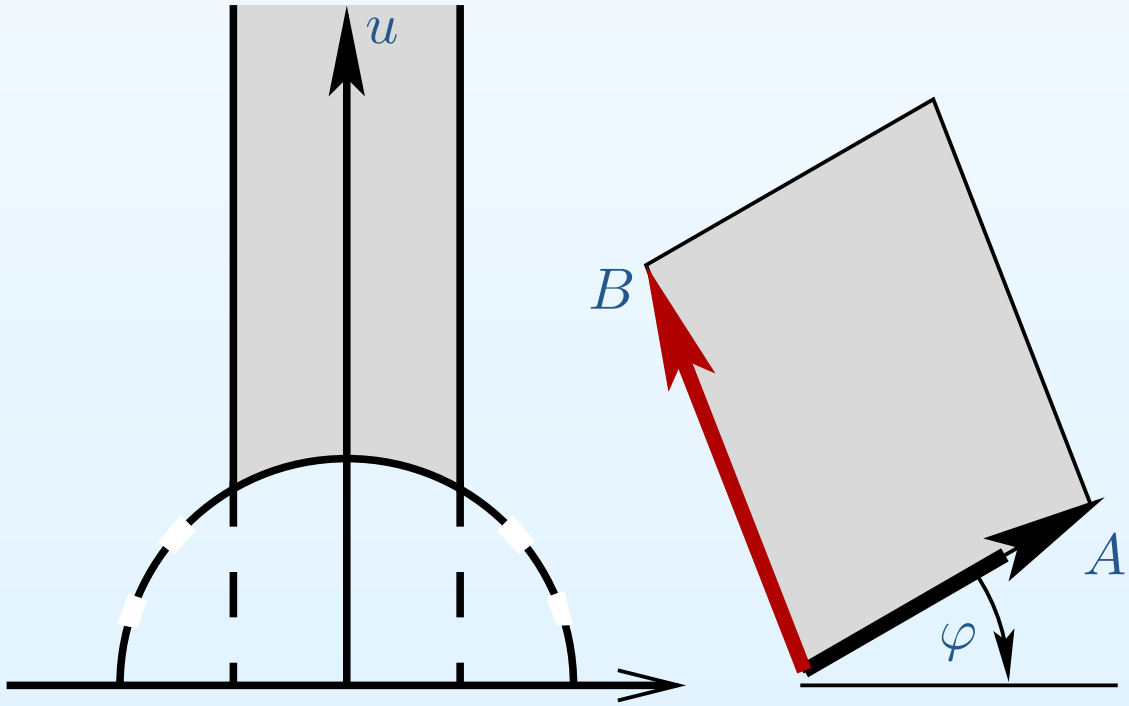
Let $A, B \in \mathbb{C}$ be periods of a holomorphic 1-form on an elliptic curve (equivalently, a pair of vectors defining a parallelogram in $\mathbb{R}^2$). Projection from the stratum $\mathcal{H}(0)$ to the modular surface $\mathrm{P}\mathcal{H}(0) = \mathcal{M}_1 = \mathbb{H}^2/\operatorname{PSL}(2,\mathbb{Z})$ corresponds to normalization of the A-period to 1 (equivalently, rescaling the parallelogram proportionally to make the length of the short side equal to 1 followed by a rotation making this side horizontal). We assume that B-period is chosen in such way, that we get directly to the fundamental domain.



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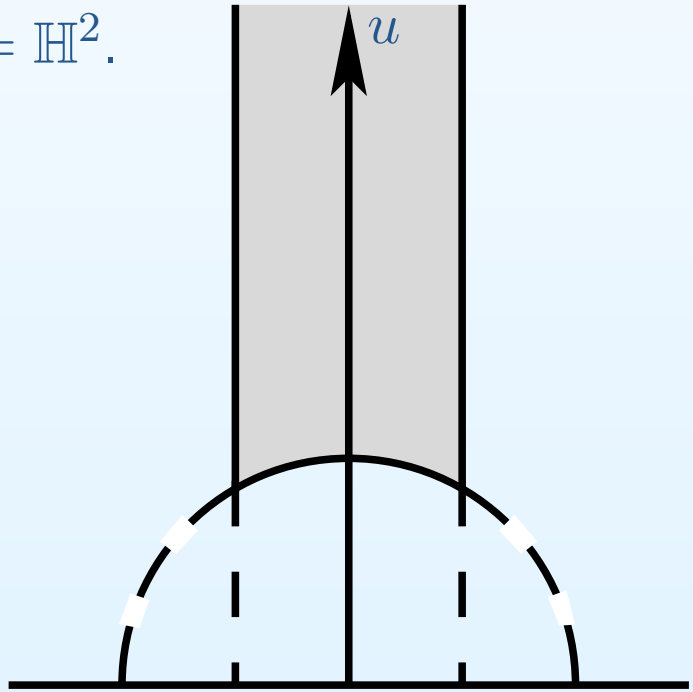

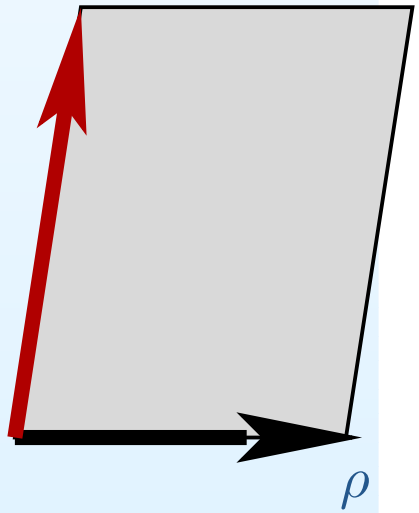

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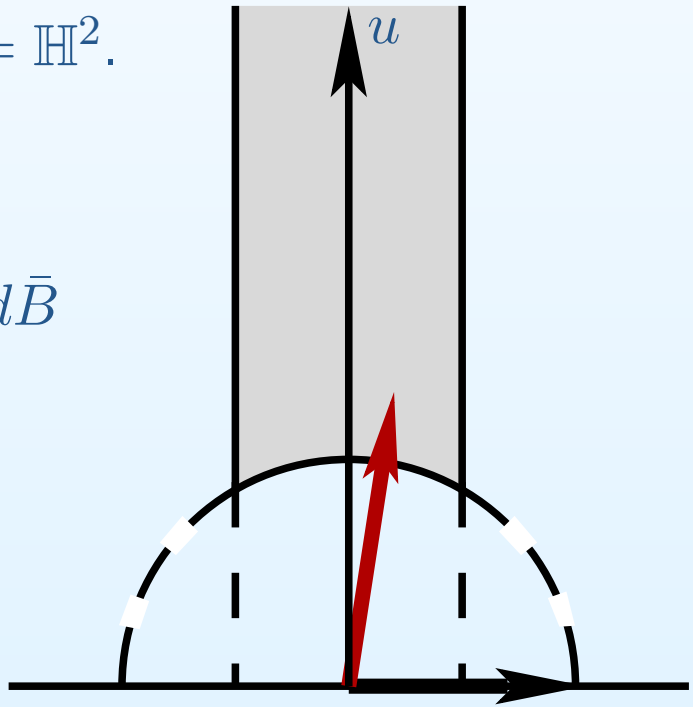
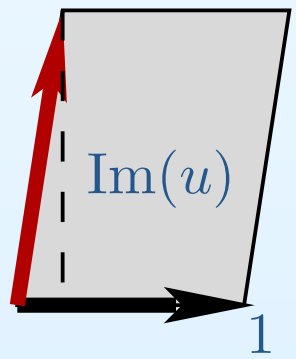


$$\mathrm{Area} = \mathrm{Im}(u) \cdot 1$$



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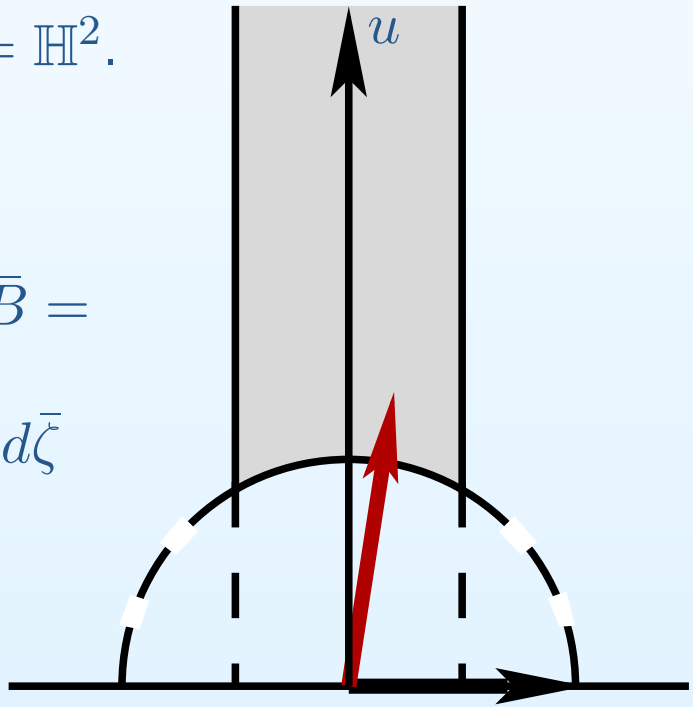

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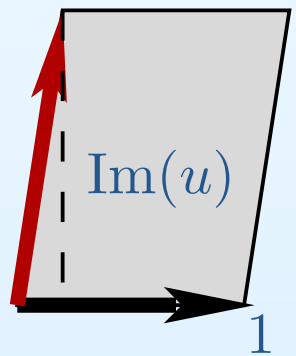

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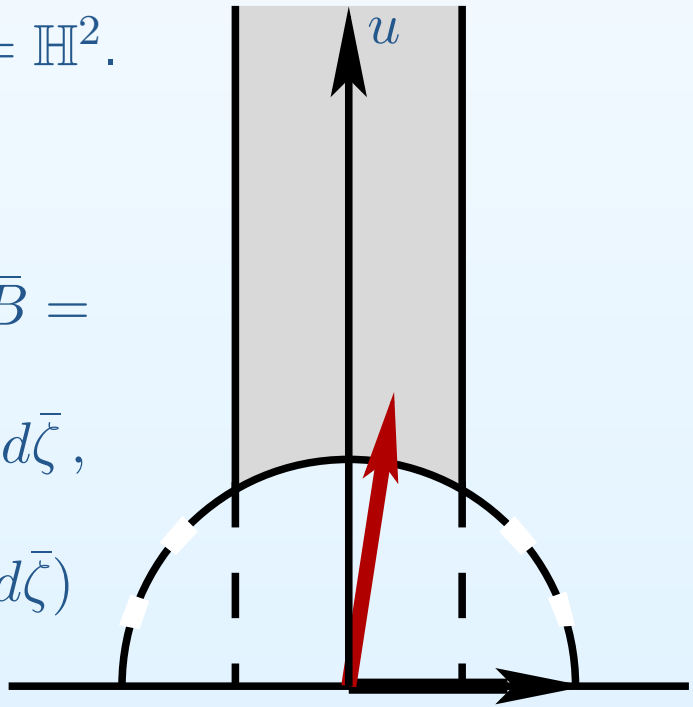
$$= -\frac{1}{4} \rho^2 du \wedge d\bar{u} \wedge (d\zeta \wedge d\bar{\zeta})$$



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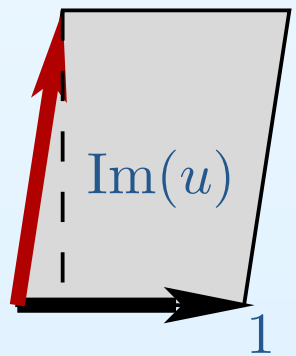

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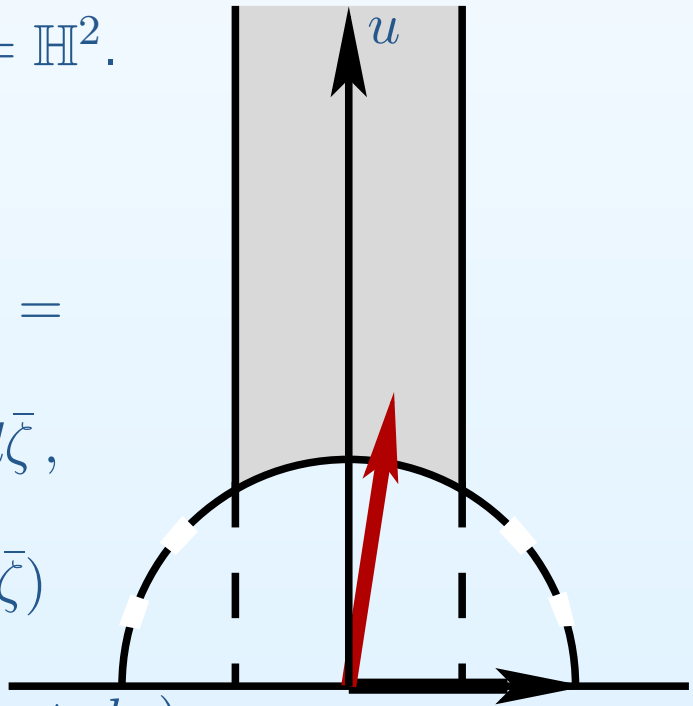

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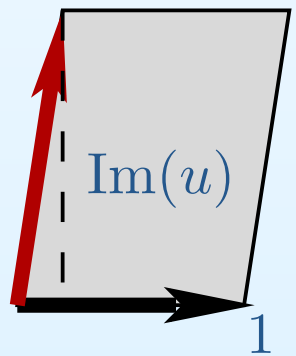

Disintegration of volume element along $\mathcal{H}(0) \to \mathcal{M}_1$

By definition of the induced volume element $d\nu_1$ on the "unit sphere" $\mathcal{H}_1(0)$ we have

$$d\nu = r^3\, dr\, d\nu_1\,.$$

where r^2 is the area of the flat torus. Recall that having rescaled the torus proportionally by a factor $\rho = |\zeta|$ we transformed its area to $\mathrm{Im}(u)$. Thus, the area of the original torus with periods A, B is $r^2 = |\zeta|^2\,\mathrm{Im}(u) = \rho^2\,\mathrm{Im}(u)$.

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$$d\nu = r^3\, dr\, d\varphi \wedge f(u,\bar{u}) du \wedge d\bar{u} = \left(\rho\sqrt{\mathrm{Im}\, u}\right)^3 \left(\sqrt{\mathrm{Im}\, u}\, d\rho\right) \wedge d\varphi \wedge f(u,\bar{u}) du \wedge d\bar{u}\,.$$

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Comparing the above expression with our original formula for $d\nu$ we get

$$\frac{i}{2}\rho^3\, d\rho \wedge d\varphi \wedge du \wedge d\bar{u} = \rho^3 (\mathrm{Im}\, u)^2 f(u,\bar{u})\, d\rho \wedge (d\varphi \wedge du \wedge d\bar{u})\ ,$$

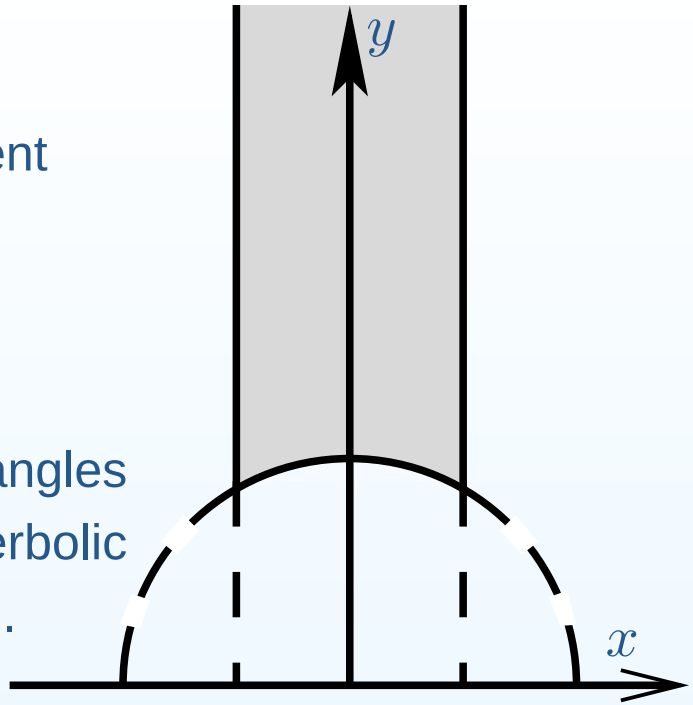
We recognize the hyperbolic volume element $f(u,\bar{u}) du \wedge d\bar{u} = \dfrac{\frac{i}{2}\, du \wedge d\bar{u}}{\mathrm{Im}^2(u)}$.

Masur–Veech volume $\mathrm{Vol}(\mathcal{H}(0))$

Letting $u = x + iy$ we get the standard volume element

$$\frac{\frac{i}{2}\, du \wedge d\bar{u}}{\mathrm{Im}^2(u)} = \frac{dx \wedge dy}{y^2}$$

in the hyperbolic half-plane. A hyperbolic triangle with angles α, β, γ, has area $\pi - (\alpha + \beta + \gamma)$. Thus, the hyperbolic area of the modular surface is $\pi - \left(\frac{\pi}{3} + \frac{\pi}{3} + 0\right) = \frac{\pi}{3}$.



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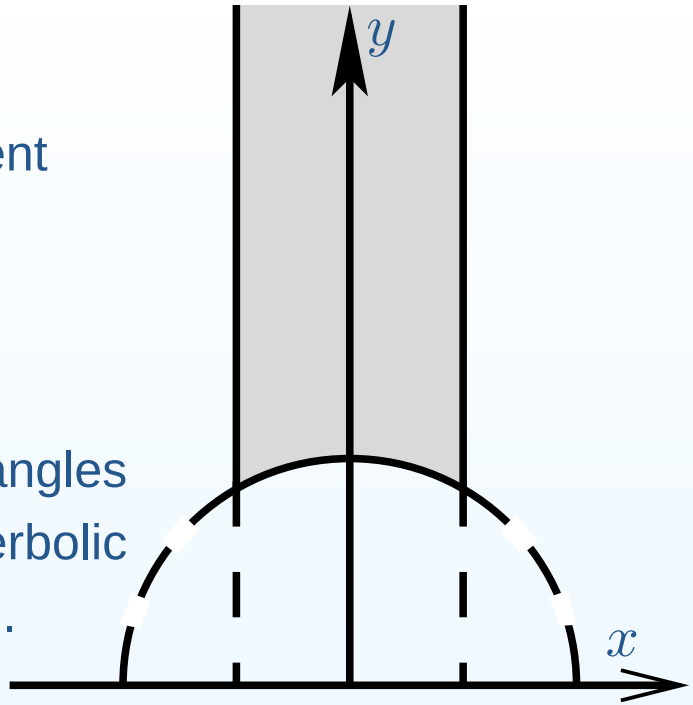
$$\frac{\frac{i}{2}\, du \wedge d\bar{u}}{\mathrm{Im}^2(u)} = \frac{dx \wedge dy}{y^2}$$

in the hyperbolic half-plane. A hyperbolic triangle with angles α, β, γ, has area $\pi - (\alpha + \beta + \gamma)$. Thus, the hyperbolic area of the modular surface is $\pi - \left(\frac{\pi}{3} + \frac{\pi}{3} + 0\right) = \frac{\pi}{3}$.



Our disintegration formula shows, that the Masur–Veech volume $\mathrm{Vol}(\mathcal{H}(0))$ equals the hyperbolic area of the modular surface times the measure of the circle S^1 responsible for the choice of the vertical direction on the torus.

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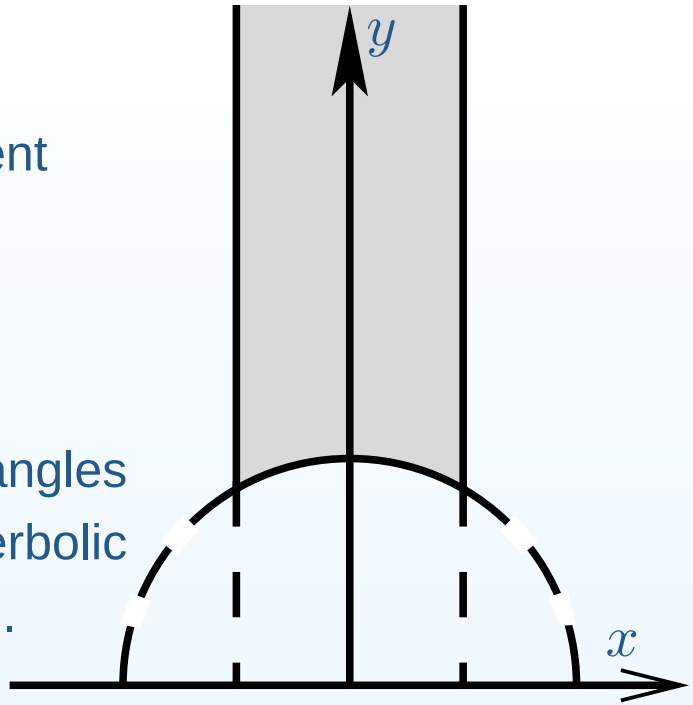

Our disintegration formula shows, that the Masur–Veech volume $\mathrm{Vol}(\mathcal{H}(0))$ equals the hyperbolic area of the modular surface times the measure of the circle S^1 responsible for the choice of the vertical direction on the torus. Observe that every flat torus admits an involution (central symmetry of its parallelogram pattern). Hence, directions ϕ and $-\phi$ give rise to isomorphic "polarized" flat tori and thus the measure of S^1 equals π and not 2π. We get

$$\mathrm{Vol}(\mathcal{H}(0)) = \frac{\pi}{3} \cdot \pi = \frac{\pi^2}{3}\,.$$