Combinatorics, Lecture 9, 2022/06/14

Let us recall three theorems we proved.

<u>Hall's Theorem</u>. For $\forall m \geq 0$, the sets $S_1, \dots, S_m$ have a SDR if and only if $S_1, \dots, S_m$ satisfy Hall's condition:

$$(*) \quad \text{for } \forall I \subseteq [m], \quad \left| \bigcup_{i \in I} S_i \right| \geq |I|$$

<u>Erdös-Ko-Rado Thm</u>. For $n \geq 2k$, the largest intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ has size $\binom{n-1}{k-1}$.

<u>Turán's Theorem</u> $ex(n, K_{r+1}) = e(T_r(n))$; moreover, the unique n-vtx K_{r+1}-free graph with $e(G) = e(T_r(n))$ is $G = T_r(n)$.

§1. A generalization of EKR.

Thm 1 Let k be a fixed integer. Let $A_1, A_2, \ldots, A_m$ be m subsets of $[n]$ such that (1) $\forall\, i \neq j \in [m]$, $A_i \not\subseteq A_j$

and $A_i \cap A_j \neq \emptyset$.

(2) $|A_i| \leq k \leq \frac{n}{2}$ for $\forall\, i \in [m]$

Then $m \leq \binom{n-1}{k-1}$.

pf: Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ and let $\ell = \min_{1 \leq i \leq m} |A_i|$. clearly $\ell \leq k$.

We will use induction on $k-\ell$.

The base case is when $k-\ell=0$.

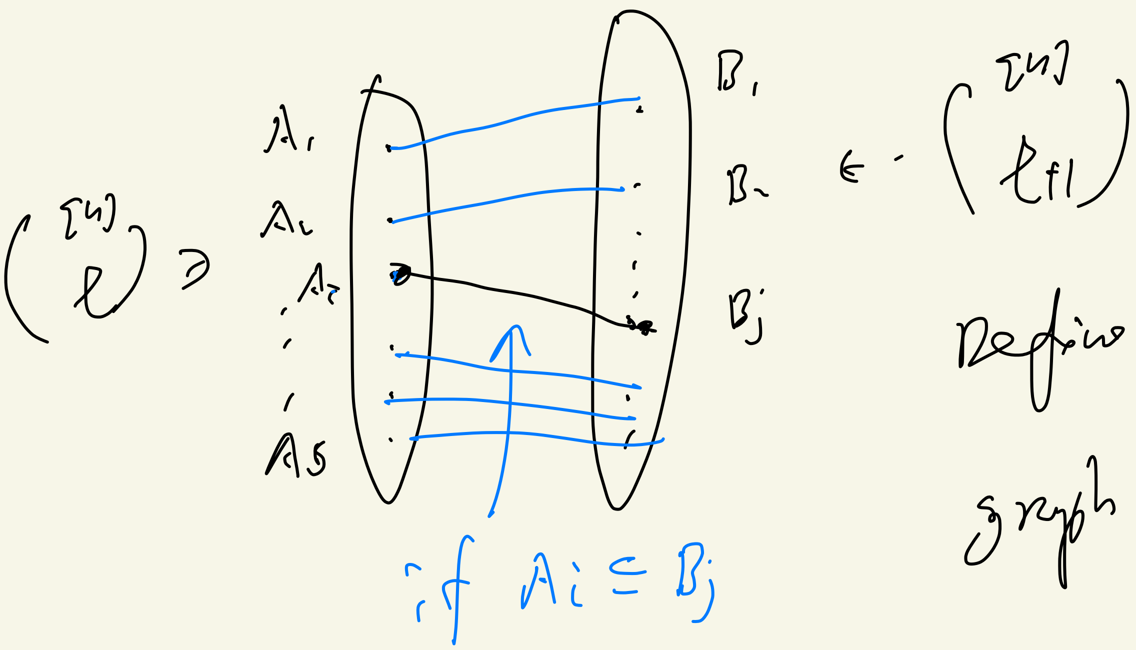
$\Rightarrow$ all A_i's have size k. This follows by the EKR Theorem.

Now assume this holds for any such $\mathcal{A}'$ with $\min_{1\le i\le m}|A_i'| < \ell$, where $\mathcal{A}' = \{A_1', \dots, A_m'\}$.

Consider $\mathcal{A} = \{A_1, \dots, A_m\}$ with min size ℓ.

Let $A_1, A_2, \dots, A_s$ be all subsets in $\mathcal{A}$ of size ℓ. Then $\boxed{\ell \le k-1 \le \frac{n}{2}-1}$

- Consider all subsets $B_j \in \binom{[n]}{\ell+1}$ which contains some A_i for $1 \le i \le s$.



Define a bipartite graph H.

Observe that

(1) Every A_i $(1 \le i \le s)$ is exactly contained in $n-\ell$ of these B_j's.

(2) Every B_j contains at most $\underline{\ell+1 \leq n-\ell}$.

$\Rightarrow$ Then this bipartite graph H satisfies the Hall's condition: $\forall\, X \subseteq \{A_1, \ldots, A_s\}$,

$$|N_H(X)| \geq |X| .$$

(Exercise)

$\Rightarrow$ H contains an $\{A_1, \ldots, A_s\}$-matching.

That is, we can find distinct $B_1, B_2, \ldots, B_s$ such that $A_i \subseteq B_i$ for $\forall\, 1 \leq i \leq s$.

Let $\mathcal{A}' = \left(\mathcal{A} \setminus \{A_1, \ldots, A_s\}\right) \cup \{B_1, \ldots, B_s\}$.

We <u>claim that</u> the new $\mathcal{A}'$ still satisfies the conditions of the theorem. (Exercise).

$\Rightarrow$ the min size of subsets in $\mathcal{A}'$ becomes $\ell+1$ $\Rightarrow$ $k-(\ell+1)$

So by induction, $m = |\mathcal{A}'| \leq \binom{n-1}{k-1}$. $\boxtimes$

§2. Partially ordered sets

Def. Let X be a finite set. Let R be a (binary) relation on X (i.e., $R \subseteq X \times X$).

If $(x, y) \in R$, then we write xRy.

Def. A partially ordered set (poset) is an ordered pair $P = (X, R)$, where R is a relation on X such that the following hold:

(1) R is reflective : xRx for $\forall x \in X$.

(2) R is antisymmetric : if xRy and yRx, then $x = y$.

(3) R is transitive : if xRy and yRz then xRz

Examples. - The poset $\left(2^{[n]}, \underset{\text{"containment"}}{\subseteq}\right)$.

$\boxed{xRy \Leftrightarrow x \subseteq y}$

- The poset $([n], |)$
  divisiblity

Def. Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be two posets. A mapping $f: X_1 \to X_2$ is an <u>embedding</u> of $(X_1, \leq_1)$ in $(X_2, \leq_2)$, if (1) f is injective;

(2) $f(x) \leq_2 f(y)$ if and only if $x \leq_1 y$.

Thm 1. For every poset $(X, \leq)$, there exists an embedding of $(X, \leq)$ in the poset $(2^X, \subseteq)$.

Pf. Consider the mapping $f: X \to 2^X$ by letting $f(x) = \{ y \in X : y \leq x \}$,

i.e. y is a predecessor
~~child~~ of x

It is easy to check that f is <u>injective</u>.

If $f(x) = f(y)$, then $x \in f(x) = f(y)$

$\Rightarrow$ $x \leq y$ (similarly, $y \leq x$)

$\Rightarrow$ $x = y$.

Now we show $x \leq y$ iff $f(x) \subseteq f(y)$.

It is clear that if $x \leq y$, then by transitive property, we have $f(x) \subseteq f(y)$.

If $f(x) \subseteq f(y)$, then $x \in f(x) \subseteq f(y)$

$\Rightarrow$ $x \leq y$. $\boxtimes$

Def. Let $(X, \leq)$ be a poset. We say x is an <u>immediate predecessor</u> of y, denoted by $x \lessdot y$, if $x < y$ and there is no other element z such that $x < z < y$.

<u>Fact</u>. For $\forall\, x, y \in (X, \leq)$, $x < y$ if and only if there exist $z_1, \ldots, z_k$ $(k \geq 0)$ such that $x \lessdot z_1 \lessdot z_2 \lessdot \cdots \lessdot z_k \lessdot y$.

<u>Pf</u>. Induction on the size of

$$M_{xy} = \{ z : x < z < y \}. \quad \blacksquare$$

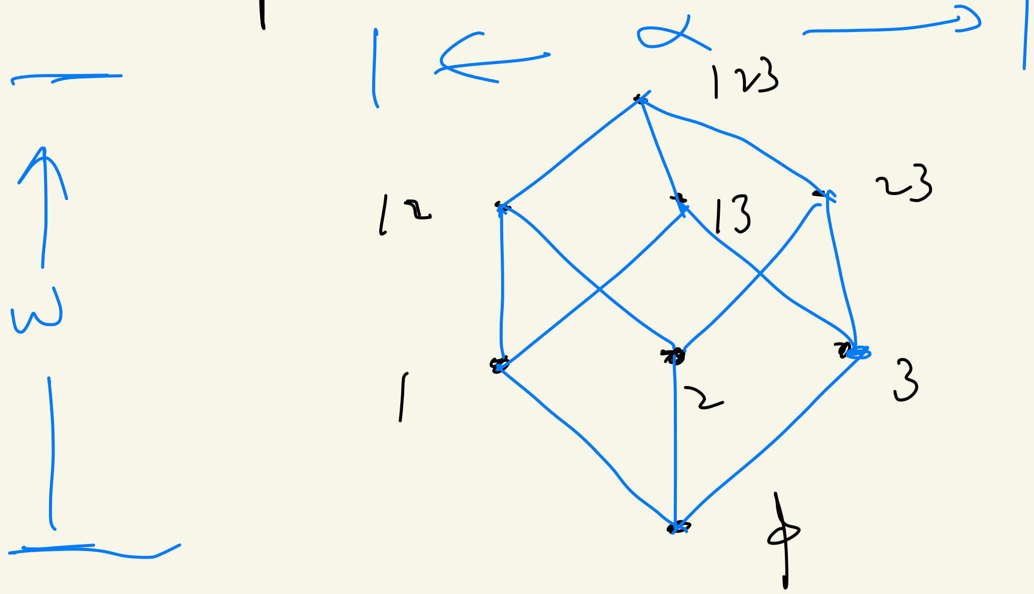
<u>Def</u>. The <u>Hasse digram</u> of a poset $(X, \leq)$ is a drawing in the plane such that

(1) every element of X is drawn as a nod,

(2) each pair $x \lessdot y$ is connected by a line segment, and

(3) if $x \lessdot y$, then the nod x must appear lower in the plane than the nod y.

i.e. poset $(2^{[3]}, \subseteq)$



Hasse diagram

Fact. $x < y$ if and only if we can find a path in the Hasse diagram from the node x to the node y, strictly from bottom to top.

Def. Let $(X, \leq)$ be a poset

(1) For $x, y \in X$, if $x < y$ or $y < x$, then x, y are comparable; otherwise, x, y are incomparable.

(2) The set $A \subseteq X$ is an antichain

if any two elements in A are incomparable.

Let $\alpha(X, \leq)$ be the maximum size of an antichain of $(X, \leq)$.

(3) The set $B \subseteq X$ is a chain, if any two elements in B are comparable.

Let $\omega(X, \leq)$ be the maximum size of a chain in $(X, \leq)$.

Thm 2. For any poset $(X, \leq)$,

$$\alpha(X, \leq) \cdot \omega(X, \leq) \geq |X|.$$

Fact. We call an element $x \in X$ minimal if x has NO predecessor. Then the set consisting of all minimal element in $(X, \leq)$ is an antichain

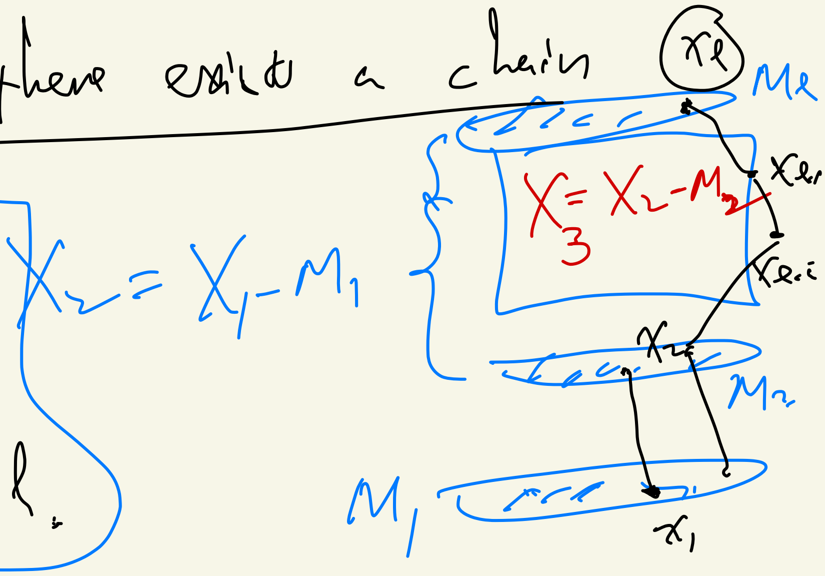
Pf of Thm 2. We will inductively define a sequence of posets $(X_i, \le)$ and a sequence of sets M_i consisting all minimal elements in $(X_i, \le)$ such that $X_{i+1} = X_i - M_i$.

In this way, we get $X_1 = X$

and $X = M_1 \cup M_2 \cup \cdots \cup M_\ell$

We claim that there exists a chain

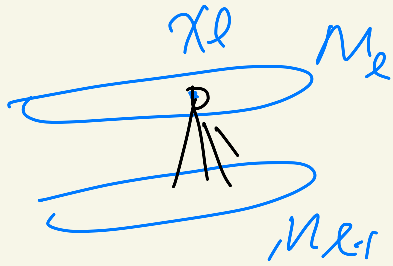
say $x_1 < x_2 < \cdots < x_\ell$ in $(X, \le)$ such that $x_i \in M_i$ for $1 \le i \le \ell$.



To see this, pick any element in M_ℓ to be x_ℓ

We assert that $\exists\ x_{\ell-1} \in M_{\ell-1}$ with $x_{\ell-1} < x_\ell$.

Otherwise, x_ℓ is the minimal element in $(X_{\ell-1}, \le)$.



We can keep this to get $x_{\ell-2} \in M_{\ell-2}$

$x_{\ell-2} < x_{\ell-1} < x_\ell$.

. . . . . . We can get $x_1 \in M_1$

with $x_1 < x_2 < \cdots < x_{\ell-1} < x_\ell$.

As $\alpha(X, \leq) \geq |M_i|$,

where M_i is an antichain of $(X_i, \leq)$, which is also an antichain of $(X, \leq)$

& $\omega(X, \leq) \geq \ell$

$$\Rightarrow |X| = \sum_{i=1}^{\ell} |M_i| \leq \ell \cdot \alpha \leq \omega \cdot \alpha$$

$\boxtimes$

Recall.

Erdős-Szekeres Thm: For any sequence $(x_1, x_2, \ldots, x_{st+1})$ of reals, there exists an increasing subsequence $(x_{i_1}, \ldots, x_{i_{s+1}})$ of X or a decreasing subsequence $(x_{i_1}, \ldots, x_{i_{t+1}})$ of X

$st+1 \Rightarrow$ $s+1$ ↗ or $t+1$ ↘.

Pf (2nd of Erdős-Szekeres) Let $X = \{1, 2, \cdots, st+1\}$

Define the poset $(X, \preceq)$ as following:

$i \preceq j$ if and only if $i \leq j$ and $x_i \leq x_j$.

First, we check that indeed this gives a poset:

(1) reflective: $i \preceq i$

(2) antisymmetric: if $i \preceq j$ & $j \preceq i$,

then $i \leq j$ & $j \leq i \Rightarrow i = j$.

(3) transitive. ✓

By Thm 2, $w(X, \preceq) \cdot \alpha(X, \preceq) \geq |X| = st+1$

so $w(X, \preceq) \geq s+1$ or $\alpha(X, \preceq) \geq t+1$.

Case 1. $w(X, \preceq) \geq s+1$.

There exist $s+1$ integers $i_1 \prec i_2 \prec \cdots \prec i_{s+1}$

$\Rightarrow i_1 < i_2 < \cdots < i_{s+1}$ & $x_{i_1} < x_{i_2} < \cdots < x_{i_{s+1}}$

i.e. an increasing subsequence of length $s+1$.

Case 2 $\alpha(X, \preceq) \geq t+1$.

There exists an antichain $\{i_1, \dots, i_{t+1}\}$.

As integers, we may let $i_1 < i_2 < \dots < i_{t+1}$

$\Rightarrow x_{i_1} > x_{i_2} > \dots > x_{i_{t+1}}$

i.e. a decreasing subsequence of length $t+1$ $\boxed{Z}$

Thm 3 (Dilworth's Thm) Let $(X, \leq)$ be a poset. Then the minimum number m of disjoint chains which together contain all elements of X is equal to $\alpha(X, \leq)$.

max of antichain of $(X, \leq)$

i.e. $X = \bigcup_{i=1}^{m} C_i$,

where C_i's are pairwisely disjoint chains in $(X, \leq)$.

Pf. (i) clearly, $m \geq \alpha(P)$. ✓

This is because in any disjoint chain decomposition

$X = \bigcup_{i=1}^{t} C_i$, $|C_i \cap A| \leq 1$,

where A denotes a max antichain.

$\Rightarrow \quad t \geq |A| = \alpha(X, \leq)$. ☒

(ii) We use induction on $|X|$ to show that $m \leq \underline{\alpha(X, \leq) \triangleq \alpha}$

The basic case is that $|X| = 0$. ✓.

Let C be a fixed maximum chain in $(X, \leq)$.

(*) If every antichain in $\underline{(X - C, \leq)}$ contains $\alpha' \leq \alpha - 1$ elements, then by induction, there exists $X - C = \bigcup_{i=1}^{\alpha'} C_i$ where C_i's are chains in $(X - C, \leq)$.

$\Rightarrow \quad X = \left(\bigcup_{i=1}^{\alpha'} C_i \right) \cup C$ has at most $\alpha' + 1 \leq \alpha$ chains ✓.

Hence, we may assume that

$\{a_1, a_2, \cdots, a_\alpha\}$ is an antichain in $(X - C, \leq)$.

Define $S^- = \{x \in X : \text{if } \exists a_i \text{ s.t. } x \le a_i\}$.

& $S^+ = \{y \in X : \text{if } \exists a_j \text{ s.t. } a_j \le y\}$.

$\Rightarrow$ $S^- \cup S^+ = X$ (O.W., we can find an anchain of size $\alpha+1$.)

& $\boxed{\{a_1, \cdots, a_\alpha\} \subseteq S^- \cap S^+}$.

$\Rightarrow$ $\boxed{\alpha(S^-, \le) = \alpha = \alpha(S^+, \le)}$.

Since C is a maximal chain, its largest element of C is NOT in S^-. $\Rightarrow |S^-| < |X|$.

By induction on S^-,

S^- is the union of $\alpha = \alpha(S^-, \le)$ disjoint chains $S_1^-, S_2^-, \cdots . S_\alpha^-$, where $a_i \in S_i^-$.

Claim: a_i is the maximal element of the chain S_i^-.

Pf of claim. O.W. $\exists\ x \in S_i^-$ with $x > a_i$.

By definition ($x \in S^-$), $\exists\ a_j$ with $x \le a_j$

$\Rightarrow \quad a_i < x \leq a_j \quad \Rightarrow \quad a_i < a_j$,

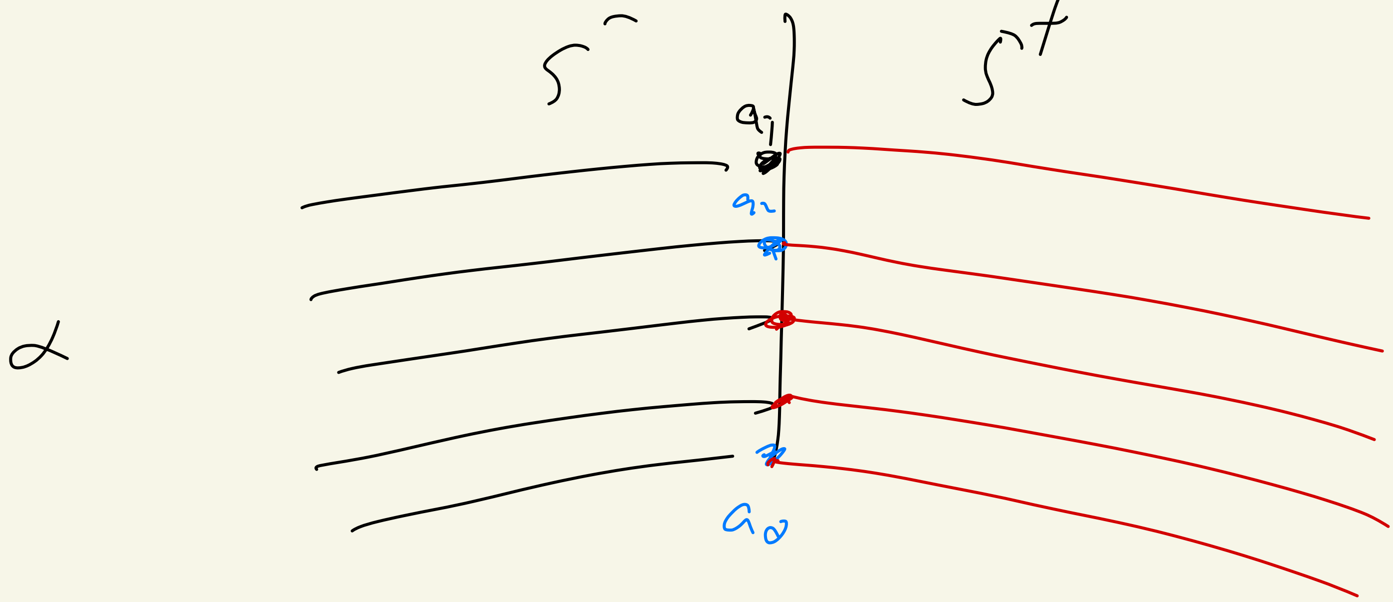
a contradiction !

Now we can do the same for S^+.

$\Rightarrow$ S^+ is the union of α disjoint chains

$S_1^+, S_2^+, \ldots, S_\alpha^+$ such that $a_i \in S_i^+$

and a_i is the minimal element of S_i^+.



We can combine $S_i^- \cup S_i^+$ to get a chain in $(X, \leq)$.

As $S^- \cup S^+ = X$, X is the union of α disjoint chains $S_i^- \cup S_i^+$, $1 \leq i \leq \alpha$.

$\Rightarrow m \leq \alpha \stackrel{\wedge}{=} \alpha(X, \leq)$. $\blacksquare$

Dilworth's Thm has a dual version

Mirsky's Theorem. Let $(X, \leq)$ be a poset. If $(X, \leq)$ contains NO chain of $m+1$ elements, then X is the union of m disjoint antichains.

Pf. By the same proof as in Thm 2. $\boxtimes$

(Exercise)