A p-ADIC MONODROMY THEOREM FOR CURVES

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Abstract. We prove that every de Rham p-adic local system on a smooth projective curve over a p-adic field is potentially semistable; that is, it becomes semistable after pulling back along a finite cover of the curve. This establishes a relative version of the classical p-adic monodromy theorem of Berger and André–Kedlaya–Mebkhout. Along the way, we show that every p-adic differential equation near a type Ⅳ point on a curve becomes trivial after a finite étale extension.

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1. Introduction

1.1. Main result.

Let p be a prime number and let K be a p-adic field, that is, a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field. Let $\mathrm{Gal}_K = \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group. The main goal of this article is to establish the following relative p-adic monodromy theorem.[1]

Theorem 1. *Let X be a smooth projective curve over K and let $\mathbb{L}$ be an étale $\mathbb{Z}_p$-local system on X. If $\mathbb{L}$ is de Rham, then $\mathbb{L}$ is potentially semistable. That is, after replacing the base field K by a finite extension, there exists a finite cover $f\colon X' \to X$ such that X' is a smooth projective curve with semistable reduction over $\mathcal{O}_K$ and $f^*\mathbb{L}$ is semistable.*

Before explaining the terminology, let us place our results into context. Consider a smooth projective curve $\overline{S}$ over $\mathbb{C}$ and let $S = \overline{S} - \{t_1, \dots, t_r\}$ be a Zariski open subset. Let $\pi\colon Y \to S$ be a smooth proper map and denote by Y_s the fiber over a point $s \in S$. For a fixed integer $i \geqslant 0$, the singular cohomology groups $H^i(Y_s, \mathbb{C})$ form a local system on S. The "local monodromy theorem" of Borel–Griffiths–Grothendieck–Katz–Landman [Gri68, Gri70, Lan66, SGA72, Kat70] asserts that, if we restrict this local system over a small punctured disk Δ^* around one of the "missing points" t_i and pick a base point $s_0 \in \Delta^*$, then the monodromy action of $\pi_1(\Delta^*, s_0)$ on $H^i(Y_{s_0}, \mathbb{C})$ is *quasi-unipotent*. More precisely, the automorphism T on $H^i(Y_{s_0}, \mathbb{C})$ induced by a generator of $\pi_1(\Delta^*, s_0)$ admits a Jordan decomposition $T = U \cdot D = D \cdot U$, where U is unipotent and D is semisimple of finite order. This result plays a fundamental role in the study of degeneration of Hodge structures (for example, see [Sch73, Ste76, Cle77, CKS86]). Moreover, it implies, rather strikingly, that all the eigenvalues of T are roots of unity.

[1] Theorem 1 is stated in the setting of $\mathbb{Z}_p$-local systems. See Theorem 7 below for a version for $\mathbb{Q}_p$-local systems.

Grothendieck's proof (cf. [SGA72]) in fact implies the following uniform statement. Let X be a smooth proper variety over K, and let $\mathbb{L}$ be an l-adic étale local system on X for a prime l different from p. Then, there exists a finite étale cover $f\colon X' \to X$ such that, for every finite extension K'/K and every $s \in X'(K')$, the inertia group $I_{K'}$ acts unipotently on $f^*\mathbb{L}|_s$. Unipotence of the inertia action on the l-adic cohomology of a variety is closely tied to semistable reduction (see, for example, Grothendieck's semistable reduction criterion for abelian varieties, as well as [RZ82]). This led Fontaine to conjecture a p-adic analogue: a p-adic representation Gal_K is de Rham if and only if it becomes semistable after some finite extension of K (see [Fon94], and also [Col03]). The conjecture of Fontaine was proved by Berger and André–Kedlaya–Mebkhout [Ber02, And02, Ked04, Meb02] and later revisited by Colmez and others (see [Col08, FF18, ABLB+25]).

Extending the p-adic local monodromy theorem to the relative geometric setting has long stood as a major open problem. Our main result provides the first such global generalization. From a more geometric perspective, our result suggests the following question.

Question 2. For a smooth proper map $Y \to X$ of projective varieties over K, up to replacing the base field K by a finite extension, does there exist a projective alteration $X' \to X$, that is, a proper generically finite étale cover with X' projective, such that $Y \times_X X' \to X'$ has (fiber-wise) semistable reduction?

The affirmative answer to this question would provide a mixed characteristic analogue of some well-known conjectures on semistable reductions in algebraic geometry, notably the semistable reduction conjecture of Abramovich–Karu [AK00, Conjecture 0.2] (also see [KKMSD73] and [ALT20]). This mixed characteristic analogue has a positive answer when Y is a family of abelian varieties over X, by considering the relative l-adic Tate module on X with $l \neq p$. Setting X to be a point leads to a natural variant: does a smooth projective variety Y over K always acquire semistable reduction after a finite étale extension? It is a folklore conjecture that this latter question has a positive answer.

1.2. p-adic local systems.

Let us turn to the notions of de Rham and semistable p-adic local systems. Let X be a smooth rigid analytic variety over K, viewed as an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. In this article, by a *p-adic local system* on X, we shall exclusively mean a $\mathbb{Z}_p$-local system, that is, a lisse $\mathbb{Z}_p$-sheaf on the étale site $X_{\text{ét}}$. One may view a p-adic local system on X as a geometric family of p-adic Galois representations parameterized by X. Indeed, for each classical point $x \in X$ with residue field $\kappa(x)$, the restriction of the local system at x gives rise to (a $\mathbb{Z}_p$-lattice inside) a p-adic representation of $\mathrm{Gal}_{\kappa(x)}$. The notion of *de Rham* local systems is defined in [Sch13] (also see [Fal02, Bri08, LZ17]).

Definition 3 ([Sch13, Definition 8.3])**.** A p-adic local system $\mathbb{L}$ on X is called *de Rham* if there exists a filtered vector bundle with integrable connection $(\mathcal{D}, \nabla, \mathrm{Fil}^\bullet)$ on X in the sense of [Sch13, Definition 7.4], together with an isomorphism

$$\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR},X} \cong \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR},X}$$

of pro-étale sheaves on X, compatible with filtrations and connections. Here $\widehat{\mathbb{L}}$ denotes the lisse $\widehat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$ associated with $\mathbb{L}$. The readers are referred to §6 of *op. cit.* for the construction of the de Rham period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR},X}$ as well as the filtration and connection on it.

It is worth pointing out that Scholze's definition [Sch13, Definition 8.3] naturally extends to $\mathbb{Q}_p$-local systems in the sense of de Jong [DJ95]. In particular, one obtains the notion of de Rham $\mathbb{Q}_p$-local systems (as used in [LZ17]; also see Remark 6 and 8). On the other hand, the notion of semistability is only defined for $\mathbb{Z}_p$-local systems in the literature, and one typically requires a semistable integral model of X to start with. This is developed in various generalities in [Fal02, Bri08, Tsu08, AI12, DLMS24, GY24]. Let $\mathfrak{X}$ be a p-adic formal scheme over $\mathcal{O}_K$ with semistable reduction and let X be its adic generic fiber. Let $\mathfrak{X}_s$ denote the special fiber of $\mathfrak{X}$ and let $\mathfrak{X}_s^{\log}$ denote the log scheme $(\mathfrak{X}_s, \alpha_s)$ where the log structure α_s is obtained from the pullback of the divisorial log structure

$$\alpha\colon \mathcal{O}_{\mathfrak{X},\text{ét}} \cap (\mathcal{O}_{\mathfrak{X},\text{ét}}[1/p])^\times \to \mathcal{O}_{\mathfrak{X},\text{ét}}$$

on $\mathfrak{X}$. The following definition is borrowed from [GY24].

Definition 4 ([GY24, Definition 4.4])**.** A $\mathbb{Z}_p$-local system $\mathbb{L}$ on X is called *semistable* if there exists a log F-isocrystal[2] $(\mathcal{E},\varphi)$ over $\mathfrak{X}_s^{\log}$, together with a Frobenius equivariant isomorphism of $\mathbb{B}_{\mathrm{crys}}$-vector bundles

$$\widehat{\mathbb{L}} \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{crys}} \cong \mathbb{B}_{\mathrm{crys}}(\mathcal{E}).$$

Here $\mathbb{B}_{\mathrm{crys}}$ is the horizontal crystalline period sheaf and $\mathbb{B}_{\mathrm{crys}}(\mathcal{E})$ is the sheaf of $\mathbb{B}_{\mathrm{crys}}$-modules on the pro-étale site $X_{\mathrm{proét}}$ associated with $(\mathcal{E},\varphi)$. For the constructions of $\mathbb{B}_{\mathrm{crys}}$ and $\mathbb{B}_{\mathrm{crys}}(\mathcal{E})$, see [GY24, Definition 3.8, Construction 4.1].

Remark 5. One can similarly define the notion of *crystalline* $\mathbb{Z}_p$-local systems. For this, we further require the p-adic formal scheme $\mathfrak{X}$ to be smooth. Then, crystalline local systems are defined in the same way as in Definition 4, simply omitting all "log" therein (see [GY24, Definition 4.4]).

Remark 6 ($\mathbb{Z}_p$-local systems vs $\mathbb{Q}_p$-local systems)**.** For rigid analytic spaces, the category of $\mathbb{Q}_p$-local systems studied by de Jong [DJ95] is, in general, strictly larger than the isogeny category of $\mathbb{Z}_p$-local systems: a $\mathbb{Q}_p$-local system need not admit a global integral lattice. In particular, de Jong constructs an example of a $\mathbb{Q}_p$-local systems of rank 2 on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{Q}_p}$ that is intrinsically analytic, whose geometric monodromy surjects onto $\mathrm{SL}_2(\mathbb{Q}_p)$. Perhaps surprisingly, the proof of our main result extends to these $\mathbb{Q}_p$-local systems, yielding an analogous result even in the absence of a global $\mathbb{Z}_p$-lattice.

Theorem 7 (p-adic monodromy for $\mathbb{Q}_p$-local systems)**.** *Let X be a smooth projective curve over K, and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on X. If $\mathbb{L}$ is de Rham, then, after replacing K by a finite extension if necessary, there exists a finite cover $f\colon X' \to X$ such that $f^*\mathbb{L}$ is* pointwise semistable*, that is, the restriction of $f^*\mathbb{L}$ to* every *classical point on X' is semistable.*

Remark 8 (Rigidity)**.** A remarkable theorem of Liu–Zhu [LZ17] asserts that de Rham $\mathbb{Q}_p$-local systems are *rigid*. That is, if a $\mathbb{Q}_p$-local system on a smooth geometrically connected rigid analytic variety is de Rham at one classical point, then it is already a de Rham local system, and hence is de Rham at every classical point. Although the most naive analogues of this rigidity phenomenon fail for semistable and crystalline local systems, it is natural to ask to what extent analogous rigidity results remain valid in these settings. In the crystalline case, some of the authors (H.D. and Z.Y.) recently established the following rigidity result.

Theorem 9 ([DY25, Theorem 1.1])**.** *Let $\mathfrak{X}$ be a geometrically connected smooth projective scheme over $\mathcal{O}_K$ and let X be its generic fiber. Let $\mathbb{L}$ be a p-adic local system on X. If there exists one classical point x_0 on X such that $\mathbb{L}|_{x_0}$ is potentially crystalline, then $\mathbb{L}$ is potentially crystalline at every classical point on X.*

Let us mention that the proof of Theorem 9 crucially relies on the main result of this article.

1.3. **Related results.**

We make a few more remarks on some relevant works in the literature. There have been several attempts to generalize the p-adic monodromy theorem to the relative setting; for example, see [Shi22, Ked25, Du26][3] and the introductions thereof. Most of these works are of local nature. Notably, Shimizu [Shi22] shows that, given a de Rham local system $\mathbb{L}$ on a smooth rigid analytic variety X over K, every classical point admits an affinoid neighborhood U such that $\mathbb{L}$ becomes semistable at all classical points on $U_{K'}$ for some finite extension K'/K. In a more recent preprint, Du [Du26] shows that there exist a dense open subset $U \subset X$ and a (generally non-finite) étale cover $V \to U$ on which $\mathbb{L}$ becomes semistable at all classical points by studying Newton polygons. In contrast, the main result of this article provides the first general p-adic monodromy theorem in the global setting. It is certainly worth mentioning that, in the global setting, Shimizu [Shi18] proves a similar result similar to ours when the local system $\mathbb{L}$ has a single Hodge–Tate weight (for a general class of analytic varieties).

Let us also mention the work of Ohkubo [Ohk13], Morita [Mor14], and Kedlaya [Ked07]. Ohkubo and Morita extend the classical p-adic monodromy theorem to more general p-adic fields; they treat complete

[2] By *log F-isocrystals*, we mean *finite locally free* isocrystals over the log crystalline site, equipped with a Frobenius structure.

[3] The proof of [Ked25, Theorem 3.2.6] relies on working with a "relative connection" in order to apply the techniques developed in [Ked22] and [Ked25, §1–2]. More precisely, the argument requires working with a relative annulus of the form $A \times_K X'$ over a base X' together with a vector bundle $\mathcal{E}$ equipped with an $\mathcal{O}_{X'}$*-linear* connection. However, one cannot expect such an $\mathcal{O}_{X'}$-linear connection to exist, because the arithmetic and geometric directions are inseparably intertwined. This appears to leave a serious gap in the argument in [Ked25, Theorem 3.2.6].

discretely valued fields with imperfect residue fields. Their results, together with an argument of Guo–Yang (cf. [GY24, §6–7]), provide what we need for the analysis near type Ⅱ points (see §5). On the other hand, Kedlaya establishes a quasi-unipotence result regarding p-adic differential equations over certain "fake Robba rings", which make a surprising appearance in our treatment of type Ⅲ points (see §1.8).

In a different direction, Berger and Colmez [BC08] prove the following result for *arithmetic families* of p-adic Galois representations: consider a reduced affinoid $\mathbb{Q}_p$-algebra S and a finite locally free S-module V equipped with a continuous S-linear Gal_K-action. If for every point x in the maximal spectrum $\mathrm{Spm}(S)$, the fiber V_x is a de Rham Gal_K-representation with Hodge–Tate weights contained in a fixed interval $[a,b]$, then all fibers become semistable over a single finite extension K'/K. It is an interesting problem to combine the approach in this article with tools developed in [BC08] to establish a p-adic monodromy result for S-valued local systems.

1.4. **Relative p-adic Hodge theory.**

Before explaining the proof of Theorem 1, it is instructive to first recall Berger's proof in the classical setting [Ber02]. By the works of Fontaine [Fon90] and Cherbonnier–Colmez [CC98], one can attach to a p-adic Gal_K-representation V a (φ,Γ_K)-module $D_K^\dagger(V)$ (resp. $D_{\mathrm{rig},K}^\dagger(V)$) over the *overconvergent* imperfect period ring $\mathbf{B}_K^\dagger$ (resp. $\mathbf{C}_K^\dagger$[4]). When V is de Rham, Berger develops a systematic procedure to remove the singularities of the differential operator on $D_{\mathrm{rig},K}^\dagger(V)$ induced by the Γ_K-action. This construction yields a p-adic differential equation $(\mathbf{N}_{\mathrm{dR}}(V),\partial_V)$ with a Frobenius structure over the Robba ring $\mathbf{C}_K^\dagger$. By the result of André–Kedlaya–Mebkhout (§3.8), such a p-adic differential equation is necessarily *quasi-unipotent*. Consequently, there exists a finite extension L/K such that the equation admits a full basis of horizontal sections after taking base change to $\mathbf{C}_L^\dagger[\log\pi]$. Berger then shows that, after a further finite extension of the base field, these horizontal sections are contained in the semistable periods of V, thus settling the classical p-adic monodromy theorem. We refer the reader to §3 for more details.

Extending this strategy to the relative setting requires a theory of relative (φ,Γ)-modules where Γ is a p-adic Lie group of the form $\Gamma\cong\Gamma_{\mathrm{geom}}\rtimes\Gamma_K$,[5] together with a relative analogue of the main results of [CC98]. This involves constructing *imperfect relative period rings* $\mathbf{A}_X^\dagger,\mathbf{B}_X^\dagger,\mathbf{C}_X^\dagger$ inside their perfect counterparts and developing a decompletion theory for relative (φ,Γ)-modules.[6] At the time of writing, the only reference in the literature on the required generality is due to Kedlaya–Liu [KL19]. However, to our best knowledge, some of the assertions and proofs in [KL19, in particular, §5] concerning (weakly) decompleting towers appear to require further justification. In particular, the general descent of relative (φ,Γ)-modules claimed in [KL19] remains difficult to verify, as the proposed argument seems to omit several essential details.

In order to have a complete and coherent theory to work with, we develop the required foundations of relative p-adic Hodge theory in a separate article [DMY26], where we systematically study imperfect relative period rings and (φ,Γ)-modules over them. The treatment in [DMY26] is inspired by [KL19] but differs substantially from *loc. cit.*, especially in how imperfect period rings are treated. A central notion introduced in [DMY26] is that of *decompleting data*, which is engineered to capture the idea of decompleting towers from [KL19, Definition 5.2.3]. This allows us to establish a relative analogue of the Fontaine–Cherbonnier–Colmez equivalence.

Theorem 10 ([DMY26, Theorem 1])**.** *Let $\mathbf{B}_\Lambda^\dagger$ and $\mathbf{C}_\Lambda^\dagger$ be the imperfect relative period rings for X associated to a decompleting data Λ. Then there are natural equivalences*

$$D^\dagger\colon \mathrm{Loc}_{\mathbb{Z}_p}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\mathbf{B}_\Lambda^\dagger} \qquad D_{\mathrm{rig}}^\dagger\colon \mathrm{Loc}_{\mathbb{Z}_p}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\mathbf{C}_\Lambda^\dagger}$$

from the isogeny category of étale $\mathbb{Z}_p$-local systems on X to the category of finite projective étale (φ,Γ)-modules over $\mathbf{B}_\Lambda^\dagger$ (resp. $\mathbf{C}_\Lambda^\dagger$).

[4]The period ring $\mathbf{B}_{\mathrm{rig},K}^\dagger$ from Berger's article [Ber02] will be denoted as $\mathbf{C}_K^\dagger$ in this article. Also see Appendix A.

[5]More precisely, we work with an affinoid rigid analytic variety X over K, equipped with a toric chart to the closed unit disk (see §4.1 for the setup). In this case, the geometric part Γ_{geom} of Γ is isomorphic to $\mathbb{Z}_p^d$ with $d=\dim X$.

[6]The corresponding *perfect relative period rings* are constructed by Kedlaya and Liu [KL15] using the theory of perfectoid spaces.

To generalize the strategy of Berger to the relative setting, another missing ingredient is a relative analogue of p-adic differential equations. With this objective in mind, we introduce the notion of *strongly decompleting data*, which axiomatizes the process of "removing singularities" in [Ber02]. This eventually allows us to construct *relative p-adic differential equations*.

Theorem 11 ([DMY26, Theorem 2], crude form)**.** *Assume that Λ is in addition strongly decompleting, and let $\mathbf{C}^\dagger_\Lambda$ be the associated imperfect relative period ring. Then there is a natural functor*

$$\mathbf{N}_{\mathrm{dR}} \colon \mathrm{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Mod}^{(\varphi,\Gamma)}_{/\mathbf{C}^\dagger_\Lambda}$$

from the isogeny category of de Rham $\mathbb{Z}_p$-local systems to the category of finite projective (φ,Γ)-modules over $\mathbf{C}^\dagger_\Lambda$, where $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ carries not only the arithmetic differential operator arising from the cyclotomic direction, but also geometric differential operators describing the variation of the local system $\mathbb{L}$ over X.

For the reader's convenience, we review the notions of decompleting and strongly decompleting data in §4, as well as some explicit examples (see §4.7). These examples turn out to be sufficient for our application towards the p-adic monodromy theorem.

1.5. Reduction of Theorem 1 to a local statement.

Finally, we explain the proof of our main result, which relies on careful analysis of (various forms of) p-adic differential equations, together with several delicate arguments in p-adic geometry. A key step is a spread-out argument (Theorem 9.2), which allows us to reduce Theorem 1 to the following local statement.

Theorem 12. *Let X be a smooth rigid analytic curve over K, viewed as an adic space over $\mathrm{Spa}(K,\mathcal{O}_K)$. Let $\mathbb{L}$ be a de Rham local system on X. Let $\xi \in X_{\mathbb{C}_K}$ be* any *adic point. Then, there exists a finite extension L/K, an open neighborhood U of the image of ξ in X_L, and a finite étale cover $f\colon V \to U$ defined over L, such that the pullback $f^*(\mathbb{L}|_U)$ is pointwise semistable (i.e., semistable at every classical point in V).*

Granting Theorem 12, we complete the proof of Theorem 1 as follows. By compactness, X admits a cover by finitely many U's as in Theorem 12.[7] Our task is to "glue" these local pieces into a global cover of X. To achieve this, we spread-out each finite étale cover $V \to U$ to a global finite cover over X and take the fiber product of all of them to form a single finite cover over X. This spread-out step is obtained using certain p-adic approximation techniques and is closely related to a result of Raynaud [Ray94], often viewed as the p-adic analogue of Runge's theorem on holomorphic functions on complex Riemann surfaces. This allows us to obtain a global cover $f\colon X' \to X$, after enlarging K if necessary, such that $f^*\mathbb{L}$ is pointwise semistable. Finally, we appeal to the following pointwise criterion of Guo–Yang [GY24] to conclude.

Theorem 13 ([GY24, Theorem 1.1])**.** *Let $\mathfrak{X}$ be a p-adic formal scheme over $\mathcal{O}_K$ with semistable reduction, and let X be its adic generic fiber. A p-adic local system $\mathbb{L}$ on X is semistable if and only if its restriction to* every *classical point on X is semistable.*

For Theorem 12, our strategy is to perform a case-by-case study based on the type of the point ξ.[8] As mentioned earlier, the case of type I points is settled by [Shi22]; while the case of type II points follows from [Ohk13, Mor14] by an argument similar to that of [GY24, §6–7]. The remaining cases are points of type III, IV, and V. For each of these types, we will need a different argument, which we summarize in the remainder of this introduction.

1.6. Type III points.

By enlarging the base field K if necessary, a type III point $\xi \in X$ admits a neighborhood isomorphic to a closed annulus inside the standard unit disk, where ξ corresponds to a circle of radius $\sigma < 1$ with $\sigma \notin p^{\mathbb{Q}}$. In this situation, a system of open neighborhoods of ξ can be chosen to consist of closed annuli $\{\sigma_1 \leqslant |T| \leqslant \sigma_2\}$ with $\sigma_1 < \sigma < \sigma_2$ and $\sigma_1, \sigma_2 \in p^{\mathbb{Q}}$. Our strategy is to consider the imperfect relative period rings

[7] Let us emphasize that it is crucial to consider *all* adic points on X. In particular, it is not enough to only consider *classical points* arising from algebraic geometry (which are certain type I points on the curve), because a union of affinoid neighborhoods taken over all classical points need not cover the entire X.

[8] See §2 for a review of the classification of points of type I–V on adic curves.

$\mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]}$ associated with these annuli and study the p-adic differential equations $\mathbf{N}_{\mathrm{dR},[\sigma_1,\sigma_2]}$ over $\mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]}$ (cf. Theorem 11). Thanks to the explicit description of $\mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]}$ in [DMY26, §11.1], a suitable completed colimit of these period rings as $\sigma_1, \sigma_2 \to \sigma$ is related to the *fake Robba rings* studied by Kedlaya in [Ked07, Ked09]. Using the analogue of the André–Kedlaya–Mebkhout quasi-unipotence theorem proved in *loc. cit.* for these fake Robba rings, we can achieve unipotence of the differential operators "at the point ξ" after pulling back to a finite étale cover of a sufficiently "thin" closed annulus. The idea is then to propagate this unipotence from the point ξ to nearby points. This propagation step relies crucially on the explicit descriptions of relative period rings associated with closed annuli. In particular, via a *monomial change of coordinates*, we show that $\mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]}$ behaves like the ring of analytic functions on a product of two closed annuli, after passing to suitable colimits.[9] Given this, we can apply the multivariable p-adic Fuchs theorem and conclude the desired propagation of unipotence. Finally, as in the classical setting (cf. [Ber02]; also see §3.8), the unipotence of the differential operators allows us to produce enough semistable periods in a neighborhood of ξ after passing to a finite étale cover.

1.7. **Type Ⅳ points.**

Type Ⅳ points are perhaps the most mysterious points on curves. They correspond to nested sequences of closed disks $\mathbb{D}_1 \supset \mathbb{D}_2 \supset \cdots$ with empty intersection, that is, $\cap_{i\geqslant 0}\mathbb{D}_i$ contains no type I point. As the disks do not share the same center, it is more difficult to analyze the relative p-adic differential equations $\mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{\mathbb{D}_i})$ in a compatible way. The idea is instead to prove the following geometric result on vector bundles with connection near a type Ⅳ point.

Theorem 14 (Theorem 7.2, crude form)**.** *Let $\xi \in X$ be a type Ⅳ point and let (M, ∇) be a vector bundle with connection on a neighborhood of ξ. Then after shrinking the neighborhood and passing to a finite étale cover, M becomes a trivial connection near ξ.*

In other words, Theorem 14 says that all p-adic differential equations near a type Ⅳ point can be solved locally after passing to a finite étale cover. A key observation is that, if a vector bundle with connection near a type Ⅳ point with trivial determinant is irreducible of rank n, then either $p|n$ or $n = 1$ (cf. Proposition 7.5). Consequently, we deduce that if a connection near a type Ⅳ point remains irreducible after passing to all possible finite étale covers, then it must be trivial of rank 1. This eventually implies Theorem 14. Applying this result directly to $D_{\mathrm{dR}}(\mathbb{L})$, we deduce that $\mathbb{L}$ is *horizontal de Rham* in the sense of [Shi22], after pulling back along a finite étale cover. Then, the main result of *loc. cit.* yields the following (stronger) local p-adic monodromy theorem around type Ⅳ points.

Theorem 15 (Theorem 7.1, crude form)**.** *Let X and $\mathbb{L}$ be as in Theorem 1 and let $\xi \in X$ be a type Ⅳ point. Then, up to replacing K by a finite extension, there exists an affinoid neighborhood U of ξ in X, and a finite étale cover $g\colon V \to U$, such that the pullback $g^*(\mathbb{L}|_U)$ is horizontal semistable in the sense of* [Shi22].

1.8. **Type Ⅴ points.**

The only remaining case is that of type Ⅴ points. Let ξ be a type Ⅴ point and let $\xi_{\mathrm{Ⅱ}}$ denote the associated type Ⅱ point. By the discussion above, we may assume that $\mathbb{L}$ is already semistable in an open neighborhood of $\xi_{\mathrm{Ⅱ}}$ (but this neighborhood might not contain ξ). With some careful analysis on integral models (see §9.3), we reduce to the special situation where ξ admits an open neighborhood U isomorphic to a closed annulus $\{\sigma \leqslant |T| \leqslant 1\}$ with finitely many open unit disks removed from the boundary circle $\{|T| = 1\}$, such that $\xi_{\mathrm{Ⅱ}}$ is the Shilov point corresponding to the boundary circle and ξ is the type Ⅴ point corresponding to the open unit disk centered at 0. We refer to such U as a "Swiss-cheese neighborhood" of ξ. In fact, these Swiss-cheese neighborhoods form a basis of open neighborhoods of ξ.

Our approach is to study the p-adic differential equations $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ over these neighborhoods. Similar to the case of type Ⅲ points, we want to achieve unipotence of the differential operators after passing to finite étale covers. Since $\mathbb{L}$ is assumed to be semistable at the type Ⅱ point $\xi_{\mathrm{Ⅱ}}$, we already have unipotence there. The idea is then to propagate this unipotence to a Swiss-cheese neighborhood. This time, however, only

[9] In fact, we further show that this behavior persists when the annulus is replaced by a certain class of finite étale covers, which suffices for our applications.

information about the differential operator in the arithmetic direction can be extracted from $\xi_{\mathbb{II}}$. Therefore, we deal with the arithmetic direction first, before turning to the geometric one (this is one essential difference between the case of type $\mathbb{III}$ points and that of type V points). More precisely, we show that $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ is arithmetically unipotent on a small enough Swiss-cheese neighborhood U. Then, we make full use of the explicit description of the relative period rings over such neighborhoods from [DMY26, §11.2] to show that, after further enlarging K and shrinking U, the arithmetic horizontal sections of $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})[\log \pi_K]$ looks like a p-adic differential equation with Frobenius structure over an annulus. Finally, we apply the classical André–Kedlaya–Mebkhout quasi-unipotence theorem to conclude.

1.9. **Notation and convention.**

Throughout this article, we follow the conventions on adic geometry in [Hub96] and [Sch12], and we adopt the following notation unless otherwise specified.

- Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field $k = k_K$. Let ϖ be a uniformizer of K. Let $K_0 = W(k)[1/p]$. Let $\mathrm{Gal}_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K.
- Fix an algebraic closure $\overline{K}$ of K and let $\mathbb{C}_K$ be its completion. Note that $\mathbb{C}_K$ is a perfectoid field; let $\mathbb{C}_K^\flat$ denote its tilt. Let $\mathcal{O}_{\mathbb{C}_K}$ (resp. $\mathcal{O}_{\mathbb{C}_K}^\flat$) denote the ring of integers in $\mathbb{C}_K$ (resp. $\mathbb{C}_K^\flat$).
- Fix a compatible system of p-power roots $\{\zeta_{p^n}\}_{n\geqslant 0}$ of unity and a compatible system of p-power roots $\{p^{1/p^n}\}_{n\geqslant 0}$ of p in $\mathcal{O}_{\mathbb{C}_K}$. In particular, this gives rise to elements $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ and $p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots)$ in $\mathcal{O}_{\mathbb{C}_K}^\flat$. Let $\overline{\pi} = \varepsilon - 1$, and let $\pi = [\varepsilon] - 1$ viewed as an element in $W(\mathcal{O}_{\mathbb{C}_K}^\flat)$ lifting $\overline{\pi}$. Let $t = \log[\varepsilon] = \log(\pi + 1)$ be "Fontaine's period" in the de Rham period ring $\mathbf{B}_{\mathrm{dR}}$.
- For each $n \geqslant 0$, let $K_n = K(\zeta_{p^n})$ and let $K_\infty = \bigcup_{n\geqslant 0} K_n$ be the cyclotomic extension. Let $\widehat{K}_\infty$ be the p-adic completion of K_∞ and let $\widehat{K}_\infty^\flat$ be its tilt. Let $\mathcal{O}_{\widehat{K}_\infty}$ (resp. $\mathcal{O}_{\widehat{K}_\infty}^\flat$) denote the ring of integers in $\widehat{K}_\infty$ (resp. $\widehat{K}_\infty^\flat$).
- Let K_0' denote the maximal unramified extension of K_0 inside K_∞. Let $\kappa = \kappa_K$ denote the residue field of K_0'.
- Let $e_K \coloneqq [K_\infty : (K_0')_\infty]$ and let $c_K \coloneqq \frac{p}{p-1} \cdot \frac{1}{e_K}$.
- The standard norm $|\cdot|$ on $\mathbb{C}_K$, normalized by $|p| = \frac{1}{p}$, induces a norm (still denoted by $|\cdot|$) on $\mathbb{C}_K^\flat$. In particular, $|\cdot|$ restricts to a norm on $\widehat{K}_\infty^\flat$. Note that $|\overline{\pi}| = |\varepsilon - 1| = p^{-\frac{p}{p-1}}$.
- Let $v\colon \mathbb{C}_K \to \mathbb{R}\cup\{+\infty\}$ (resp. $v\colon \mathbb{C}_K^\flat \to \mathbb{R}\cup\{+\infty\}$) denote the valuation given by $v(y) \coloneqq -\log_p |y|$. We have $v(\overline{\pi}) = v(\varepsilon - 1) = \frac{p}{p-1}$.[10]
- Let $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ be the Galois group of K_∞, which equals the kernel of the cyclotomic character $\chi\colon \mathrm{Gal}_K \to \mathbb{Z}_p^\times$.
- Let $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Note that $\Gamma_K \cong \mathrm{Gal}_K / H_K$.
- Let $\mathbb{D}_K(\alpha, \sigma)$ (resp., $\mathbb{D}_K(\alpha, < \sigma)$) denote the closed disk (resp., open disk) defined over K centered at α with radius σ. (See Definition 2.5 for precise definitions.)
- For a complete non-archimedean field L and an (open, closed, or half-open) interval $I \subset \mathbb{R}_{>0}$ with end-points in $p^{\mathbb{Q}}$, we denote by $A_L(I)$ the annulus over L (viewed as an adic space) centered at 0 with radius $|T| \in I$. When $I = [\sigma, \sigma]$ consists of a singleton, the annulus $A_L[\sigma, \sigma]$ is referred to as a *circle*. We write $\Gamma_L(I) = \mathcal{O}(A_L(I))$ for the ring of analytic functions on $A_L(I)$. Sometimes we emphasize the coordinate T for the annulus by writing $A_L(I; T)$ and $\Gamma_L(I; T)$ instead of $A_L(I)$ and $\Gamma_L(I)$.
- Throughout the article, for all the period rings $\widetilde{\mathbf{A}}_K^r, \widetilde{\mathbf{B}}_K^r, \widetilde{\mathbf{C}}_K^r, \widetilde{\mathbf{C}}_K^{[s,r]}$ and $\mathbf{A}_K^r, \mathbf{B}_K^r, \mathbf{C}_K^r, \mathbf{C}_K^{[s,r]}$, etc., as well as their relative counterparts, we always assume $r, s \in \mathbb{Q}$ in the superscripts.
- Moreover, we adopt the following notation for rings arising in the theory of Robba rings. Suppose A is a K_0'-algebra equipped with a norm $|\cdot|$ extending the p-adic norm on K_0', and let A° be the

[10] In [Ber02] and [Col08], the valuation v is denoted by $v_{\mathbf{E}}$, the ring $\mathcal{O}_{\mathbb{C}_K}^\flat$ (resp. $\mathbb{C}_K^\flat$) is denoted by $\widetilde{\mathbf{E}}^+$ (resp. $\widetilde{\mathbf{E}}$), and the ring $\mathcal{O}_{\widehat{K}_\infty}^\flat$ (resp. $\widehat{K}_\infty^\flat$) is denoted by $\widetilde{\mathbf{E}}_K^+$ (resp. $\widetilde{\mathbf{E}}_K$).

subring of power-bounded elements. For any $r > 0$, let

$$\mathscr{R}_A^{\mathrm{int},r} := \left\{ \sum_{i\in\mathbb{Z}} a_i x^i \;\middle|\; a_i \in A^\circ,\ |a_i| \cdot p^{-ir\cdot c_K} \to 0 \text{ as } i \to -\infty \right\}$$

and let $\mathscr{R}_A^{\mathrm{int}} := \varinjlim_{r\to 0^+} \mathscr{R}_A^{\mathrm{int},r}$. For any $0 < s \leqslant r$, let

$$\mathscr{R}_A^{[s,r]} := \left\{ \sum_{i\in\mathbb{Z}} a_i x^i \;\middle|\; a_i \in A,\ |a_i| \cdot p^{-it\cdot c_K} \to 0 \text{ as } i \to \pm\infty,\ \text{for all } t \in [s,r] \right\}.$$

For $r > 0$, we write $\mathscr{R}_A^r := \varprojlim_{s\to 0^+} \mathscr{R}_A^{[s,r]}$ and $\mathscr{R}_A := \varinjlim_{r\to 0^+} \mathscr{R}_A^r$. We refer to $\mathscr{R}_A^{\mathrm{int}}$ (resp. $\mathscr{R}_A$) as the *integral Robba ring* over A (resp. *Robba ring* over A). In practice, the variable x will mostly be taken to be π_K, except in §8.5 where we will take the variable to be $[T^\flat]$. Note that when $A = K_0'$ and $r, s \in \mathbb{Q}$, we have $\mathscr{R}_{K_0'}^{[s,r]} = \Gamma_{K_0'}([p^{-r\cdot c_K}, p^{-s\cdot c_K}]; x)$ and $\mathscr{R}_{K_0'}^r = \Gamma_{K_0'}([p^{-r\cdot c_K}, 1); x)$.

Acknowledgement. The authors would like to thank Tomoyuki Abe, Tinhinane Azzouz, Alexander Beilinson, Dori Bejleri, Miaofen Chen, Heng Du, Matthew Emerton, Lei Fu, Haoyang Guo, Mark Kisin, Shizhang Li, Kevin Lin, Alexander Petrov, Lie Qian, Peter Scholze, Xu Shen, Koji Shimizu, Arnaud Vanhaecke, Junyi Xie, and Ziquan Yang for helpful discussions during the preparation of this article, during which, the proof of our main result has gone through several iterations. In an earlier draft of this manuscript, the authors claimed a result for annuli that was too strong to be correct. We thank Peter Scholze for pointing this out. We are grateful to Junyi Xie for discussions around Proposition 9.3 and also on type Ⅳ points on Berkovich curves. We thank Tinhinane Azzouz for discussions around Corollary 7.4. During the project, H.D. was partially supported by the National Natural Science Foundation of China (Grant No. 12422101) and the National Key R&D Program of China (Grant Nos. 2023YFA1009703 and 2021YFA1000704). Y.S.M. was partially supported by the Beijing Natural Science Foundation (Grant No. IS25030). Finally, Z.Y. wishes to thank Hausdorff Institute of Mathematics and Yau Mathematical Sciences Center at Tsinghua University for hospitality during his stays while completing this project.

2. Preliminaries on p-adic geometry

In this section, we recall some basic results from p-adic geometry that will be used throughout the article, including a classification of points on a smooth rigid analytic curve (regarded as an adic space). We adopt the notation introduced in §1.9. Although this article only concerns the case where K is a p-adic field, all the results in §2 remain valid for an arbitrary discretely valued non-archimedean field equipped with a nontrivial valuation.

2.1. Classification of points on closed unit disks.

We start with the classification of points on a closed unit disk over $\mathbb{C}_K$. We refer the reader to [Wed19, Example 7.57] (also see [Sch12, Example 2.20]) for a detailed exposition on this subject. Consider the closed unit disk centered at 0

$$\mathbb{D}_{\mathbb{C}_K} := \mathrm{Spa}\,(\mathbb{C}_K\langle T\rangle, \mathcal{O}_{\mathbb{C}_K}\langle T\rangle) = \mathbb{D}_{\mathbb{C}_K}(0,1)$$

over $\mathbb{C}_K$, with coordinate T. For any $\alpha \in \mathcal{O}_{\mathbb{C}_K}$ and any $0 < \sigma \leqslant 1$, let $\mathbb{D}_{\mathbb{C}_K}(\alpha,\sigma) = \{|T-\alpha| \leqslant \sigma\} \subset \mathbb{D}_{\mathbb{C}_K}$ denote the closed disk of radius r centered at α. Similarly, let $\mathbb{D}_{\mathbb{C}_K}(\alpha, <\sigma) = \{|T-\alpha| < \sigma\}$ denote the open disk of radius σ. The points on $\mathbb{D}_{\mathbb{C}_K}$ fall into the following five types.

Type I. Each $\alpha \in \mathcal{O}_{\mathbb{C}_K}$ gives rise to a continuous valuation $x_\alpha \colon \mathbb{C}_K\langle T\rangle \to \mathbb{R}_{\geqslant 0}$ sending $f \mapsto |f(\alpha)|$. Such a valuation is referred to as a *type I* point. There is a one-to-one correspondence between $\mathcal{O}_{\mathbb{C}_K}$ and the type I points on $\mathbb{D}_{\mathbb{C}_K}$.

Type Ⅱ and Ⅲ. For each closed disk $\mathbb{D}_{\mathbb{C}_K}(\alpha,\sigma)$ with $\alpha \in \mathcal{O}_{\mathbb{C}_K}$ and $0 < \sigma \leqslant 1$, consider the continuous valuation

$$x_{\alpha,\sigma} \colon \mathbb{C}_K\langle T\rangle \to \mathbb{R}_{\geqslant 0}, \quad f \mapsto \sup_{\substack{\beta\in\mathcal{O}_{\mathbb{C}_K} \\ |\beta-\alpha|\leqslant\sigma}} |f(\beta)|.$$

Such a point $x_{\alpha,\sigma}$ is *type II* if $\sigma \in |\mathbb{C}_K^\times| = p^{\mathbb{Q}}$; otherwise $x_{\alpha,\sigma}$ is *type III*. In the case of type II, the point $x_{\alpha,\sigma}$ is called the *Gaussian point* of the disk $\mathbb{D}_{\mathbb{C}_K}(\alpha,\sigma)$.

Type IV. Consider a sequence η of closed disks $\mathbb{D}_1 \supset \mathbb{D}_2 \supset \cdots$ in $\mathbb{D}_{\mathbb{C}_K}$ such that their intersection $\cap_{i\in\mathbb{Z}_{>0}}\mathbb{D}_i$ contains no type I point. Then η gives rise to a continuous valuation

$$x_\eta \colon \mathbb{C}_K\langle T\rangle \to \mathbb{R}_{\geqslant 0}, \quad f \mapsto \inf_{i\in\mathbb{Z}_{>0}} \sup_{\substack{\beta\in\mathcal{O}_{\mathbb{C}_K}\\ \beta\in\mathbb{D}_i}} |f(\beta)|.$$

Such a point is called *type IV*. We often write $x_\eta =$ "$\cap_{i\in\mathbb{Z}_{>0}}\mathbb{D}_i$" and view the intersection of these nested disks as the type IV point itself.

Type V. These are the points of rank-2 valuations (and thus do not appear in the theory of Berkovich spaces). Let us start with the type II point $x_{\alpha,r}$ corresponding to the closed disk $\mathbb{D}_{\mathbb{C}_K}(\alpha,\sigma)$. For a sign $? \in \{>,<\}$, consider the totally ordered abelian group

$$\Gamma_{?\sigma} = \begin{cases} \Gamma_{<\sigma} = \mathbb{R}_{>0}\times\gamma^{\mathbb{Z}} & \text{where } \gamma = \sigma^-, \text{ i.e., } \sigma' < \gamma < \sigma \text{ for all } \sigma' < \sigma \\ \Gamma_{>\sigma} = \mathbb{R}_{>0}\times\gamma^{\mathbb{Z}} & \text{where } \gamma = \sigma^+, \text{ i.e., } \sigma < \gamma < \sigma' \text{ for all } \sigma' > \sigma \end{cases}.$$

(When $\sigma = 1$, we only allow $? =$ "$<$".) Let $\beta \in \mathcal{O}_{\mathbb{C}_K}$ such that $|\beta - \alpha| \leqslant \sigma$. In particular, β corresponds to a type I point x_β in $\mathbb{D}_{\mathbb{C}_K}(\alpha,\sigma)$. Notice that every element $f \in \mathbb{C}_K\langle T\rangle$ can be uniquely written as $f = \sum_{i=0}^\infty a_i(T-\beta)^i$ with $a_i \in \mathbb{C}_K$. Consider the continuous rank-2 valuation $\mathbb{C}_K\langle T\rangle \to \Gamma_{?\sigma}\cup\{0\}$ sending

$$f = \sum_{i=0}^\infty a_i(T-\beta)^i \longmapsto \sup_i |a_i|\gamma^i.$$

Such a valuation is referred to as a *type V* point.[11] When $?$ is "$>$", the valuation does not depend on the choice of x_β; it only depends on the disk $\mathbb{D}_{\mathbb{C}_K}(\alpha,\sigma)$. We denote this type V point by $x_{\alpha,>\sigma}$. On the other hand, when $?$ is "$<$", we denote the corresponding point by $x_{\beta,<\sigma}$. This point $x_{\beta,<\sigma}$ only depends on the open disk $\mathbb{D}_{\mathbb{C}_K}(\beta,<\sigma) \subset \mathbb{D}_{\mathbb{C}_K}(\alpha,r)$; it does not depend on the specific choice of x_β in this open disk.

Remark 2.1. Points of type I, III, IV, and V are closed. The closure of a type II point $x_{\alpha,\sigma}$ precisely consists of $x_{\alpha,\sigma}$ and the type V points $x_{\beta,<\sigma}$ and $x_{\alpha,>\sigma}$ described above; this closure is homeomorphic to $\mathbb{P}^1_{\tilde{k}}$ when $0 < \sigma < 1$ (resp., $\mathbb{A}^1_{\tilde{k}}$ when $\sigma = 1$), with $x_{\alpha,\sigma}$ being the generic point.

Next, we describe points on the closed unit disk (centered at 0)

$$\mathbb{D}_K := \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) = \mathbb{D}_K(0,1)$$

defined over K. The following lemma and corollary allow us to identify $\mathbb{D}_K$ as a quotient of $\mathbb{D}_{\mathbb{C}_K}$.

Lemma 2.2. Let A be a Banach K-algebra. Let L/K be a finite normal extension and consider the Banach L-algebra $A_L := A\otimes_K L$. Let A° and A_L° be the subring of power-bounded elements in A and A_L, respectively. Let $\pi\colon \mathrm{Spa}(A_L, A_L^\circ) \to \mathrm{Spa}(A, A^\circ)$ be the natural map (of sets) induced by the inclusion $A \to A_L$ sending $a \mapsto a\otimes 1$.

(1) If L/K is separable, there is a natural action of $\mathrm{Gal}(L/K)$ on $\mathrm{Spa}(A_L, A_L^\circ)$ and π induces a homeomorphism $\mathrm{Spa}(A_L, A_L^\circ)/\mathrm{Gal}(L/K) \cong \mathrm{Spa}(A, A^\circ)$, where the left-hand side is equipped with the quotient topology.
(2) If L/K is purely inseparable, π is a homeomorphism; namely, $\mathrm{Spa}(A_L, A_L^\circ) \cong \mathrm{Spa}(A, A^\circ)$.

Proof. These are likely well-known to experts. For completeness, we include a proof (also see [Ber90, Proposition 1.3.5] for a proof in the setup of Berkovich spectra). For part (1), the natural action of $\mathrm{Gal}(L/K)$ on A_L preserves A_L° and hence induces an action on $\mathrm{Spa}(A_L, A_L^\circ)$. Notice that

$$\left(A_L^{\mathrm{Gal}(L/K)}, (A_L^\circ)^{\mathrm{Gal}(L/K)}\right) = (A, A^\circ).$$

The desired result follows from [Han16, Theorem 1.2]. For part (2), we construct an inverse of π. By assumption, we must have $\mathrm{char}(K) = p > 0$ for some prime number p, and there exists an integer $m \geqslant 1$ such that $x^{p^m} \in A$ for all $x \in A_L$. Let $|\cdot|_A \in \mathrm{Spa}(A, A^\circ)$; namely, $|\cdot|_A\colon A \to \Gamma\cup\{0\}$ is a continuous

[11] We have excluded the case $\sigma = 1$ and $? =$ "$>$". In this case, the valuation does not live in $\mathbb{D}_{\mathbb{C}_K}$, but rather lives in the closure of $\mathbb{D}_{\mathbb{C}_K}$ in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_K}$.

valuation such that $|x|_A \leqslant 1$ for all $x \in A^\circ$. By enlarging Γ if necessary, we may assume that $\Gamma \cup \{0\}$ contains $\sqrt[p^m]{|A|_A}$. Consider a continuous valuation $|\cdot|_{A_L} : A_L \to \Gamma \cup \{0\}$ given by $|x|_{A_L} := (|x^{p^m}|_A)^{\frac{1}{p^m}}$. Then $|\cdot|_{A_L}$ is a continuous valuation on A_L satisfying $|x|_{A_L} \leqslant 1$ for all $x \in A_L^\circ$. Using the multiplicativity of valuations, one sees that $|\cdot|_A \mapsto |\cdot|_{A_L}$ is indeed the inverse of π. □

Corollary 2.3. Let A be a Banach K-algebra and let $A_{\mathbb{C}_K}$ be the completed tensor $A\widehat{\otimes}_K\mathbb{C}_K$. Let A° and $A^\circ_{\mathbb{C}_K}$ be the subring of power-bounded elements in A and $A_{\mathbb{C}_K}$, respectively. Then the natural map $\pi\colon \operatorname{Spa}(A_{\mathbb{C}_K}, A^\circ_{\mathbb{C}_K}) \to \operatorname{Spa}(A, A^\circ)$ induces a homeomorphism

$$\operatorname{Spa}(A_{\mathbb{C}_K}, A^\circ_{\mathbb{C}_K})/\operatorname{Gal}_K \cong \operatorname{Spa}(A, A^\circ).$$

Proof. Note that the natural map

$$\operatorname{Spa}(A_{\mathbb{C}_K}, A^\circ_{\mathbb{C}_K}) \to \varprojlim_{L/K} \operatorname{Spa}(A_L, A_L^\circ)$$

is a homeomorphism, where the limit runs through all finite normal extensions L/K inside $\overline{K}$. Then the desired statement follows from the previous lemma. □

Applying Corollary 2.3 to our situation, the natural map $\pi\colon \mathbb{D}_{\mathbb{C}_K} \to \mathbb{D}_K$ is surjective and induces a homeomorphism $\mathbb{D}_{\mathbb{C}_K}/\operatorname{Gal}_K \cong \mathbb{D}_K$; in particular, the points on $\mathbb{D}_K$ can be identified as the Gal_K-orbits of the points on $\mathbb{D}_{\mathbb{C}_K}$.

Definition 2.4. For $\star \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}\}$, we say that a point on $\mathbb{D}_K$ has *type* $\star$ if it is the image of a point on $\mathbb{D}_{\mathbb{C}_K}$ of type $\star$.

For later use, we also introduce the following notation for closed/open disks over K.

Definition 2.5. For $\alpha \in \mathcal{O}_{\mathbb{C}_K}$ and $0 < \sigma \leqslant 1$, let $\mathbb{D}_K(\alpha, \sigma)$ (resp. $\mathbb{D}_K(\alpha, < \sigma)$) denote the image of $\mathbb{D}_{\mathbb{C}_K}(\alpha, \sigma)$ (resp. $\mathbb{D}_{\mathbb{C}_K}(\alpha, < \sigma)$) under the projection $\pi\colon \mathbb{D}_{\mathbb{C}_K} \to \mathbb{D}_K$. We still refer to them as *closed* (resp. *open*) *disks* of radius σ centered at α.

We warn the reader that, even when $\sigma \in |K^\times|$, the disk $\mathbb{D}_K(\alpha, \sigma)$ is not necessarily isomorphic to $\mathbb{D}_K(0, \sigma)$ as rigid analytic spaces over K; in fact, $\mathbb{D}_K(\alpha, \sigma) \cong \mathbb{D}_K(0, \sigma)$ if and only if $\mathbb{D}_{\mathbb{C}_K}(\alpha, \sigma)$ contains some x_β with $\beta \in \mathcal{O}_K$.

2.2. **Points on smooth curves.**

Now, we describe the points on a general smooth curve. Let Y be a smooth rigid analytic curve over K, viewed as an adic space over $\operatorname{Spa}(K, \mathcal{O}_K)$. By Corollary 2.3, there is a natural surjection $\pi\colon Y_{\mathbb{C}_K} \to Y$ and the points on Y can be identified as the Gal_K-orbits of the points on $Y_{\mathbb{C}_K}$.

Definition 2.6. For $\star \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}\}$, we say that a point $x \in Y_{\mathbb{C}_K}$ has *type* $\star$ if for an/any étale map $f\colon U \to \mathbb{D}_{\mathbb{C}_K}$ from an open neighborhood U of x in $Y_{\mathbb{C}_K}$, the image $f(x) \in \mathbb{D}_{\mathbb{C}_K}$ has type $\star$. We say that a point on Y has *type* $\star$ if it is the image of a point on $Y_{\mathbb{C}_K}$ of type $\star$ under the surjection $\pi\colon Y_{\mathbb{C}_K} \to Y$.

Note that an étale morphism between smooth rigid analytic curves over K sends a point to a point of the same type.

Definition 2.7 (K-rational points)**.** Let Y be a smooth rigid analytic curve over K, viewed as an adic space over $\operatorname{Spa}(K, \mathcal{O}_K)$. Recall that a point on Y corresponds to a Gal_K-orbit of points on $Y_{\mathbb{C}_K}$.

(1) We say that a point on $Y_{\mathbb{C}_K}$ is *K-rational* if its Gal_K-orbit consists of a single element.
(2) We say that a point on Y is *K-rational* if it is the image of a K-rational point on $Y_{\mathbb{C}_K}$. More generally, we say that a point on Y is *K'-rational* for some finite extension K'/K if it is the image of a K'-rational point on $Y_{\mathbb{C}_K}$.
(3) We say that a type I point on Y is *classical* if it is K'-rational for some finite extension K'/K. Classical points are precisely those points on Y that belong to the underlying rigid analytic space.

Remark 2.8. If x is a K-rational point on Y, and K'/K is a finite extension, then the pre-image of x under the projection $Y_{K'} \to Y$ consists of a single element; thus, it can also be viewed as a K'-rational point on $Y_{K'}$.

Example 2.9. Consider the closed unit disk $\mathbb{D}_K = \mathbb{D}_K(0,1)$ over K. A type I point on $\mathbb{D}_K$ is a Gal_K-orbit of a type I point x_α on $\mathbb{D}_{\mathbb{C}_K}$, with $\alpha \in \mathcal{O}_{\mathbb{C}_K}$. This point is K-rational precisely when $\alpha \in \mathcal{O}_K$ (in which case the Gal_K-orbit consists of just x_α itself), and it is K'-rational if it is the Gal_K-orbit of some x_α where $\alpha \in \mathcal{O}_{K'}$. A point of type II or III, given by the Gal_K-orbit of $x_{\alpha,\sigma} \in \mathbb{D}_{\mathbb{C}_K}$, is K'-rational if one can choose $\alpha \in \mathcal{O}_{K'}$. A similar remark holds for type V points given by the Gal_K-orbit of $x_{\alpha,>\sigma}$ or $x_{\beta,<\sigma}$. In particular, every point of type II, III, or V is K'-rational for some finite extension K'/K.

Example 2.10. More generally, for a smooth rigid analytic curve Y over K, every point $x \in Y$ of type II, III, or V is K'-rational for some finite extension K'/K. In this case, all points in the pre-image of this x under the projection $Y_{K'} \to Y$ are K'-rational.

2.3. **Stalks and residue fields.**

For any point x on a smooth rigid analytic curve Y over K (viewed as an adic space), we write $\mathcal{O}_{Y,x}$ for the stalk of the structure sheaf at x, and denote the residue field of $\mathcal{O}_{Y,x}$ by $\mathbf{k}(x)$. The continuous (multiplicative) valuation corresponding to x extends to a valuation v_x on $\mathbf{k}(x)$. The completion of $\mathbf{k}(x)$ with respect to v_x is denoted by $\mathscr{H}(x)$. The field $\mathbf{k}(x)$ (resp. $\mathscr{H}(x)$) is referred to as the *residue field* (resp. *completed residue field*) at x.

Example 2.11 (Stalks of non-type I points)**.** For a non-type I point ξ on a smooth rigid analytic curve Y over K, its stalk $\mathcal{O}_{Y,\xi}$ is a field and is equal to $\mathbf{k}(\xi)$. Hence, the completed stalk agrees with $\mathscr{H}(\xi)$. Moreover, suppose ξ is a type V point and ξ_{II} is the unique type II point specializing to ξ, then we have an injection $\iota\colon \mathbf{k}(\xi) \hookrightarrow \mathbf{k}(\xi_{\mathrm{II}})$ by [Hub96, Lemma 1.1.10]. In fact, ι is a homeomorphism onto its image and $\iota(\mathbf{k}(\xi))$ is dense in $\mathbf{k}(\xi_{\mathrm{II}})$ (also see [Mor19, Theorem I.1.5.4(b)]). Passing to completions, we obtain an isomorphism $\mathscr{H}(\xi) \cong \mathscr{H}(\xi_{\mathrm{II}})$.

The following lemma will be used in the proof of the main theorem.

Lemma 2.12. Let Y be a smooth rigid analytic curve over K, viewed as an adic space. Let $\xi \in Y$ be a type V point and let ξ_{II} be the unique type II generalization of ξ. Then the inclusion

$$\iota\colon \mathcal{O}_{Y,\xi} = \mathbf{k}(\xi) \hookrightarrow \mathcal{O}_{Y,\xi_{\mathrm{II}}} = \mathbf{k}(\xi_{\mathrm{II}})$$

induces an equivalence between the category of finite extensions of $\mathcal{O}_{Y,\xi}$ and that of $\mathcal{O}_{Y,\xi_{\mathrm{II}}}$.

Proof. For simplicity, we identify $\mathbf{k}(\xi)$ as a subfield of $\mathbf{k}(\xi_{\mathrm{II}})$ via ι. We first fix the following notation.

- Let v denote the rank-2 (multiplicative) valuation on $\mathbf{k}(\xi)$ and let w denote the rank-1 (multiplicative) valuation on $\mathbf{k}(\xi_{\mathrm{II}})$.
- Let $u \coloneqq w|_{\mathbf{k}(\xi)}$ denote the restriction of w to $\mathbf{k}(\xi)$, which is a nontrivial rank-1 valuation on $\mathbf{k}(\xi)$.
- Let $\mathbf{k}^+(\xi) = \mathcal{O}^+_{Y,\xi} \coloneqq \{a \in \mathcal{O}_{Y,\xi} \,|\, v(a) \leqslant 1\}$, $\mathbf{k}^+(\xi_{\mathrm{II}}) = \mathcal{O}^+_{Y,\xi_{\mathrm{II}}} \coloneqq \{a \in \mathcal{O}_{Y,\xi_{\mathrm{II}}} \,|\, w(a) \leqslant 1\}$, and $\mathbf{k}^+_u(\xi) \coloneqq \{a \in \mathcal{O}_{Y,\xi} \,|\, u(a) \leqslant 1\}$ be the corresponding valuation subrings. For simplicity, we write $V = \mathbf{k}^+(\xi)$ and $W = \mathbf{k}^+_u(\xi)$.

Note that $W = \mathbf{k}(\xi) \cap \mathbf{k}^+(\xi_{\mathrm{II}})$ and $V \subset W$. Also note that (W, u) is a rank-1 valuation ring, while (V, v) is a rank-2 valuation ring. Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) denote the unique height-1 prime in V (resp. in W, that is, the maximal ideal in W).

We claim that the non-archimedean valued field $(\mathbf{k}(\xi), u)$ is *quasi-complete* in the sense of [Ber93, Definition 2.3.1]. By [Tem, Theorem 1.9.13], this is equivalent to claiming that $(W, \mathfrak{q})$ is a Henselian pair. It follows from [Mor19, Corollary I.1.4.4] that $W = V_{\mathfrak{p}}$ and $\mathfrak{q} = \mathfrak{p}V_{\mathfrak{p}}$. In fact, since V is a valuation ring, we must have $\mathfrak{p} = \mathfrak{p}V_{\mathfrak{p}}$ as subsets of $\mathbf{k}(\xi)$. Therefore, by Gabber's criterion (cf. [Sta26, Tag 09XI (5)]), it suffices to show that $(V, \mathfrak{p})$ is Henselian. Next, we observe that $\mathfrak{p} = \mathrm{rad}(\varpi)$. To see this, note that for every prime ideal $\mathfrak{q}' \subsetneq \sqrt{\mathfrak{p}}$ properly contained in the radical of $\mathfrak{p}$, we have $\varpi^n \notin \mathfrak{q}$ for all n, and hence $\mathfrak{q}' \subset (\varpi^n)$ for all n. This forces $\mathfrak{q}' = (0)$ by the topological nilpotence of ϖ. Since being a Henselian pair is insensitive to passing to radicals of the relevant ideal, it suffices to show that $(V, (\varpi))$ is Henselian, but this is [Mor19, Proposition III.6.3.7] (also see [Bha17, Proposition 7.5.5]). This finishes the proof of the claim. In fact, the same argument implies that $(\mathbf{k}(\xi_{\mathrm{II}}), w)$ is also quasi-complete.

Let $\mathbf{k}(\xi)^\wedge_u$ denote the completion of $\mathbf{k}(\xi)$ with respect to u, and similarly define $\mathbf{k}(\xi)^\wedge_v$ and $\mathbf{k}(\xi_{\mathrm{II}})^\wedge_w$. Note that $\mathbf{k}(\xi)^\wedge_v$ (resp. $\mathbf{k}(\xi_{\mathrm{II}})^\wedge_w$) is precisely the completed residue field $\mathscr{H}(\xi)$ (resp. $\mathscr{H}(\xi_{\mathrm{II}})$). We have

$$\mathbf{k}(\xi)^\wedge_v = \mathbf{k}(\xi)^\wedge_u = \mathbf{k}(\xi_{\mathrm{II}})^\wedge_w = \mathscr{H}(\xi) = \mathscr{H}(\xi_{\mathrm{II}}) \tag{2.1}$$

by Example 2.11. Now, by [Ber93, Proposition 2.4.1], taking base change along the inclusion $\mathbf{k}(\xi) \hookrightarrow \mathbf{k}(\xi)^\wedge_u$ (resp. $\mathbf{k}(\xi_{\mathrm{II}}) \hookrightarrow \mathbf{k}(\xi_{\mathrm{II}})^\wedge_w$) induces an equivalence between finite extensions of $\mathbf{k}(\xi)$ and that of $\mathbf{k}(\xi)^\wedge_u$ (resp. finite extensions of $\mathbf{k}(\xi_{\mathrm{II}})$ and that of $\mathbf{k}(\xi_{\mathrm{II}})^\wedge_w$). Combining this with (2.1), we are done. □

2.4. **Comparison with Berkovich curves.**

Let Y be a smooth affinoid rigid analytic curve over K associated with an affinoid K-algebra $\mathscr{A}$. Let Y^{Ber} (resp. Y^{ad}) be its associated Berkovich space (resp. adic space). Note that Y^{Ber} is identified as the maximal Hausdorff quotient of Y^{ad} as a topological space. Let $\mathcal{O}_{Y^{\mathrm{Ber}}}$ denote the structure sheaf of the Berkovich space. For an open subset[12] $U \subset Y^{\mathrm{Ber}}$, the global sections are given by $\Gamma(U, \mathcal{O}_{Y^{\mathrm{Ber}}}) = \lim_{V \subset U} \mathscr{A}_V$, where the limit is taken over all $V \subset U$ that are finite unions of affinoid subdomains (that is, *special subsets*) in Y, and $\mathscr{A}_V$ is the ring of analytic functions on V (see [Ber90, 2.2.6]). Let x be a point on Y^{Ber}, which we also view as a point on Y^{ad} (it has type I, II, III, or IV). By the Gerritzen–Grauert Theorem (see [BGR84, 7.3.5/3] and [Ber90, Proposition 2.2.3]), the Berkovich stalk $\mathcal{O}_{Y^{\mathrm{Ber}},x}$ can be computed as $\mathcal{O}_{Y^{\mathrm{Ber}},x} = \operatorname*{colim}_{x \in U} \lim_{V \subset U} \mathscr{A}_V$, where the colimit is taken over all open neighborhoods U of x (in the canonical topology), and for each such U the limit is taken over all *rational subsets* V contained in U. On the other hand, the stalk of the structure sheaf $\mathcal{O}_{Y^{\mathrm{ad}}}$ of the adic space Y^{ad} is given by $\mathcal{O}_{Y^{\mathrm{ad}},x} = \operatorname*{colim}_{x \in V} \mathscr{A}_V$ where the colimit is taken over all rational subsets $V \subset Y$ that contain the point x. Thus, we have a natural inclusion

$$\mathcal{O}_{Y^{\mathrm{Ber}},x} \hookrightarrow \mathcal{O}_{Y^{\mathrm{ad}},x}$$

of local rings. This inclusion is an isomorphism when x is a type I point but *not* necessarily an isomorphism in general. When x has type II, III, or IV, both the Berkovich stalk $\mathcal{O}_{Y^{\mathrm{Ber}},x}$ and adic stalk $\mathcal{O}_{Y^{\mathrm{ad}},x}$ are fields (cf. Example 2.11). The valuation corresponding to x extends to a valuation v_x on both stalks, making them topological fields. We claim that, in this case, the inclusion $\mathcal{O}_{Y^{\mathrm{Ber}},x} \hookrightarrow \mathcal{O}_{Y^{\mathrm{ad}},x}$ has a dense image and thus becomes an isomorphism upon taking completions, which we identify and simply denote by $\mathscr{H}(x)$ as in Example 2.11. To see the claim, let $h \in \Gamma(V, \mathcal{O}_{Y^{\mathrm{ad}}})$ be an analytic function on a rational subset V containing x, we want to show that for any $\epsilon \in \mathbb{R}_{>0}$, there exists an open subset $U \subset Y^{\mathrm{Ber}}$ (in the canonical topology) containing V and an analytic function $h' \in \Gamma(U, \mathcal{O}_{Y^{\mathrm{Ber}}})$, such that $v_x(h - h') < \epsilon$. Without loss of generality, we may write

$$V = Y\left(\frac{f_1, \dots, f_n}{g}\right) = \left\{ y \in Y \;\middle|\; v_y(f_i) \leqslant v_y(g),\ \forall i \right\}$$

where $f_1, \dots, f_n \in A$ generates the unit ideal, $g \in A$, and $f_n = \varpi^N$ for some sufficiently large integer N (see [Sch12, Remark 2.8]). By the construction of rational subsets, there exists some h' of the form

$$h' = \sum_{\substack{I=(i_1,\dots,i_n) \\ \text{finite sum}}} a_I \cdot (f_1/g)^{i_1} (f_2/g)^{i_2} \cdots (f_n/g)^{i_n}, \qquad a_I \in A,$$

such that $|h - h'|_V < \epsilon$, where $|\cdot|_V$ denotes the spectral norm on V. In particular, $v_x(h - h') < \epsilon$. Since h' is defined on the Zariski open subset $Y \backslash V(g)$, it is defined on some open subset $U \subset Y^{\mathrm{Ber}}$ (in the canonical topology) that contains V. This justifies our claim.

Finally, let us remark that, when x has type III or IV, the inclusion $\mathcal{O}_{Y^{\mathrm{Ber}},x} \hookrightarrow \mathcal{O}_{Y^{\mathrm{ad}},x}$ is in fact an isomorphism since Berkovich neighborhoods are cofinal among adic neighborhoods at a point of type III or IV (cf. [Ber07, Proposition 2.2.1]).

3. (φ, Γ_K)-modules and Galois representations

In this section, we review the classical theory on how to associate overconvergent (φ, Γ_K)-modules and p-adic differential equations to (de Rham) p-adic Galois representations, following [CC98, Ber02, Col08]. We largely follow the notation of Kedlaya–Liu [KL15] on period rings although the content in this section predates *op. cit.* Experts familiar with [CC98] and [Ber02] should feel free to skip most of this section.

With much regret, we remind the reader that standard references in the literature use slightly different notations for period rings. To orient the reader, we compile a list of these conventions in Appendix A.

[12] This is an open subset in the Hausdorff topology of Y^{Ber}, also referred to as the *canonical* topology.

3.1. **Perfect period rings.**

Let $\widetilde{\mathbf{A}} \coloneqq W(\mathbb{C}_K^\flat)$ and $\widetilde{\mathbf{A}}^+ \coloneqq W(\mathcal{O}_{\mathbb{C}_K}^\flat)$, equipped with natural Gal_K- and Frobenius actions. Let $\widetilde{\mathbf{A}}_K \coloneqq (\widetilde{\mathbf{A}})^{H_K} = W(\widehat{K}_\infty^\flat)$ and $\widetilde{\mathbf{A}}_K^+ \coloneqq (\widetilde{\mathbf{A}}^+)^{H_K} = W(\mathcal{O}_{\widehat{K}_\infty^\flat})$ denote the H_K-invariants. Every element $x \in \widetilde{\mathbf{A}}$ (resp. $x \in \widetilde{\mathbf{A}}_K$) can be uniquely expressed as $x = \sum_{n=0}^\infty p^n[y_n]$ with $y_n \in \mathbb{C}_K^\flat$ (resp. $y_n \in \widehat{K}_\infty^\flat$). We refer the reader to [Fon94] for the construction of period rings $\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{crys}}, \mathbf{B}_{\mathrm{st}}$ and their variants.

Definition 3.1 ([Ber02, §2.1], [Col08, §5.2])**.** (1) For every $r > 0$, define

$$\widetilde{\mathbf{A}}^r \coloneqq \left\{ x = \sum_{n=0}^\infty p^n[y_n] \in \widetilde{\mathbf{A}} \;\middle|\; \lim_{n\to\infty} v(y_n) + \frac{n}{r} = +\infty \right\}$$

and define $\widetilde{\mathbf{A}}_K^r \coloneqq (\widetilde{\mathbf{A}}^r)^{H_K}$.

(2) For any $s \in (0, r]$, we define a norm λ_s on $\widetilde{\mathbf{A}}^r$ by

$$\lambda_s(x) \coloneqq \sup_{n\geqslant 0} \{p^{-n} \cdot |y_n|^s\}.$$

This restricts to a norm λ_s on $\widetilde{\mathbf{A}}_K^r$. Note that $\widetilde{\mathbf{A}}^r$ and $\widetilde{\mathbf{A}}_K^r$ are complete with respect to λ_r.

(3) Define $\widetilde{\mathbf{A}}^\dagger \coloneqq \varinjlim_{r\geqslant 0} \widetilde{\mathbf{A}}^r$ and $\widetilde{\mathbf{A}}_K^\dagger \coloneqq \varinjlim_{r>0} \widetilde{\mathbf{A}}_K^r$, where the transition maps are natural inclusions.

(4) The period rings $\widetilde{\mathbf{B}}, \widetilde{\mathbf{B}}^+, \widetilde{\mathbf{B}}^r, \widetilde{\mathbf{B}}^\dagger, \widetilde{\mathbf{B}}_K, \widetilde{\mathbf{B}}_K^+, \widetilde{\mathbf{B}}_K^r$, and $\widetilde{\mathbf{B}}_K^\dagger$ are obtained from the corresponding period rings of type $\mathbf{A}$ with p inverted.

One checks that $\widetilde{\mathbf{A}}$ (resp. $\widetilde{\mathbf{A}}_K$) is the p-adic completion of $\widetilde{\mathbf{A}}^\dagger$ (resp. $\widetilde{\mathbf{A}}_K^\dagger$).

Definition 3.2 ([Ber02, §2.3, Definition 2.16], [Col08, §5.4])**.**

(1) For $0 < s \leqslant r$, we define $\widetilde{\mathbf{C}}^{[s,r]}$ (resp. $\widetilde{\mathbf{C}}_K^{[s,r]}$) to be the completion of $\widetilde{\mathbf{B}}^r$ (resp. of $\widetilde{\mathbf{B}}_K^r$) with respect to $\max\{\lambda_s, \lambda_r\}$ (or equivalently, completion with respect to the family of norms $\{\lambda_t\}_{t\in[s,r]}$).

(2) For every $r > 0$, we define $\widetilde{\mathbf{C}}^r$ (resp. $\widetilde{\mathbf{C}}_K^r$) to be the Fréchet completion of $\widetilde{\mathbf{B}}^r$ (resp. $\widetilde{\mathbf{B}}_K^r$) with respect to the family of norms $\{\lambda_s\}_{s\in(0,r]}$.

(3) Define $\widetilde{\mathbf{C}}^\dagger$ and $\widetilde{\mathbf{C}}_K^\dagger$ to be the union $\widetilde{\mathbf{C}}^\dagger \coloneqq \varinjlim_{r>0} \widetilde{\mathbf{C}}^r$ and $\widetilde{\mathbf{C}}_K^\dagger \coloneqq \varinjlim_{r>0} \widetilde{\mathbf{C}}_K^r$.

Remark 3.3. (i) For each $r > 0$, we have $\widetilde{\mathbf{A}}_K^r = (\widetilde{\mathbf{A}}^r)^{H_K} = \widetilde{\mathbf{A}}^r \cap \widetilde{\mathbf{A}}_K$ as subrings of $\widetilde{\mathbf{A}}$. Similarly we have $\widetilde{\mathbf{B}}_K^r = (\widetilde{\mathbf{B}}^r)^{H_K}$ and $\widetilde{\mathbf{C}}_K^r = (\widetilde{\mathbf{C}}^r)^{H_K}$, and therefore $\widetilde{\mathbf{A}}_K^\dagger = (\widetilde{\mathbf{A}}^\dagger)^{H_K}, \widetilde{\mathbf{B}}_K^\dagger = (\widetilde{\mathbf{B}}^\dagger)^{H_K}$, and $\widetilde{\mathbf{C}}_K^\dagger = (\widetilde{\mathbf{C}}^\dagger)^{H_K}$. In particular, our definitions agree with the ones in [Ber02] and [Col08].

(ii) For each $r > 0$, the natural map $\widetilde{\mathbf{C}}^r \to \varprojlim_{0<s\leqslant r} \widetilde{\mathbf{C}}^{[s,r]}$ is an isomorphism. A similar isomorphism holds for $\widetilde{\mathbf{C}}_K^r$ and $\widetilde{\mathbf{C}}_K^{[s,r]}$.

Remark 3.4 (Frobenius structures)**.** The Frobenius maps on $\widetilde{\mathbf{A}}^+$ and $\widetilde{\mathbf{A}}$ extend to Frobenius maps on $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$, which we denote by φ. Moreover, for $0 < s \leqslant r$, we have Frobenius isomorphisms $\varphi\colon \widetilde{\mathbf{A}}^r \xrightarrow{\sim} \widetilde{\mathbf{A}}^{r/p}$, $\varphi\colon \widetilde{\mathbf{B}}^r \xrightarrow{\sim} \widetilde{\mathbf{B}}^{r/p}$, $\varphi\colon \widetilde{\mathbf{C}}^r \xrightarrow{\sim} \widetilde{\mathbf{C}}^{r/p}$, $\varphi\colon \widetilde{\mathbf{C}}^{[s,r]} \xrightarrow{\sim} \widetilde{\mathbf{C}}^{[s/p,\, r/p]}$ extending the Frobenius on $\widetilde{\mathbf{A}}^+$. Passing to colimits, we obtain Frobenius maps φ on the period rings $\widetilde{\mathbf{A}}^\dagger, \widetilde{\mathbf{B}}^\dagger$, and $\widetilde{\mathbf{C}}^\dagger$. Similarly, there are natural Frobenius structures on $\widetilde{\mathbf{A}}_K^r, \widetilde{\mathbf{B}}_K^r, \widetilde{\mathbf{C}}_K^r, \widetilde{\mathbf{C}}_K^{[s,r]}, \widetilde{\mathbf{A}}_K^\dagger, \widetilde{\mathbf{B}}_K^\dagger$, and $\widetilde{\mathbf{C}}_K^\dagger$.

3.2. **Imperfect period rings.**

Let $\mathbf{E} \coloneqq k(\!(\overline{\pi})\!)^{\mathrm{sep}}$ be the separable closure of $k(\!(\overline{\pi})\!)$ inside $\mathbb{C}_K^\flat$. Let $\mathbf{E}_K \coloneqq (\mathbf{E})^{H_K}$, which is a discretely-valued field with residue field κ_K. Let $\mathbf{E}_K^+$ be its ring of integers. By choosing a uniformizer $\overline{\pi}_K$ of $\mathbf{E}_K^+$, we have $\mathbf{E}_K \cong \kappa_K(\!(\overline{\pi}_K)\!)$ and $\mathbf{E}_K^+ \cong \kappa_K[\![\overline{\pi}_K]\!]$. (Note that when $K = K_0$, we have $\mathbf{E}_{K_0} \cong k(\!(\overline{\pi})\!)$.) Moreover, the theory of fields of norms supplies an isomorphism

$$\varprojlim_n \mathcal{O}_{K_n}/\mathfrak{a} \xrightarrow{\sim} \mathbf{E}_K^+$$

where $\mathfrak{a}$ is the ideal consisting of elements x satisfying $|x| \leqslant p^{-\frac{1}{p}}$ and the inverse limit is taken over the Frobenius maps $\mathcal{O}_{K_{n+1}}/\mathfrak{a} \to \mathcal{O}_{K_n}/\mathfrak{a}$ sending $x \mapsto x^p$. By [Col08, Proposition 4.12], $\mathbf{E}_K$ is a totally ramified extension of $\mathbf{E}_{K_0'} = \kappa_K(\!(\overline{\pi})\!)$ of degree $e_K = [K_\infty : (K_0')_\infty]$ as in §1.9; thus, we have $|\overline{\pi}_K| = p^{-\frac{p}{p-1}\cdot\frac{1}{e_K}}$.

There are subrings $\mathbf{A} \subset \widetilde{\mathbf{A}}$ and $\mathbf{A}_K \subset \widetilde{\mathbf{A}}_K$, which are both p-adically complete and satisfy

$$\mathbf{A}/(p) = \mathbf{E}, \qquad \mathbf{A}_K/(p) = \mathbf{E}_K \cong \kappa_K(\!(\overline{\pi}_K)\!).$$

Moreover, $\mathbf{A}$ is stable under the φ- and Gal_K-actions on $\widetilde{\mathbf{A}}$, and we have $\mathbf{A}_K = (\mathbf{A})^{H_K}$, which thus carries an action of Γ_K. We omit the constructions here and refer the reader to [Col08, §6.1] for details, noting only that, for a finite unramified extension $L_0 = W(k_0)[1/p]$ over K_0, the ring $\mathbf{A}_{L_0}$ is precisely the p-adic completion of $W(k_0)[\![\pi]\!][1/\pi]$ in $\widetilde{\mathbf{A}}$. We write $\mathbf{A}^+_{L_0} \subset \mathbf{A}_{L_0}$ for the subring $W(k_0)[\![\pi]\!]$.

Construction 3.5 ([Col08, §6.2])**.** We construct a lift $\pi_K \in \mathbf{A}_K$ of $\overline{\pi}_K$ as follows. Let $\overline{P}_K \in \mathbf{E}^+_{K'_0}[X] = \kappa_K[\![\overline{\pi}]\!][X]$ be the minimal polynomial of $\overline{\pi}_K$ over $\mathbf{E}_{K'_0}$. Choose a monic polynomial $P_K \in \mathbf{A}^+_{K'_0}[X]$ of the same degree that reduces to $\overline{P}_K$ modulo p. By Hensel's Lemma, P_K has a unique root π_K in $\mathbf{A}_K$ lifting $\overline{\pi}_K$. Moreover, $\mathbf{A}_K$ is a free $\mathbf{A}_{K'_0}$-module with basis $\{1, \pi_K, \dots, \pi_K^{e_K-1}\}$.

Using [Col08, Proposition 7.5], one can identify $\mathbf{A}_K$ with the p-adic completion of $\mathcal{O}_{K'_0}[\![\pi_K]\!][\frac{1}{\pi_K}]$. More explicitly, we have

$$\mathbf{A}_K = \left\{ \sum_{i\in\mathbb{Z}} c_i \pi_K^i \,\middle|\, c_i \in \mathcal{O}_{K'_0} \text{ and } c_i \to 0 \text{ } p\text{-adically as } i \to -\infty \right\}.$$

Definition 3.6 (Classical imperfect period rings)**.**
(1) For $r > 0$, define $\mathbf{A}^r := \mathbf{A} \cap \widetilde{\mathbf{A}}^r$ and $\mathbf{A}^r_K := \mathbf{A}_K \cap \widetilde{\mathbf{A}}^r_K$, where the intersection is taken inside $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}_K$ respectively.
(2) Define $\mathbf{A}^\dagger$ and $\mathbf{A}^\dagger_K$ as the unions $\mathbf{A}^\dagger := \varinjlim_{r>0} \mathbf{A}^r$ and $\mathbf{A}^\dagger_K := \varinjlim_{r>0} \mathbf{A}^r_K$.
(3) The period rings $\mathbf{B}, \mathbf{B}_K$, $\mathbf{B}^r$, $\mathbf{B}^r_K$, $\mathbf{B}^\dagger$, and $\mathbf{B}^\dagger_K$ are obtained from the corresponding period rings of type $\mathbf{A}$ with p inverted.
(4) For every $0 < s \leqslant r$, we define $\mathbf{C}^{[s,r]}$ (resp. $\mathbf{C}^{[s,r]}_K$) to be the completion of $\mathbf{B}^r$ (resp. $\mathbf{B}^r_K$) with respect to $\max\{\lambda_s, \lambda_r\}$ (or equivalently, completion with respect to the family of norms $\{\lambda_t\}_{t\in[s,r]}$) inside $\widetilde{\mathbf{C}}^{[s,r]}$ (resp. $\widetilde{\mathbf{C}}^{[s,r]}_K$). Note that, for any $r' \in [s,r]$, the norm $\lambda_{r'}$ extends to $\mathbf{C}^{[s,r]}$ and $\mathbf{C}^{[s,r]}_K$.
(5) For every $r > 0$, we define $\mathbf{C}^r$ (resp. $\mathbf{C}^r_K$) to be the Fréchet completion of $\mathbf{B}^r$ (resp. $\mathbf{B}^r_K$) inside $\widetilde{\mathbf{C}}^r$ (resp. $\widetilde{\mathbf{C}}^r_K$) with respect to the family of norms $\{\lambda_s\}_{s\in(0,r]}$.
(6) Finally, define $\mathbf{C}^\dagger$ and $\mathbf{C}^\dagger_K$ as the unions $\mathbf{C}^\dagger := \varinjlim_{r>0} \mathbf{C}^r$ and $\mathbf{C}^\dagger_K := \varinjlim_{r>0} \mathbf{C}^r_K$.

Remark 3.7.
(i) Since $\widetilde{\mathbf{A}}_K$ is the p-adic completion of $\widetilde{\mathbf{A}}^\dagger_K$, we deduce that $\mathbf{A}_K$ is the p-adic completion of $\mathbf{A}^\dagger_K$.
(ii) For each $r > 0$, the natural map $\mathbf{C}^r \to \varprojlim_{0<s\leqslant r} \mathbf{C}^{[s,r]}$ is an isomorphism. A similar isomorphism holds for $\mathbf{C}^r_K$ and $\mathbf{C}^{[s,r]}_K$.
(iii) Remark 3.3 continues to hold for imperfect period rings defined above. In particular, we have $\mathbf{A}^\dagger_K = (\mathbf{A}^\dagger)^{H_K}$, $\mathbf{B}^\dagger_K = (\mathbf{B}^\dagger)^{H_K}$, and $\mathbf{C}^\dagger_K = (\mathbf{C}^\dagger)^{H_K}$.
(iv) Finally, we remind the reader again that the rings $\mathbf{C}^\dagger$ and $\mathbf{C}^\dagger_K$ are denoted by $\mathbf{B}^\dagger_{\mathrm{rig}}$ and $\mathbf{B}^\dagger_{\mathrm{rig},K}$ in [Ber02] and [Col08] (see Appendix A).

Remark 3.8 (Topology)**.** We endow $\widetilde{\mathbf{A}}^r_K$ with the topology induced by the λ_r-norm and endow $\widetilde{\mathbf{A}}^\dagger_K$ with the inductive limit topology (i.e., the finest topology making $\widetilde{\mathbf{A}}^r_K \hookrightarrow \widetilde{\mathbf{A}}^\dagger_K$ continuous for all $r > 0$). Inverting p, we obtain natural topologies on $\widetilde{\mathbf{B}}^r_K$ and $\widetilde{\mathbf{B}}^\dagger_K$. The ring $\widetilde{\mathbf{C}}^{[s,r]}_K$ is endowed with the topology induced by $\max\{\lambda_r, \lambda_s\}$, which then induces a Fréchet topology on $\widetilde{\mathbf{C}}^r_K$. Finally, we equip $\widetilde{\mathbf{C}}^\dagger_K$ with the inductive limit topology (i.e., the LF topology). The topologies on the imperfect period rings are defined similarly.

Remark 3.9 ((φ, Γ_K)-structures)**.**
(i) The Frobenius φ on $\widetilde{\mathbf{A}}_K$ (resp. on $\widetilde{\mathbf{A}}^\dagger_K$) preserves $\mathbf{A}_K$ (resp. $\mathbf{A}^\dagger_K$). For $0 < s \leqslant r$, we also have Frobenius maps $\varphi\colon \mathbf{A}^r_K \to \mathbf{A}^{r/p}_K, \varphi\colon \mathbf{B}^r_K \to \mathbf{B}^{r/p}_K, \varphi\colon \mathbf{C}^r_K \to \mathbf{C}^{r/p}_K, \varphi\colon \mathbf{C}^{[s,r]}_K \to \mathbf{C}^{[s/p,\, r/p]}_K$ as well as Frobenius maps on $\mathbf{B}^\dagger_K$ and $\mathbf{C}^\dagger_K$. We caution the reader that $\varphi\colon \mathbf{A}^r_K \to \mathbf{A}^{r/p}_K$ is no longer bijective.
(ii) All period rings $\widetilde{\mathbf{A}}^r_K$, $\widetilde{\mathbf{A}}^\dagger_K$, $\widetilde{\mathbf{B}}^r_K$, $\widetilde{\mathbf{B}}^\dagger_K$, $\widetilde{\mathbf{C}}^{[s,r]}_K$, $\widetilde{\mathbf{C}}^r_K$, $\widetilde{\mathbf{C}}^\dagger_K$, as well as their imperfect counterparts, are equipped with continuous actions of Γ_K, where Γ_K is endowed with the profinite topology. The Γ_K-actions commute with the Frobenius actions.

Notation 3.10. Let $t = \log(1+\pi)$. This is an element that lies in $\mathbf{C}_K^\dagger$ but not in $\mathbf{B}_K^\dagger$.

Lemma 3.11 ([Col08, Lemme 6.5])**.** Let

$$r_K = \begin{cases} (2v(\mathfrak{d}_{\mathbf{E}_K/\mathbf{E}_{K_0}}))^{-1} & \text{if } e_K > 1, \\ 1 & \text{if } e_K = 1. \end{cases}$$

Then for all $0 < r < r_K$, we have $\pi_K \in \mathbf{A}_K^r$ and the element $u = \frac{\pi_K}{[\overline{\pi}_K]}$ is a unit in the ring of integers of $\widetilde{\mathbf{A}}_K^r$ (with respect to the norm λ_r). More precisely, if we write $u = \sum_{n=0}^{\infty} p^n[u_n]$ with $u_n \in \widehat{K}_\infty^\flat$, then $u_0 = 1$ and $rv(u_n) + n > 0$ for all $n \geqslant 1$.

In particular, for $0 < r < r_K$, we know that $\lambda_r(\pi_K) = |\overline{\pi}_K|^r = p^{-\frac{p}{p-1}\cdot\frac{r}{e_K}}$.

3.3. Explicit (geometric) descriptions.

There are explicit descriptions of imperfect period rings in terms of double-ended power series subject to certain convergence conditions.

Proposition 3.12. If $0 < r < r_K$, there is an isomorphism

$$\mathscr{R}_{K_0'}^{\mathrm{int},r} = \left\{ \sum_{i\in\mathbb{Z}} c_i x^i \,\middle|\, c_i \in \mathcal{O}_{K_0'},\ |c_i| \cdot p^{-\frac{p}{p-1}\cdot\frac{ir}{e_K}} \to 0 \text{ as } i \to -\infty \right\} \xrightarrow{\sim} \mathbf{A}_K^r$$

sending $f \mapsto f(\pi_K)$. Moreover, the isomorphism preserves the norms on both sides, where the norm on the left-hand side is $\sup_{i\in\mathbb{Z}} \left\{ |c_i| \cdot p^{-\frac{p}{p-1}\cdot\frac{ir}{e_K}} \right\}$ and the norm on the right-hand side is λ_r.

Proof. This is [CC98, Proposition II.2.1] (also see [Ber02, Proposition 1.4] and [Col08, Proposition 7.5(i)]). □

Remark 3.13. We have similar explicit descriptions of $\mathbf{C}_K^r$ and $\mathbf{C}_K^{[s,r]}$. For example, for $0 < r < r_K$, the isomorphism in Proposition 3.12 induces an isomorphism

$$\mathscr{R}_{K_0'}^{r} = \left\{ \sum_{i\in\mathbb{Z}} c_i x^i \,\middle|\, c_i \in K_0',\ |c_i| \cdot p^{-\frac{p}{p-1}\cdot\frac{ir'}{e_K}} \to 0 \text{ as } i \to \pm\infty \text{ for any } r' \in (0,r] \right\} \xrightarrow{\sim} \mathbf{C}_K^r$$

sending $f \mapsto f(\pi_K)$. In other words, we can identify $\mathbf{C}_K^r$ with the ring of analytic functions $\Gamma_{K_0'}([\delta,1);\pi_K)$ on the half-open annulus $A_{K_0'}([\delta,1);\pi_K)$ with open outer boundary at radius 1 and closed inner boundary at radius $\delta = p^{-\frac{p}{p-1}\cdot\frac{r}{e_K}}$. Likewise, we have an identification $\mathbf{C}_K^{[s,r]} = \Gamma_{K_0'}([\delta_1,\delta_2];\pi_K)$ where $\delta_1 = p^{-\frac{p}{p-1}\cdot\frac{r}{e_K}}$ and $\delta_2 = p^{-\frac{p}{p-1}\cdot\frac{s}{e_K}}$. Consequently, the ring $\mathbf{C}_K^\dagger$ can be identified with the Robba ring $\mathscr{R}_{K_0'}$ with coefficients in K_0'. We refer the reader to [Col08, Proposition 7.5 & 7.6] and [DMY26, Proposition 3.25] for more details.

3.4. The rings $\mathbf{C}_{\log,K}^\dagger$, $\mathbf{C}_{\log}^\dagger$, and $\widetilde{\mathbf{C}}_{\log}^\dagger$.

To study semistable representations, we also need the "logarithmic variants" of the imperfect period rings.

Definition 3.14. For $0 < s \leqslant r$, we define

$$\mathbf{C}_{\log,K}^r \coloneqq \mathbf{C}_K^r[\log\pi], \quad \mathbf{C}_{\log,K}^\dagger \coloneqq \mathbf{C}_K^\dagger[\log\pi] = \varinjlim_{r>0} \mathbf{C}_{\log,K}^r$$

equipped with commuting φ and Γ_K-actions extending the ones on $\mathbf{C}_K^\dagger$. The Frobenius map φ on $\log\pi$ is induced from $\varphi(\pi) = (\pi+1)^p - 1$ and can be written as $\varphi(\log\pi) = p\log\pi + \log(\varphi(\pi)/\pi^p)$. The Γ_K-action on $\log\pi$ is given by $\gamma(\log\pi) = \log\pi + \log(\gamma(\pi)/\pi)$ for $\gamma \in \Gamma_K$. The rings $\mathbf{C}_{\log}^\dagger$ and $\widetilde{\mathbf{C}}_{\log}^\dagger$ are defined in similar ways, by adjoining $\log\pi$ to $\mathbf{C}^\dagger$ and $\widetilde{\mathbf{C}}^\dagger$, respectively.[13]

[13]The rings $\mathbf{C}_{\log,K}^\dagger$, $\mathbf{C}_{\log}^\dagger$, and $\widetilde{\mathbf{C}}_{\log}^\dagger$ are denoted by $\mathbf{B}_{\log,K}^\dagger$, $\mathbf{B}_{\log}^\dagger$, and $\widetilde{\mathbf{B}}_{\log}^\dagger$ in [Ber02], respectively.

3.5. **Overconvergent (φ,Γ_K)-modules.**

For every $\mathbb{Q}_p$-representation V of Gal_K, one can associate a (φ,Γ_K)-module $D_K^\dagger(V) \coloneqq (V\otimes_{\mathbb{Q}_p}\mathbf{B}^\dagger)^{H_K}$ over $\mathbf{B}_K^\dagger$, as well as (φ,Γ_K)-modules over $\mathbf{C}_K^\dagger$ and $\mathbf{C}_{\log,K}^\dagger$ given by

$$D_{\mathrm{rig},K}^\dagger(V) \coloneqq D_K^\dagger(V)\otimes_{\mathbf{B}_K^\dagger}\mathbf{C}_K^\dagger, \qquad D_{\log,K}^\dagger(V) \coloneqq D_K^\dagger(V)\otimes_{\mathbf{B}_K^\dagger}\mathbf{C}_{\log,K}^\dagger.$$

Theorem 3.15 ([CC98], [KL15, Theorem 9.5.6])**.** *The functors $D_K^\dagger$ and $D_{\mathrm{rig},K}^\dagger$ induce equivalences of categories*

$$\begin{aligned}\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K) &\xrightarrow{\sim} \{\text{étale } (\varphi,\Gamma_K)\text{-modules over } \mathbf{B}_K^\dagger\}\\ &\xrightarrow{\sim} \{\text{étale } (\varphi,\Gamma_K)\text{-modules over } \mathbf{C}_K^\dagger\}\end{aligned}$$

between the category of finite dimensional continuous Gal_K-representations over $\mathbb{Q}_p$-vector spaces and the category of (finite free) étale (φ,Γ_K)-modules over $\mathbf{B}_K^\dagger$ (resp. $\mathbf{C}_K^\dagger$).[14]

Remark 3.16. One can also consider (φ,Γ_K)-modules over perfect period rings. Indeed, we have equivalences of categories between the category $\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_K)$ of continuous Gal_K-representations over finite free $\mathbb{Z}_p$-modules, that of (φ,Γ_K)-modules over $\widetilde{\mathbf{A}}_K^\dagger$, and that of (φ,Γ_K)-modules over $\widetilde{\mathbf{A}}_K$. Similarly, we have equivalences of categories between $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, the category of étale (φ,Γ_K)-modules over $\widetilde{\mathbf{B}}_K^\dagger$, and that of étale (φ,Γ_K)-modules over $\widetilde{\mathbf{C}}_K^\dagger$. For $V\in\mathrm{Rep}_{\mathbb{Z}_p}(\mathrm{Gal}_K)$ or $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, let $\widetilde{D}_K^{\mathrm{int}}(V)$, $\widetilde{D}_K^{\mathrm{int},\dagger}(V)$, $\widetilde{D}_K^\dagger(V)$, $\widetilde{D}_{\mathrm{rig},K}^\dagger(V)$ denote the associated (φ,Γ_K)-modules over $\widetilde{\mathbf{A}}_K$, $\widetilde{\mathbf{A}}_K^\dagger$, $\widetilde{\mathbf{B}}_K^\dagger$, $\widetilde{\mathbf{C}}_K^\dagger$, respectively. In particular, $\widetilde{D}_K^\dagger(V)$ (resp. $\widetilde{D}_{\mathrm{rig},K}^\dagger(V)$) is the base change of $D_K^\dagger(V)$ (resp. $D_{\mathrm{rig},K}^\dagger(V)$) along $\mathbf{B}_K^\dagger\to\widetilde{\mathbf{B}}_K^\dagger$ (resp. $\mathbf{C}_K^\dagger\to\widetilde{\mathbf{C}}_K^\dagger$). We refer the reader to [CC98] and [KL15] for more details.

3.6. **Differential modules.**

Before proceeding further, let us recall some basic notions on p-adic differential equations. Firstly, a *differential ring* is a commutative ring R equipped with a derivation $d\colon R\to R$ (cf. [Ked22, §5]), that is, an additive map satisfying the Leibniz rule $d(ab)=ad(b)+bd(a)$ for all $a,b\in R$. A *differential module* over a differential ring (R,d) is an R-module M equipped with a differential operator $D\colon M\to M$ compatible with d, that is, an additive map satisfying $D(ax)=aD(x)+d(a)x$ for all $a\in R, x\in M$. When R is equipped with more than one differential operator $d_1,\dots,d_m$, we abuse the terminology and say that $(M,D_1,\dots,D_m)$ is a *differential module* over $(R,d_1,\dots,d_m)$ if (M,D_i) is a differential module over (R,d_i) for all $i=1,\dots m$.

We are primarily interested in the Robba ring $\mathbf{C}_K^\dagger$ on which the action of the Lie algebra of Γ_K induces a differential operator ∇ defined by $\nabla=\log(\gamma)/\log(\chi(\gamma))$ for any γ sufficiently close to 1, where $\chi\colon\Gamma_K\to\mathbb{Z}_p^\times$ is the usual cyclotomic character. By [Ber02, Lemme 4.2], one has $\nabla = t\cdot\partial = t\cdot(1+\pi)\cdot\frac{d}{d\pi}$, where t is defined in Notation 3.10 and $\partial=(1+\pi)\cdot\frac{d}{d\pi}$ is a well-defined differential operator on $\mathbf{C}_K^\dagger$. Then, a differential module (M,∂_M) over $(\mathbf{C}_K^\dagger,\partial)$ can be viewed as a connection

$$\nabla_M\colon M\to M\otimes_{\mathbf{C}_K^\dagger}\Omega^1_{\mathbf{C}_K^\dagger}$$

where $\nabla_M(x)=\partial_M(x)\cdot d\log(1+\pi)=\frac{\partial_M(x)}{1+\pi}\cdot d\pi$.[15] A finite free differential module over $(\mathbf{C}_K^\dagger,\partial)$ is known as a *p-adic differential equation* in [Ber02].

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[14] Recall that a (φ,Γ_K)-module over $\mathbf{B}_K^\dagger$ is a finite free $\mathbf{B}_K^\dagger$-module M with a semilinear Frobenius action φ inducing an isomorphism $\varphi\otimes\mathrm{id}\colon M\otimes_{\mathbf{B}_K^\dagger,\varphi}\mathbf{B}_K^\dagger\xrightarrow{\sim}M$ and a continuous semilinear Γ_K-action that commutes with φ. There is a similar notion over $\mathbf{C}_K^\dagger$. Such a (φ,Γ_K)-module over (either $\mathbf{B}_K^\dagger$ or $\mathbf{C}_K^\dagger$) is *étale* if there is a finite free module M_0 over $\mathbf{A}_K^\dagger$ equipped with a continuous semilinear Γ_K-action and a Frobenius semilinear endomorphism $\varphi\colon M_0\to M_0$ commuting with Γ_K, such that the induced linear map $\varphi\otimes\mathrm{id}\colon M_0\otimes_{\mathbf{A}_K^\dagger,\varphi}\mathbf{A}_K^\dagger\xrightarrow{\sim}M_0$ is an isomorphism, and that M is the base change of M_0 as a (φ,Γ_K)-module.

[15] Note that this is a connection extending $dx=\frac{\partial x}{1+\pi}d\pi$.

Remark 3.17. In practice, one may replace $(\mathbf{C}_K^\dagger, \partial)$ with the differential ring $(\mathbf{C}_K^\dagger, \frac{d}{d\pi_K})$ without changing the underlying theory of p-adic differential equations, as $\partial = u_0 \cdot \frac{d}{d\pi_K}$ for some nonzero element $u_0 \in \mathbf{B}_K^\dagger$ (hence, a unit in $\mathbf{C}_K^\dagger$).

Definition 3.18. Let (M, ∂_M) be a finite free differential module over $(\mathbf{C}_K^\dagger, \partial)$. A *Frobenius structure* on M is a φ-semilinear map $\Phi\colon M \to M$ such that

(1) Φ induces an isomorphism $\varphi^*(M) \xrightarrow{\sim} M$, i.e., $\Phi(M)$ generates M as a $\mathbf{C}_K^\dagger$-module;
(2) Φ is compatible with the differential structures; namely, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\nabla_M} & M \otimes_{\mathbf{C}_K^\dagger} \Omega^1_{\mathbf{C}_K^\dagger} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi\otimes d\varphi} \\ M & \xrightarrow{\nabla_M} & M \otimes_{\mathbf{C}_K^\dagger} \Omega^1_{\mathbf{C}_K^\dagger} \end{array}$$

commutes. Equivalently, Φ satisfies $\partial_M \circ \Phi = p\,\Phi \circ \partial_M$.

3.7. p-adic differential equations attached to de Rham representations.

Let $D^\dagger_{\mathrm{rig},K}(V)$ be the étale (φ, Γ_K)-module over $\mathbf{C}_K^\dagger$ associated with some $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$. The action of the Lie algebra of Γ_K induces a differential operator $\nabla_V = \log(\gamma)/\log(\chi(\gamma))$ on $D^\dagger_{\mathrm{rig},K}(V)$. To obtain a differential module over $(\mathbf{C}_K^\dagger, \partial)$, we consider

$$\partial_V = \frac{1}{t}\nabla_V = \frac{1}{\log(1+\pi)}\nabla_V.$$

In general, this operator does not preserve $D^\dagger_{\mathrm{rig},K}(V)$, as it might introduce poles at the vanishing locus of t. The main technical result of [Ber02] states that, when V is de Rham, these poles can be removed systematically to produce an actual p-adic differential equation over $(\mathbf{C}_K^\dagger, \partial)$.

Theorem 3.19 ([Ber02, Théorème 5.10])**.** *Let V be a de Rham representation of Gal_K of dimension d with non-negative Hodge–Tate weights.*[16] *Then there exists a unique free $\mathbf{C}_K^\dagger$-submodule $\mathbf{N}_{\mathrm{dR}}(V) \subset D^\dagger_{\mathrm{rig},K}(V)$ of rank d that is stable under ∂_V. Moreover, $\mathbf{N}_{\mathrm{dR}}(V)$ is stable under φ and Γ_K-actions,*[17] *and, consequently, it is a finite free differential module over $(\mathbf{C}_K^\dagger, \partial)$ with a Frobenius structure. Furthermore, the inclusion $\mathbf{N}_{\mathrm{dR}}(V) \subset D^\dagger_{\mathrm{rig},K}(V)$ induces an isomorphism after inverting t.*

Remark 3.20. The construction of $\mathbf{N}_{\mathrm{dR}}(V)$ can be interpreted as a functor

$$\mathbf{N}_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\mathrm{Gal}_K) \longrightarrow \{(\varphi, \Gamma_K)\text{-modules over } \mathbf{C}_K^\dagger\},$$

from the category of all de Rham representations of Gal_K, without any assumption on Hodge-Tate weights. We refer the reader to [DMY26, §3.4.2] for details.

3.8. The classical p-adic monodromy theorem.

To conclude this section, we recall the proof of the classical p-adic monodromy theorem, which asserts that every de Rham representation is potentially semistable. The first step is the following result of Berger, which recognizes the F-isocrystal $D_{\mathrm{st}}(V)$ from the (φ, Γ_K)-module $D^\dagger_{\log,K}(V)$.

Theorem 3.21 ([Ber02, Théorème 3.6])**.** *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}_K)$, we have*

$$D_{\mathrm{crys}}(V) = \left(D^\dagger_{\mathrm{rig},K}(V)[\tfrac{1}{t}]\right)^{\Gamma_K} \qquad \text{and} \qquad D_{\mathrm{st}}(V) = \left(D^\dagger_{\log,K}(V)[\tfrac{1}{t}]\right)^{\Gamma_K}$$

as subspaces of $(V \otimes_{\mathbb{Q}_p} \widetilde{\mathbf{C}}^\dagger[\frac{1}{t}])^{\mathrm{Gal}_K}$ and $(V \otimes_{\mathbb{Q}_p} \widetilde{\mathbf{C}}^\dagger_{\log}[\frac{1}{t}])^{\mathrm{Gal}_K}$ respectively.

[16] For general de Rham representations, one can first twist the representations by powers of cyclotomic character to make the Hodge-Tate weights non-negative.

[17] Note that, as a (φ, Γ_K)-module, $\mathbf{N}_{\mathrm{dR}}(V)$ is not necessarily étale.

The second ingredient (perhaps the most important one) is the following quasi-unipotence theorem, independently proved by André, Kedlaya, and Mebkhout (see [And02, Ked04, Meb02]). To state the theorem, let us first record the following remark on the theory of Robba rings.

Remark 3.22. Recall that $\mathbf{A}_K^\dagger$ and $\mathbf{C}_K^\dagger$ can be identified with $\mathscr{R}_{K_0'}^{\mathrm{int}}$ and $\mathscr{R}_{K_0'}$, respectively, and that $\mathbf{A}_K^\dagger/p = \mathscr{R}_{K_0'}^{\mathrm{int}}/p = \kappa(\!(\overline{\pi}_K)\!)$. By the basic theory of Robba rings, any finite separable extension $\kappa_1/\kappa(\!(\overline{\pi}_K)\!)$ can be uniquely lifted to a finite étale extension $\mathbf{A}_K^\dagger = \mathscr{R}_{K_0'}^{\mathrm{int}} \to \mathscr{R}^{\mathrm{int}\prime}$. We then obtain a finite étale extension $\mathbf{C}_K^\dagger = \mathscr{R}_{K_0'} \to \mathscr{R}'$ where $\mathscr{R}' \coloneqq \mathscr{R}_{K_0'} \otimes_{\mathscr{R}_{K_0'}^{\mathrm{int}}} \mathscr{R}^{\mathrm{int}\prime}$. It turns out $\mathscr{R}'$ is itself a Robba ring with coefficients in certain finite unramified extension of K_0'. In what follows, we often use $\mathscr{R}(\kappa_1)$ to denote $\mathscr{R}'$ to emphasize the role of κ_1. Note that the differential operator ∂ uniquely extends to $\mathscr{R}(\kappa_1)$.

Theorem 3.23 (André, Kedlaya, Mebkhout)**.** *Let (M, ∂_M) be a differential module over $(\mathbf{C}_K^\dagger, \partial)$ equipped with a Frobenius structure. Then there exists a finite separable extension $\kappa_1/\kappa(\!(\overline{\pi}_K)\!)$ with induced finite étale extension $\mathbf{C}_K^\dagger = \mathscr{R}_{K_0'} \to \mathscr{R}(\kappa_1)$ such that $M \otimes_{\mathbf{C}_K^\dagger} \mathscr{R}(\kappa_1)$ becomes unipotent, that is, it is a successive extension of trivial differential modules over $(\mathscr{R}(\kappa_1), \partial)$.*

Corollary 3.24. Let (M, ∂_M) be a differential module over $(\mathbf{C}_K^\dagger, \partial)$ equipped with a Frobenius structure. Then there exists a finite extension L/K such that $M \otimes_{\mathbf{C}_K^\dagger} \mathbf{C}_L^\dagger$ becomes unipotent. In particular, (M, ∂_M) admits a full basis of solutions in $\mathbf{C}_{\log,L}^\dagger = \mathbf{C}_L^\dagger[\log \pi]$.

Now, we are ready to finish the proof.

Theorem 3.25 (The classical p-adic monodromy theorem, [Ber02, Théorème 0.7])**.** *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\mathrm{Gal}_K)$. Then, there exists a finite extension L/K such that $V|_{\mathrm{Gal}_L}$ is a semistable representation of Gal_L.*

Proof. Without loss of generality, assume that V is a de Rham representation of dimension d with non-negative Hodge–Tate weights. Let $(\mathbf{N}_{\mathrm{dR}}(V), \partial_V)$ be the rank d differential module over $(\mathbf{C}_K^\dagger, \partial)$ with a Frobenius structure as in Theorem 3.19 and Remark 3.20. By Corollary 3.24, there exists a finite extension L/K such that

$$(\mathbf{N}_{\mathrm{dR}}(V) \otimes_{\mathbf{C}_K^\dagger} \mathbf{C}_{\log,L}^\dagger)^{\partial_V = 0}$$

has full rank. Therefore, there exists a large enough m such that, after passing to the finite extension $L_m = L(\zeta_{p^m})$ over L, the Γ_{L_m}-invariant subspace

$$(\mathbf{N}_{\mathrm{dR}}(V) \otimes_{\mathbf{C}_K^\dagger} \mathbf{C}_{\log,L_m}^\dagger)^{\Gamma_{L_m}}$$

has full rank. By Theorem 3.21 and the fact that $\mathbf{N}_{\mathrm{dR}}(V)$ coincides with $D_{\mathrm{rig},K}^\dagger(V)$ after inverting t, we conclude that $D_{\mathrm{st}}(V|_{\mathrm{Gal}_{L_m}})$ has full rank and thus $V|_{\mathrm{Gal}_{L_m}}$ is semistable. □

4. Relative period rings and relative (φ, Γ)-modules

The goal of this section is to summarize the results from [DMY26] on imperfect relative period rings and relative (φ, Γ)-modules over such rings. These generalize the (φ, Γ_K)-module theory in the classical setting. To maintain the flow of the exposition, we present only those definitions and theorems required for this article and refer the reader to *loc. cit.* for complete proofs. As this section is somewhat technical, the reader may wish to skip the technical details on a first reading and return to them later if necessary. This section is organized as follows.

- In §4.1 and §4.2, we recall the notion of toric tower and perfect relative period rings following [KL15].
- In §4.3 and §4.4, we introduce imperfect relative period rings associated with *decompleting data*, as well as the notion of (φ, Γ)-modules over such period rings. We also collect some descent results under the assumption of decompleting data, including the main descent result in [DMY26] for (φ, Γ)-modules over $\mathbf{C}$-type period rings. (cf. [DMY26, §5 – 7].)
- In §4.5 and §4.6, we extend Berger's $\mathbf{N}_{\mathrm{dR}}$ functor to the relative setting for *strongly decompleting data* (cf. [DMY26, §8 – 10]) and discuss the effect of passing to finite étale covers.

- In §4.7, we provide an explicit description of imperfect period rings in the case of (punctured) annuli (cf. [DMY26, §11].).
- Finally, in §4.8, we provide a criterion for semistability of p-adic local systems in terms of Γ-invariants of relative (φ,Γ)-modules.

Although this article only requires the theory of relative (φ,Γ)-modules over a rational subdomain of an annulus, we present the theory in a more general setting for the sake of completeness.

4.1. **Toric chart and toric tower.**

Throughout §4, let A_X be a reduced affinoid K-algebra and let $X=\mathrm{Spa}(A_X,A_X^+)$ be the associated affinoid adic space over $\mathrm{Spa}(K,\mathcal{O}_K)$. In particular, $A_X^+=A_X^\circ$. For any integer $d\geqslant 1$, let $\mathbb{D}_K^d$ denote the d-dimensional closed unit polydisk over K; that is, $\mathbb{D}_K^d\coloneqq\mathrm{Spa}(K\langle T_1,\dots,T_d\rangle,\mathcal{O}_K\langle T_1,\dots,T_d\rangle)$. Recall from [DMY26, §4.1] that a *toric chart* of X is an étale morphism

$$h\colon X\to\mathbb{D}_K^d$$

of adic spaces that is a composition of rational localizations and finite étale maps, such that the image of h does not intersect the locus $\{T_1\cdots T_d=0\}$. Let us fix such a toric chart h. For each $n\geqslant 0$, consider

$$\mathbb{D}_{K,n}^d\coloneqq\mathrm{Spa}\left(K_n\langle T_1^{\frac{1}{p^n}},\dots,T_d^{\frac{1}{p^n}}\rangle,\mathcal{O}_{K_n}\langle T_1^{\frac{1}{p^n}},\dots,T_d^{\frac{1}{p^n}}\rangle\right).$$

This is a d-dimensional closed unit polydisk over K_n, with coordinates $T_1^{\frac{1}{p^n}},\dots,T_d^{\frac{1}{p^n}}$. When the base field K and dimension d are understood, we often write $\mathbb{D}_n=\mathbb{D}_{K,n}^d$ to simplify the notation. For each $n\geqslant 0$, let $X_n\coloneqq X\times_{\mathbb{D}_K^d}\mathbb{D}_n$. The natural maps $\psi_n\colon\mathbb{D}_n\to\mathbb{D}_K^d$ induced by the inclusion of rings give rise to a tower of finite étale maps $\psi=\{\psi_n\colon X_n\to X\}$ over X, which we refer to as the *toric tower* over X relative to the toric chart h. This tower is Galois with Galois group $\Gamma\cong\Gamma_{\mathrm{geom}}\rtimes\Gamma_K$, where Γ_{geom} is the Galois group of

$$X_\infty=\varprojlim_{n\geqslant 0}X_n\longrightarrow\varprojlim_{n\geqslant 0}X\times_{\mathrm{Spa}(K,\mathcal{O}_K)}\mathrm{Spa}(K_n,\mathcal{O}_{K_n}).$$

In particular, Γ_{geom} acts on K_∞ trivially and permutes the p-power roots of T_i's. On the other hand, Γ_K acts on the p-power roots of unity via the usual cyclotomic character χ and acts trivially on the prescribed p-power roots T_i^{1/p^n}'s of T_i.[18] Note that $\Gamma_{\mathrm{geom}}\cong\mathbb{Z}_p^{\oplus d}$ and $\Gamma_K\subset\mathbb{Z}_p^\times$ is an open subgroup; for any $\sigma\in\Gamma_K$ and $\tau\in\Gamma_{\mathrm{geom}}$, we have $\sigma\tau\sigma^{-1}=\tau^{\chi(\sigma)}$. Finally, let us note that the tower $X_\infty=\varprojlim_n X_n\to X$ is an affinoid perfectoid object in the pro-étale site $X_{\mathrm{pro\acute{e}t}}$ of X with associated perfectoid space $\widehat{X}_\infty$.

Definition 4.1. (1) For every $n\geqslant 0$, write $X_n=\mathrm{Spa}(A_{X,n},A_{X,n}^+)$. Each $A_{X,n}$ is equipped with the spectral norm $|\cdot|$, which extends to a norm $|\cdot|$ on $A_{X,\infty}\coloneqq\bigcup_{n\geqslant 0}A_{X,n}$.

(2) Let $\widehat{A}_{X,\infty}$ denote the completion of $A_{X,\infty}$ with respect to the norm $|\cdot|$, and define

$$\widehat{A}_{X,\infty}^+\coloneqq\left\{x\in\widehat{A}_{X,\infty}\,\middle|\,|x|\leqslant 1\right\}.$$

Then $\widehat{A}_{X,\infty}$ is a perfectoid affinoid $\mathbb{Q}_p$-Banach algebra, $(\widehat{A}_{X,\infty},\widehat{A}_{X,\infty}^+)$ is a perfectoid Huber pair, and we have $\widehat{X}_\infty=\mathrm{Spa}(\widehat{A}_{X,\infty},\widehat{A}_{X,\infty}^+)$.

(3) Denote the tilt of $(\widehat{A}_{X,\infty},\widehat{A}_{X,\infty}^+)$ by

$$(\widetilde{R}_X,\widetilde{R}_X^+)\coloneqq(\widehat{A}_{X,\infty}^\flat,\widehat{A}_{X,\infty}^{\flat,+}).$$

We again write $|\cdot|$ for the norm on $\widetilde{R}_X$ induced from the norm $|\cdot|$ on $\widehat{A}_{X,\infty}$.

(4) Define a valuation $v\colon\widetilde{R}_X\to\mathbb{R}\cup\{+\infty\}$ by setting $v(y)\coloneqq-\log_p|y|$, extending the valuation v on $\widehat{K}_\infty^\flat\subset\widetilde{R}_X$ defined at the beginning of §3.

(5) Note that ϖ is a pseudouniformizer in $A_{X,\infty}^+$. We fix a tilt $\varpi^\flat\in\widetilde{R}_X^+$ of ϖ.

Remark 4.2. Throughout the paper, we often need to "extend the base field", that is, replacing K by a finite extension L. Whenever we do so, we implicitly replace the toric chart $h\colon X\to\mathbb{D}_K^d$ by its base change $h\colon X_L\to\mathbb{D}_L^d$. In particular, the group $\Gamma=\Gamma_{\mathrm{geom}}\rtimes\Gamma_K$ is replaced by $\Gamma_{\mathrm{geom}}\rtimes\Gamma_L$.

[18] In fact, a choice of compatible system of p-power roots of T_i's determines a splitting $\Gamma\cong\Gamma_{\mathrm{geom}}\rtimes\Gamma_K$.

4.2. **Perfect relative period rings.**

We briefly recall the constructions of perfect period rings in the relative setting following [KL15]. These are natural extensions of the classical perfect period rings reviewed in §3.1.

Definition 4.3. Let $h\colon X \to \mathbb{D}_K^d$ be a toric chart.

(1) Define $\widetilde{\mathbf{A}}_X \coloneqq W(\widetilde{R}_X)$ and $\widetilde{\mathbf{A}}_X^+ \coloneqq W(\widetilde{R}_X^+)$.
(2) For each $r > 0$, we define a norm λ_r on $\widetilde{\mathbf{A}}_X^+$ by $\lambda_r\left(\sum_{n=0}^\infty p^n[x_n]\right) = \sup_n \{p^{-n}|x_n|^r\}$, which extends the norm λ_r on $\widetilde{\mathbf{A}}_K^+$.
(3) For each $r > 0$, define $\widetilde{\mathbf{A}}_X^r$ to be the completion of $\widetilde{\mathbf{A}}_X^+[[\varpi^\flat]^{-1}]$ under the norm λ_r. Equivalently,
$$\widetilde{\mathbf{A}}_X^r \coloneqq \left\{ \sum_{n=0}^\infty p^n[x_n] \in \widetilde{\mathbf{A}}_X \;\middle|\; p^{-n}|x_n|^r \to 0 \text{ as } n \to +\infty \right\}.$$
Let $\widetilde{\mathbf{A}}_X^\dagger$ be the union $\widetilde{\mathbf{A}}_X^\dagger \coloneqq \varinjlim_{r>0} \widetilde{\mathbf{A}}_X^r$.
(4) The period rings $\widetilde{\mathbf{B}}_X$, $\widetilde{\mathbf{B}}_X^r$, and $\widetilde{\mathbf{B}}_X^\dagger$ are obtained from the corresponding period rings of type $\mathbf{A}$ with p inverted.
(5) For every $0 < s \leqslant r$, define $\widetilde{\mathbf{C}}_X^{[s,r]}$ to be the completion of the period ring $\widetilde{\mathbf{B}}_X^r$ with respect to $\max\{\lambda_s, \lambda_r\}$ (or equivalently, completion with respect to the family of norms $\{\lambda_t\}_{t\in[s,r]}$).
(6) For every $r > 0$, define $\widetilde{\mathbf{C}}_X^r$ to be the Fréchet completion of $\widetilde{\mathbf{B}}_X^r$ with respect to the family of norms $\{\lambda_s\}_{s\in(0,r]}$. Finally, define $\widetilde{\mathbf{C}}_X^\dagger$ to be the union $\widetilde{\mathbf{C}}_X^\dagger \coloneqq \varinjlim_{r>0} \widetilde{\mathbf{C}}_X^r$.

Remark 4.4 (Topology)**.** We endow $\widetilde{\mathbf{A}}_X^r$ with the topology induced by the λ_r-norm and endow $\widetilde{\mathbf{A}}_X^\dagger$ with the inductive limit topology (i.e., the finest topology making $\widetilde{\mathbf{A}}_X^r \hookrightarrow \widetilde{\mathbf{A}}_X^\dagger$ continuous for all $r > 0$). Inverting p, we obtain natural topologies on $\widetilde{\mathbf{B}}_X^r$ and $\widetilde{\mathbf{B}}_X^\dagger$. The ring $\widetilde{\mathbf{C}}_X^{[s,r]}$ is endowed with the topology induced by $\max\{\lambda_r, \lambda_s\}$, which then induces a Fréchet topology on $\widetilde{\mathbf{C}}_X^r$. Finally, we equip $\widetilde{\mathbf{C}}_X^\dagger$ with the inductive limit topology (i.e., the LF topology).

Remark 4.5.
(i) Strictly speaking, the perfect period rings $\widetilde{\mathbf{A}}_X$, $\widetilde{\mathbf{A}}_X^r$ etc. depend on the choice of the toric chart $h\colon X \to \mathbb{D}_K^d$. In the cases relevant to this article, however, the toric chart is evident from the context.
(ii) Remark 3.4 applies in the relative setting. In other words, the Frobenius maps on $\widetilde{\mathbf{A}}_X^+$ and $\widetilde{\mathbf{A}}_X$ extend to Frobenius structures on $\widetilde{\mathbf{A}}_X^r$, $\widetilde{\mathbf{A}}_X^\dagger$, $\widetilde{\mathbf{B}}_X$, $\widetilde{\mathbf{B}}_X^r$, $\widetilde{\mathbf{B}}_X^\dagger$, $\widetilde{\mathbf{C}}_X^{[s,r]}$, $\widetilde{\mathbf{C}}_X^r$, and $\widetilde{\mathbf{C}}_X^\dagger$, which we again denote as φ.
(iii) The natural Γ-action on $\widetilde{R}_X$ induces an action on $\widetilde{\mathbf{A}}_X$. Notice that the λ_r-norm is preserved under the Γ-action. Consequently, for every $r > 0$, the subspace $\widetilde{\mathbf{A}}_X^r \subset \widetilde{\mathbf{A}}_X$ is stable under the Γ-action and the action is isometric with respect to λ_r. This in turn induces a continuous Γ-action on $\widetilde{\mathbf{A}}_X^\dagger$. Similarly, there are natural isometric (resp. continuous) Γ-actions on $\widetilde{\mathbf{B}}_X^r$ and $\widetilde{\mathbf{C}}_X^{[s,r]}$ (resp. on $\widetilde{\mathbf{B}}_X^\dagger$, $\widetilde{\mathbf{C}}_X^r$, and $\widetilde{\mathbf{C}}_X^\dagger$).

4.3. **Imperfect relative period rings, I: decompleting data.**

As mentioned in the introduction, our construction of imperfect relative period rings is phrased in terms of *decompleting data* introduced in [DMY26]. Given a toric chart (X, h) as in §4.1, a *decompleting datum* consists of a pair $\Lambda = (R_\Lambda, \mathbf{A}_\Lambda)$, where

- $R_\Lambda \subset \widetilde{R}_X$ is a φ-stable $\kappa(\!(\overline{\pi}_K)\!)$-subalgebra such that R_Λ is complete with respect to the norm $|\cdot|$ inherited from $\widetilde{R}_X$;
- $\mathbf{A}_\Lambda \subset \widetilde{\mathbf{A}}_X$ is a φ-stable $\mathbf{A}_K$-subalgebra, which is complete with respect to the p-adic topology, such that the image of the composition
$$\mathbf{A}_\Lambda/p \to \widetilde{\mathbf{A}}_X/p \cong \widetilde{R}_X$$
lies in R_Λ;

satisfying an extensive list of axioms; in particular, they acquire (φ,Γ)-actions. Since most of the axioms are of technical nature, we refer the reader to [DMY26, Definitions 5.6 & 6.3] for the precise definition, noting only that R_Λ and $\mathbf{A}_\Lambda$ intuitively play the roles of $\mathbf{E}_K$ and $\mathbf{A}_K$ in the classical setting (cf. §3.2). From such a pair $(R_\Lambda,\mathbf{A}_\Lambda)$, one can further build imperfect relative period rings

$$\mathbf{A}^r_\Lambda,\mathbf{A}^\dagger_\Lambda,\mathbf{B}^r_\Lambda,\mathbf{B}^\dagger_\Lambda,\mathbf{C}^{[s,r]}_\Lambda,\mathbf{C}^r_\Lambda,\mathbf{C}^\dagger_\Lambda$$

for $r>0$ sufficiently small and for $0<s\leqslant r$, all equipped with suitable (φ,Γ)-actions, in a fashion similar to Definition 3.6. For the convenience of the reader, let us recall the definition.

Definition 4.6. Let $\Lambda=(R_\Lambda,\mathbf{A}_\Lambda)$ be a decompleting datum for (X,h).

(1) For any $r>0$, let $\mathbf{A}^r_\Lambda\coloneqq\mathbf{A}_\Lambda\cap\widetilde{\mathbf{A}}^r_X$ equipped with the inherited λ_r-norm from $\widetilde{\mathbf{A}}^r_X$. Let $\mathbf{B}^r_\Lambda\coloneqq\mathbf{A}^r_\Lambda[\frac{1}{p}]$.
(2) Let $\mathbf{A}^\dagger_\Lambda$ be the union $\mathbf{A}^\dagger_\Lambda\coloneqq\varinjlim_{r>0}\mathbf{A}^r_\Lambda$ and $\mathbf{B}^\dagger_\Lambda\coloneqq\mathbf{A}^\dagger_\Lambda[\frac{1}{p}]$. It follows that $\mathbf{A}^\dagger_\Lambda=\mathbf{A}_\Lambda\cap\widetilde{\mathbf{A}}^\dagger_X$.
(3) For $0<s\leqslant r$, let $\mathbf{C}^{[s,r]}_\Lambda$ be the completion of $\mathbf{B}^r_\Lambda$ with respect to $\max\{\lambda_r,\lambda_s\}$ (or equivalently, completion with respect to the family of norms $\{\lambda_t\}_{t\in[s,r]}$). Let $\mathbf{C}^r_\Lambda$ be the Fréchet completion of $\mathbf{B}^r_\Lambda$ with respect to the family of norms $\{\lambda_s\}_{s\in(0,r]}$ and let $\mathbf{C}^\dagger_\Lambda\coloneqq\varinjlim_{r>0}\mathbf{C}^r_\Lambda$.

These imperfect relative period rings form relative analogues of the classical imperfect period rings $\mathbf{A}^r_K$, $\mathbf{A}^\dagger_K,\mathbf{B}^r_K,\mathbf{B}^\dagger_K,\mathbf{C}^{[s,r]}_K,\mathbf{C}^r_K,\mathbf{C}^\dagger_K$. For later use, let us record several properties enjoyed by the imperfect relative period rings associated with a decompleting datum.

4.3.1. *(φ,Γ)-structures.* The Frobenius map $\varphi\colon\widetilde{\mathbf{A}}^r_X\xrightarrow{\sim}\widetilde{\mathbf{A}}^{r/p}_X$ induces a Frobenius $\varphi\colon\mathbf{A}^r_\Lambda\to\mathbf{A}^{r/p}_\Lambda$ for all r. Similarly, we have Frobenius maps $\varphi\colon\mathbf{B}^r_\Lambda\to\mathbf{B}^{r/p}_\Lambda,\varphi\colon\mathbf{C}^r_\Lambda\to\mathbf{C}^{r/p}_\Lambda,\varphi\colon\mathbf{C}^{[s,r]}_\Lambda\to\mathbf{C}^{[s/p,\,r/p]}_\Lambda$. Moreover, $\mathbf{A}^\dagger_\Lambda,\mathbf{B}^\dagger_\Lambda,\mathbf{C}^\dagger_\Lambda$ are all stable under Frobenius. Moreover, all of the imperfect period rings $\mathbf{A}^r_\Lambda,\mathbf{A}^\dagger_\Lambda,\mathbf{B}_\Lambda,\mathbf{B}^r_\Lambda,\mathbf{B}^\dagger_\Lambda$, $\mathbf{C}^{[s,r]}_\Lambda$, $\mathbf{C}^r_\Lambda,\mathbf{C}^\dagger_\Lambda$ are stable under the Γ-actions.

4.3.2. *Topology.* The imperfect period ring $\mathbf{A}^r_\Lambda$ (resp. $\mathbf{A}^\dagger_\Lambda$, $\mathbf{B}^r_\Lambda$, $\mathbf{B}^\dagger_\Lambda$, $\mathbf{C}^{[s,r]}_\Lambda$, $\mathbf{C}^r_\Lambda$, $\mathbf{C}^\dagger_\Lambda$) is naturally identified as a subring of $\widetilde{\mathbf{A}}^r_X$ (resp. $\widetilde{\mathbf{A}}^\dagger_X$, $\widetilde{\mathbf{B}}^r_X$, $\widetilde{\mathbf{B}}^\dagger_X$, $\widetilde{\mathbf{C}}^{[s,r]}_X$, $\widetilde{\mathbf{C}}^r_X$, $\widetilde{\mathbf{C}}^\dagger_X$). We equip these period rings with the corresponding subspace topologies (cf. Remark 4.4).

4.3.3. *Affinoid structure.* Part of the axioms requires that, for r sufficiently small and for each $0<s\leqslant r$ with $s,r\in\mathbb{Q}$, the Banach K_0'-algebra $\mathbf{C}^{[s,r]}_\Lambda$ is a reduced affinoid K_0'-algebra. Moreover, the norm $\max\{\lambda_r,\lambda_s\}$ is a complete power-multiplicative K_0'-algebra norm on $\mathbf{C}^{[s,r]}_\Lambda$, and agrees with the spectral norm on $\mathbf{C}^{[s,r]}_\Lambda$.

4.3.4. *Intersection properties.* There exists a sufficiently small $r_0>0$ such that the following statements hold.

(1) For any $r\in(0,r_0]$, the natural map $\mathbf{C}^r_\Lambda\to\varprojlim_{0<s\leqslant r}\mathbf{C}^{[s,r]}_\Lambda$ is an isomorphism.
(2) $\widetilde{\mathbf{A}}^s_X\cap\mathbf{C}^{[s,r]}_\Lambda=\mathbf{A}^r_\Lambda$, for any $0<s\leqslant r\leqslant r_0$, where the intersection is taken inside $\widetilde{\mathbf{C}}^\dagger_X$. Similarly, $\widetilde{\mathbf{A}}^\dagger_X\cap\mathbf{C}^\dagger_\Lambda=\mathbf{A}^\dagger_\Lambda$ where the intersection is taken inside $\widetilde{\mathbf{C}}^\dagger_X$.
(3) We also have $\mathbf{C}^{[s,r]}_\Lambda\cap\mathbf{C}^{[s',r']}_\Lambda=\mathbf{C}^{[s,r']}_\Lambda$ for any $0<s\leqslant s'\leqslant r\leqslant r'<r_0$, where the intersections are taken inside $\mathbf{C}^{[s',r]}_\Lambda$.

We refer the reader to [DMY26, Propositions 5.25 & 5.26] for more details.

4.3.5. *Normalized differential operators.* Recall the splitting $\Gamma\cong\Gamma_{\mathrm{geom}}\rtimes\Gamma_K$ as in §4.1 and choose topological generators $\tau_1,\dots,\tau_d\in\Gamma_{\mathrm{geom}}$ such that $\tau_i(T_i^{\frac{1}{p^n}})=\zeta_{p^n}T_i^{\frac{1}{p^n}}$ for each i and $\tau_i(T_j^{\frac{1}{p^n}})=T_j^{\frac{1}{p^n}}$ for all $j\neq i$. This determines an isomorphism $\eta=(\eta_1,\dots,\eta_d)\colon\Gamma_{\mathrm{geom}}\xrightarrow{\sim}\mathbb{Z}_p^d$. For r sufficiently small and for each $0<s\leqslant r$, $\mathbf{C}^{[s,r]}_\Lambda$ is equipped with $(d+1)$ differential operators

$$\nabla_0\coloneqq\frac{\log(\gamma_0)}{\log\chi(\gamma_0)},\qquad\nabla_i\coloneqq\frac{\log(\tau_i^{p^m})}{\eta_i(\tau_i^{p^m})}=p^{-m}\cdot\log(\tau_i^{p^m}),$$

which are well-defined for $\gamma_0\in\Gamma_K$ sufficiently close to 1 and for m sufficiently large. Passing to the limit, we obtain differential operators $\nabla_0,\nabla_1,\dots,\nabla_d$ on $\mathbf{C}^r_\Lambda$ and $\mathbf{C}^\dagger_\Lambda$. Let us further define the following normalized differential operators

$$d_0\coloneqq t^{-1}\cdot\nabla_0\quad\text{ and }\quad d_i\coloneqq t^{-1}\cdot\nabla_i,\ \ i=1,\dots,d$$

on these rings.

4.4. Relative φ-modules and (φ,Γ)-modules over imperfect relative period rings.

The axioms of a decompleting datum ensure that there is a robust theory of finite projective (φ,Γ)-modules over these imperfect relative period rings. Throughout §4.4, let $\Lambda=(R_\Lambda,\mathbf{A}_\Lambda)$ be a decompleting datum for a toric chart (X,h). Let $\mathbf{A}_\Lambda^r,\mathbf{A}_\Lambda^\dagger,\mathbf{B}_\Lambda^r,\mathbf{B}_\Lambda^\dagger,\mathbf{C}_\Lambda^{[s,r]},\mathbf{C}_\Lambda^r,\mathbf{C}_\Lambda^\dagger$ be the associated imperfect period rings. We make precise the notion of φ-modules and (φ,Γ)-modules over these rings.

Definition 4.7. (1) Let $\mathbf{D}^r$ be one of the rings from $\{\mathbf{A}_\Lambda^r,\mathbf{B}_\Lambda^r,\mathbf{C}_\Lambda^r,\widetilde{\mathbf{A}}_X^r,\widetilde{\mathbf{B}}_X^r,\widetilde{\mathbf{C}}_X^r\}$ for some $r>0$. A *φ-module* over $\mathbf{D}^r$ is a finite projective $\mathbf{D}^r$-module M together with an isomorphism

$$\varphi_M\colon M\otimes_{\mathbf{D}^r,\varphi}\mathbf{D}^{r/p}\xrightarrow{\sim}M\otimes_{\mathbf{D}^r,\iota}\mathbf{D}^{r/p}$$

of $\mathbf{D}^{r/p}$-modules, where $\varphi\colon\mathbf{D}^r\to\mathbf{D}^{r/p}$ is the Frobenius map (cf. §4.3), and ι is the inclusion map.

(2) Similarly, let $\mathbf{D}$ be one of the rings from $\{\mathbf{A}_\Lambda,\mathbf{A}_\Lambda^\dagger,\mathbf{B}_\Lambda,\mathbf{B}_\Lambda^\dagger,\mathbf{C}_\Lambda^\dagger,\widetilde{\mathbf{A}}_X,\widetilde{\mathbf{A}}_X^\dagger,\widetilde{\mathbf{B}}_X,\widetilde{\mathbf{B}}_X^\dagger,\widetilde{\mathbf{C}}_X^\dagger\}$. A *$\varphi$-module* over $\mathbf{D}$ is a finite projective $\mathbf{D}$-module M together with an isomorphism $\varphi_M\colon M\otimes_{\mathbf{D},\varphi}\mathbf{D}\xrightarrow{\sim}M$ of $\mathbf{D}$-modules.

Definition 4.8. (1) Let $\mathbf{D}$ be one of the rings from Definition 4.7. A *Γ-module* over $\mathbf{D}$ is a finite projective $\mathbf{D}$-module equipped with a continuous semilinear action of Γ.[19] A *(φ,Γ)-module* over $\mathbf{D}$ is a φ-module M over $\mathbf{D}$, equipped with a continuous semilinear action of Γ that commutes with the action of φ.

(2) A φ-module (resp. (φ,Γ)-module) over one of the following rings $\{\mathbf{B}_\Lambda,\mathbf{B}_\Lambda^\dagger,\mathbf{C}_\Lambda^\dagger,\widetilde{\mathbf{B}}_X,\widetilde{\mathbf{B}}_X^\dagger,\widetilde{\mathbf{C}}_X^\dagger\}$ is *étale* if it is the base change of a φ-module (resp. (φ,Γ)-module) over $\{\mathbf{A}_\Lambda,\mathbf{A}_\Lambda^\dagger,\mathbf{A}_\Lambda^\dagger,\widetilde{\mathbf{A}}_X,\widetilde{\mathbf{A}}_X^\dagger,\widetilde{\mathbf{A}}_X^\dagger\}$ respectively.

(3) Write $\mathrm{Mod}_{/\mathbf{D}}^{\varphi}$ (resp. $\mathrm{Mod}_{/\mathbf{D}}^{\Gamma}$, resp. $\mathrm{Mod}_{/\mathbf{D}}^{(\varphi,\Gamma)}$) for the category of φ-modules (resp. the category of Γ-modules, resp. the category of (φ,Γ)-modules) over $\mathbf{D}$. Similarly, write $\mathrm{Mod}_{/\mathbf{D}}^{\varphi,\text{ét}}$ (resp. $\mathrm{Mod}_{/\mathbf{D}}^{(\varphi,\Gamma),\text{ét}}$) for the category of étale φ-modules (resp. the category of étale (φ,Γ)-modules) over $\mathbf{D}$ whenever appropriate.

Remark 4.9. Our notion of étale (φ,Γ)-modules requires the corresponding lattice over period rings of type $\mathbf{A}$ to be stable under both φ and Γ. This is similar to [CC98, Definition I.4.2]. In contrast, [KL19] only requires the lattice to be stable under φ. In other words, the definition we use is more restrictive, but seems more suitable for the purpose of decompletion and descent.

The following result is one of the main results of [DMY26] (cf. Theorem 10).

Theorem 4.10 ([DMY26, Theorem 1, Theorem 7.18, Corollary 7.20])**.** *Given a decompleting datum $\Lambda=(R_\Lambda,\mathbf{A}_\Lambda)$ for (X,h), the base change functors*

$$\mathrm{Mod}_{/\mathbf{A}_\Lambda^\dagger}^{(\varphi,\Gamma)}\longrightarrow\mathrm{Mod}_{/\widetilde{\mathbf{A}}_X^\dagger}^{(\varphi,\Gamma)},\qquad\mathrm{Mod}_{/\mathbf{B}_\Lambda^\dagger}^{(\varphi,\Gamma),\text{ét}}\longrightarrow\mathrm{Mod}_{/\widetilde{\mathbf{B}}_X^\dagger}^{(\varphi,\Gamma),\text{ét}},\qquad\mathrm{Mod}_{/\mathbf{C}_\Lambda^\dagger}^{(\varphi,\Gamma),\text{ét}}\longrightarrow\mathrm{Mod}_{/\widetilde{\mathbf{C}}_X^\dagger}^{(\varphi,\Gamma),\text{ét}}$$

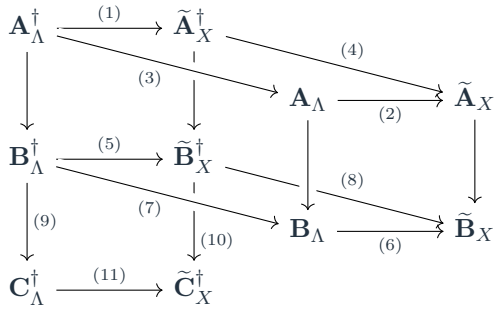
are all equivalences of categories. Moreover, the same statements hold if we replace Γ by an open subgroup.

Remark 4.11. Given a decompleting datum $\Lambda=(R_\Lambda,\mathbf{A}_\Lambda)$ for (X,h), we frequently need to understand the behavior of the base change functors for (φ,Γ)-modules across the following diagram of period rings (for

[19] For a finite projective $\mathbf{D}$-module M, we may identify M as a direct summand of some finite free $\mathbf{D}$-module and equip M with the subspace topology. This topology is independent of the choice of the presentation. On the other hand, Γ is equipped with the usual profinite topology.

instance, whether these functors are equivalences or fully faithful).

(4.1)



The horizontal arrows correspond to so-called "decompletion questions" while the diagonal arrows correspond to "overconvergence questions". The theorem above provides equivalences of categories of (étale) (φ,Γ)-modules across the arrows (1), (5), (11) in the diagram.

- For arrows (2) and (4), we have equivalences $\mathrm{Mod}^{(\varphi,\Gamma)}_{/\mathbf{A}_\Lambda} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma)}_{/\widetilde{\mathbf{A}}_X}$ and $\mathrm{Mod}^{(\varphi,\Gamma)}_{/\widetilde{\mathbf{A}}^\dagger_X} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma)}_{/\widetilde{\mathbf{A}}_X}$;
- For arrows (6) and (8), we also have $\mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\mathbf{B}_\Lambda} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\widetilde{\mathbf{B}}_X}$ and $\mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\widetilde{\mathbf{B}}^\dagger_X} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\widetilde{\mathbf{B}}_X}$;
- For arrow (9), we have a fully faithful functor $\mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\mathbf{B}^\dagger_\Lambda} \longrightarrow \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\mathbf{C}^\dagger_\Lambda}$;
- Arrow (10) induces an equivalence $\mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\widetilde{\mathbf{B}}^\dagger_X} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\widetilde{\mathbf{C}}^\dagger_X}$.

Notably, with the exception of arrows (1), (5), and (11) (which are established by Theorem 4.10), all preceding equivalences and full faithfulness properties remain valid under milder hypotheses on Λ, formalized as *weakly decompleting data* in [DMY26]. In fact, some of descent assertions already hold for (étale) φ-modules. On the other hand, the proof of the essential surjectivity of the functor on étale (φ,Γ)-modules along arrow (11) crucially relies on Γ-structures and is a key innovation of *loc. cit.* We refer the reader to [DMY26, §7] for a comprehensive list of such descent statements as well as their proofs.

Remark 4.12. It is unclear whether the functor $\mathrm{Mod}^{(\varphi,\Gamma)}_{/\mathbf{C}^\dagger_\Lambda} \longrightarrow \mathrm{Mod}^{(\varphi,\Gamma)}_{/\widetilde{\mathbf{C}}_X}$ (on general, not necessarily étale, (φ,Γ)-modules) is fully faithful or essentially surjective. Nevertheless, we have the following partial result for a decompleting datum $\Lambda = (R_\Lambda, \mathbf{A}_\Lambda)$: a descent of a finite free (φ,Γ)-module over $\widetilde{\mathbf{C}}^\dagger_X$ to a (φ,Γ)-module over $\mathbf{C}^\dagger_\Lambda$, if exists, is necessarily unique. This is [DMY26, Theorem 7.17].

Remark 4.13 (Relation with local systems)**.** Let $\mathrm{Loc}_{\mathbb{Z}_p}(X)$ denote the category of étale $\mathbb{Z}_p$-local systems on X and let $\mathrm{Loc}_{\mathbb{Z}_p}(X)_{\mathbb{Q}}$ denote its isogeny category. By [KL15, Theorem 8.5.5], there is an equivalence of categories

$$\widetilde{D}^{\mathrm{int}} : \mathrm{Loc}_{\mathbb{Z}_p}(X) \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma)}_{/\widetilde{\mathbf{A}}_X}$$

which reduces to the Riemann–Hilbert correspondence of Katz after reducing modulo p (cf. [Kat73, Proposition 4.1.1], also see [BS23, Proposition 3.4]). Consequently, under the assumption of decompleting data, $\mathrm{Loc}_{\mathbb{Z}_p}(X)$ (resp. $\mathrm{Loc}_{\mathbb{Z}_p}(X)_{\mathbb{Q}}$) is equivalent to the categories of (φ,Γ)-modules across all the arrows (1)-(4) (resp. categories of étale (φ,Γ)-modules across all the arrows (5)-(11)). In particular, we obtain equivalences of categories

$$\begin{aligned}
\widetilde{D}^{\mathrm{int},\dagger} &: \mathrm{Loc}_{\mathbb{Z}_p}(X) \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma)}_{/\widetilde{\mathbf{A}}^\dagger_X}.\\
D^{\mathrm{int},\dagger} &: \mathrm{Loc}_{\mathbb{Z}_p}(X) \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma)}_{/\mathbf{A}^\dagger_\Lambda}\\
\widetilde{D}^\dagger_{\mathrm{rig}} &: \mathrm{Loc}_{\mathbb{Z}_p}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\widetilde{\mathbf{C}}^\dagger_X}.\\
D^\dagger_{\mathrm{rig}} &: \mathrm{Loc}_{\mathbb{Z}_p}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Mod}^{(\varphi,\Gamma),\text{ét}}_{/\mathbf{C}^\dagger_\Lambda}
\end{aligned}$$

where the last equivalence can be viewed as a relative version of the base change functor $D^\dagger_{\mathrm{rig},K}$ in §3.5.

4.5. **Imperfect relative period rings, II: strongly decompleting data.**

Going one step further, we need a more restrictive condition called *strongly decompleting data*, which is a decompleting datum $\Lambda = (R_\Lambda, \mathbf{A}_\Lambda)$ that satisfies an additional set of axioms. We refer the reader to [DMY26, Definition 8.8] for the precise definition. These axioms will allow us to extend Berger's $\mathbf{N}_{\mathrm{dR}}$ functor to the relative setting.

Theorem 4.14 ([DMY26, Theorem 2])**.** *Given a strongly decompleting datum $\Lambda = (R_\Lambda, \mathbf{A}_\Lambda)$ for (X, h), there is a natural $\otimes$-functor*

$$\mathbf{N}_{\mathrm{dR}} \colon \mathrm{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Mod}^{(\varphi,\Gamma)}_{/\mathbf{C}^\dagger_\Lambda}$$

from the isogeny category of de Rham $\mathbb{Z}_p$-local systems to the category of finite projective (φ, Γ)-modules over $\mathbf{C}^\dagger_\Lambda$, such that the following properties hold for every $\mathbb{L} \in \mathrm{Loc}^{\mathrm{dR}}_{\mathbb{Z}_p}(X)_{\mathbb{Q}}$:

1. *$\mathbf{N}_{\mathrm{dR}}(\mathbb{L})[1/t]$ coincides with $D^\dagger_{\mathrm{rig}}(\mathbb{L})[1/t]$ after inverting Fontaine's period t;*
2. *$\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ is equipped with commuting differential operators $D_0, D_1, \ldots, D_d$, making $(\mathbf{N}_{\mathrm{dR}}(\mathbb{L}), D_i)$ a differential module over the differential ring $(\mathbf{C}^\dagger_\Lambda, d_i)$ for all $i = 0, \ldots, d$, where d_i is the normalized differential operator on $\mathbf{C}^\dagger_\Lambda$ defined in §4.3.5.*

Remark 4.15. In the case where X is a suitable affinoid subdomain of a closed annulus, we will give an explicit description of the differential operators on $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ (cf. §4.7). In particular, as we shall see, $\mathbf{C}^\dagger_\Lambda$ carries the "correct" differential operators $\frac{d}{d\pi_K}$ and $\frac{d}{d[T^\flat]}$, up to units in the imperfect relative period rings. See Remark 4.28.

4.6. **Imperfect relative period rings, III: passing to finite étale extensions.**

Next, let us briefly discuss the behavior of (strongly) decompleting data under finite étale extensions. Let $h \colon X \to \mathbb{D}^d_K$ be a toric chart and let $\Lambda = (R_\Lambda, \mathbf{A}_\Lambda)$ be a decompleting datum for (X, h). Consider a finite étale map $g \colon X' \to X$. Let $h' \colon X' \to \mathbb{D}^d_K$ be the composition $h \circ g$. Note that the finite étale map $g \colon X' \to X$ induces a finite étale morphism $\widehat{A}_{X,\infty} \to \widehat{A}_{X',\infty}$, which corresponds to some $R' \in (R_\Lambda)_{\mathrm{fét}}$ and $\mathbf{A}' \in (\mathbf{A}_\Lambda)_{\mathrm{fét}}$ via the following natural equivalences of categories of finite étale algebras

$$(\mathbf{A}_\Lambda)_{\mathrm{fét}} \cong (R_\Lambda)_{\mathrm{fét}} \cong \left(\widehat{A}_{X,\infty}\right)_{\mathrm{fét}}$$

from [DMY26, Proposition 5.28].

Proposition 4.16 ([DMY26, Theorem 5.30, Theorem 8.13])**.** If $\Lambda = (R_\Lambda, \mathbf{A}_\Lambda)$ is a decompleting (resp. strongly decompleting) datum for (X, h), then $(R', \mathbf{A}')$ is a decompleting (resp. strongly decompleting) datum for (X', h').

Remark 4.17 (Passing to p-power Kummer cover)**.** For the proof of Theorem 1, we often need to "replace X by a p-power Kummer cover". More precisely, assume that K contains ζ_{p^n} for an integer $n \geqslant 1$ and we consider p^n-th Kummer cover $g \colon X_n \to X$. The new toric chart becomes $h' \colon X_n \to \mathbb{D}^d_K$ which is the composition of $g \colon X_n \to X$ and $h \colon X \to \mathbb{D}^d_K$ while Γ stays the same. Note that $\Gamma_n \coloneqq \mathrm{Gal}(X_\infty / X_n)$ is an open normal subgroup of Γ of index p^{nd}. A key observation is that

$$(X_n)_\infty = X_n \times_{h', \mathbb{D}^d_K} \mathbb{D}_\infty \cong X_\infty \times (\mathbb{Z}/p^n\mathbb{Z})^d$$

is isomorphic to p^{nd} copies of X_∞. Moreover, the Γ-action on $(X_n)_\infty$ is given by the action of Γ_n on X_∞ and the finite subgroup $\Gamma/\Gamma_n \cong (\mathbb{Z}/p^n\mathbb{Z})^d$ permutes the p^{nd} copies of X_∞. If $\Lambda = (R_\Lambda, \mathbf{A}_\Lambda)$ is a strongly decompleting datum for (X, h) and $(R', \mathbf{A}')$ is the associated strongly decompleting datum for (X_n, h'), then R' (resp. $\mathbf{A}'$) is isomorphic to the direct product of p^{nd} copies of R_Λ (resp. $\mathbf{A}_\Lambda$). The Γ-action on R' (resp. $\mathbf{A}'$) is given by the Γ_n-action on R_Λ (resp. $\mathbf{A}_\Lambda$) and the finite subgroup $\Gamma/\Gamma_n \cong (\mathbb{Z}/p^n\mathbb{Z})^d$ permutes the p^{nd} copies. The same assertions hold for all the induced perfect and imperfect relative period rings.

4.7. **Examples of strongly decompleting data.**

In this subsection, we recall from [DMY26, §11] two important classes of examples of strongly decompleting data: one for closed annuli and the other for "closed annuli with punctures" (more precisely, closed annuli of outer radius 1 with finitely many open unit disks removed from the boundary circle $|T| = 1$). In particular, we provide explicit descriptions of the associated relative period rings, which will play essential roles in the remainder of the article.

4.7.1. *Closed annuli.*

We first consider the case of closed annuli. Let $[\sigma_1, \sigma_2] \subset (0, 1]$ be a closed interval with $\sigma_1, \sigma_2 \in p^{\mathbb{Q}}$, and consider the closed annulus $A_K([\sigma_1, \sigma_2]; T)$. Let $\mathbb{D}_K$ be the closed unit disk and take the natural inclusion $\iota\colon A_K([\sigma_1, \sigma_2]; T) \hookrightarrow \mathbb{D}_K$ to be the toric chart. To simplify the notation, in the subscripts of period rings we write $[\sigma_1, \sigma_2]$ instead of $A_K([\sigma_1, \sigma_2]; T)$. The constructions in §4.2 produce perfectoid rings $\widehat{A}_{[\sigma_1,\sigma_2],\infty}$ and $\widetilde{R}_{[\sigma_1,\sigma_2]}$ and yield perfect period rings

$$\widetilde{\mathbf{A}}_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{A}}^r_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{A}}^\dagger_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{B}}_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{B}}^r_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{B}}^\dagger_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{C}}^{[s,r]}_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{C}}^r_{[\sigma_1,\sigma_2]}, \widetilde{\mathbf{C}}^\dagger_{[\sigma_1,\sigma_2]}.$$

Explicitly, let $T^\flat := (T, T^{\frac{1}{p}}, T^{\frac{1}{p^2}}, \dots) \in \widetilde{R}_{\mathbb{D}_K}$ using the prescribed p-power roots of T, then $\widetilde{R}_{[\sigma_1,\sigma_2]}$ can be identified as the rational localization of

$$\widetilde{R}_{\mathbb{D}_K} = \widehat{K}^\flat_\infty\langle (T^\flat)^{\frac{1}{p^\infty}}\rangle$$

given by $\sigma_1 \leqslant |T^\flat| \leqslant \sigma_2$; namely, $\widetilde{R}_{[\sigma_1,\sigma_2]}$ consists of $\sum_{j\in\mathbb{Z}[\frac{1}{p}]} a_j (T^\flat)^j$ with $a_j \in \widehat{K}^\flat_\infty$ such that for any $\varepsilon > 0$, there are only finitely many indices j such that $|a_j| \cdot \sigma_1^j \geqslant \varepsilon$ and likewise only finitely many j satisfying $|a_j| \cdot \sigma_2^j \geqslant \varepsilon$. All these period rings are equipped with commuting actions of φ and $\Gamma \cong \Gamma_{\mathrm{geom}} \rtimes \Gamma_K$. The choices of p-power roots of T and p-power roots of unity induce an isomorphism $\eta\colon \Gamma_{\mathrm{geom}} \cong \mathbb{Z}_p$ and the cyclotomic character $\chi\colon \Gamma_K \hookrightarrow \mathbb{Z}_p^\times$. Let $[T^\flat]$ denote the Teichmüller lift of $T^\flat$ in $\widetilde{\mathbf{A}}_{[\sigma_1,\sigma_2]} = W(\widetilde{R}_{[\sigma_1,\sigma_2]})$.

Definition 4.18. (1) Let $R_{[\sigma_1,\sigma_2]} = \Gamma_{\kappa((\overline{\pi}_K))}([\sigma_1, \sigma_2]; T^\flat)$ (cf. §1.9) as a subring of $\widetilde{R}_{[\sigma_1,\sigma_2]}$. In other words, $R_{[\sigma_1,\sigma_2]}$ is the ring of functions on the closed annulus

$$A_{\kappa((\overline{\pi}_K))}([\sigma_1, \sigma_2]; T^\flat) \subset \mathbb{D}_{\kappa((\overline{\pi}_K))}.$$

(2) For any $0 < r < r_K$, define $\mathbf{A}^r_{[\sigma_1,\sigma_2]}$ to be the ring of formal sums $\sum_{i\in\mathbb{Z}} b_i[T^\flat]^i$ with $b_i \in \mathbf{A}^r_K$ such that $\lambda_r(b_i) \cdot \sigma^{ri} \to 0$ as $i \to \pm\infty$, for all $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. Define a norm $\tilde{\lambda}_r$ on $\mathbf{A}^r_{[\sigma_1,\sigma_2]}$ by

$$\tilde{\lambda}_r\left(\sum_{i\in\mathbb{Z}} b_i[T^\flat]^i\right) = \sup_{\sigma\in[\sigma_1,\sigma_2]} \sup_{i\in\mathbb{Z}} \left\{\lambda_r(b_i) \cdot \sigma^{ri}\right\}.$$

(3) Let $\mathbf{A}^\dagger_{[\sigma_1,\sigma_2]}$ be the union $\mathbf{A}^\dagger_{[\sigma_1,\sigma_2]} := \varinjlim_{0<r<r_K} \mathbf{A}^r_{[\sigma_1,\sigma_2]}$ and let $\mathbf{A}_{[\sigma_1,\sigma_2]}$ be the p-adic completion of $\mathbf{A}^\dagger_{[\sigma_1,\sigma_2]}$. Note that every element in $\mathbf{A}_{[\sigma_1,\sigma_2]}$ can be uniquely expressed as $\sum_{i\in\mathbb{Z}} b_i[T^\flat]^i$ with $b_i \in \mathbf{A}_K$.

(4) Let $\mathbf{B}_{[\sigma_1,\sigma_2]}$, $\mathbf{B}^r_{[\sigma_1,\sigma_2]}$, $\mathbf{B}^\dagger_{[\sigma_1,\sigma_2]}$ be the rings $\mathbf{A}_{[\sigma_1,\sigma_2]}$, $\mathbf{A}^r_{[\sigma_1,\sigma_2]}$, $\mathbf{A}^\dagger_{[\sigma_1,\sigma_2]}$ with p inverted.

(5) For $0 < s \leqslant r < r_K$, let $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$ be the completion of $\mathbf{B}^r_{[\sigma_1,\sigma_2]}$ with respect to the norm $\max\{\tilde{\lambda}_r, \tilde{\lambda}_s\}$ (or equivalently, with respect to the family of norms $\{\tilde{\lambda}_t\}_{t\in[s,r]}$). Let

$$\mathbf{C}^r_{[\sigma_1,\sigma_2]} := \varprojlim_{0<s\leqslant r} \mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}, \qquad \text{and} \qquad \mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} = \varinjlim_{0<r<r_K} \mathbf{C}^r_{[\sigma_1,\sigma_2]}.$$

Proposition 4.19 ([DMY26, Proposition 11.11, Lemma 11.2])**.**

(1) The pair $\left(R_{[\sigma_1,\sigma_2]}, \mathbf{A}_{[\sigma_1,\sigma_2]}\right)$ forms a strongly decompleting datum. The associated imperfect period rings are precisely

$$\mathbf{A}^r_{[\sigma_1,\sigma_2]}, \mathbf{A}^\dagger_{[\sigma_1,\sigma_2]}, \mathbf{B}_{[\sigma_1,\sigma_2]}, \mathbf{B}^r_{[\sigma_1,\sigma_2]}, \mathbf{B}^\dagger_{[\sigma_1,\sigma_2]}, \mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}, \mathbf{C}^r_{[\sigma_1,\sigma_2]}, \mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}.$$

(2) The norm $\tilde{\lambda}_r$ coincides with λ_r, for all $0 < r < r_K$.

We will also need the following explicit descriptions of $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$ and $\mathbf{C}^r_{[\sigma_1,\sigma_2]}$.

Lemma 4.20 ([DMY26, Corollary 11.5])**.**

(1) For $0 < s \leqslant r < r_K$, the ring $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$ consists of $\sum_{i\in\mathbb{Z}} b_i[T^\flat]^i$ with $b_i \in \mathbf{C}^{[s,r]}_K$ such that $\lambda_{r'}(b_i)\cdot\sigma^{r'i} \to 0$ as $i \to \pm\infty$, for all $r' \in [s,r]$ and for all $\sigma \in [\sigma_1,\sigma_2]$.[20]

(2) For $0 < r < r_K$, the ring $\mathbf{C}^{r}_{[\sigma_1,\sigma_2]}$ consists of $\sum_{i\in\mathbb{Z}} b_i[T^\flat]^i$ with $b_i \in \mathbf{C}^{r}_K$ such that $\lambda_{r'}(b_i)\cdot\sigma^{r'i} \to 0$ as $i \to \pm\infty$, for all $r' \in (0,r]$ and for all $\sigma \in [\sigma_1,\sigma_2]$.

Remark 4.21. By Remark 3.13, $\mathbf{C}^{[s,r]}_K$ can be identified with the ring of analytic functions $\Gamma_{K'_0}([\delta_1,\delta_2];\pi_K)$ on the closed annulus $A_{K'_0}([\delta_1,\delta_2];\pi_K)$ where $\delta_1 = p^{-\frac{p}{p-1}\cdot\frac{r}{e_K}}$, $\delta_2 = p^{-\frac{p}{p-1}\cdot\frac{s}{e_K}}$. Moreover, for any $r' \in [s,r]$, the $\lambda_{r'}$-norm on $\mathbf{C}^{[s,r]}_K$ coincides with the ρ-Gauss norm on $A_{K'_0}([\delta_1,\delta_2];\pi_K)$ with $\rho = p^{-\frac{p}{p-1}\cdot\frac{r'}{e_K}}$. Therefore, by Lemma 4.20, the ring $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$ consists of two-variable power series $\sum_{i,j\in\mathbb{Z}} c_{i,j}\pi_K^j[T^\flat]^i$ with $c_{i,j} \in K'_0$ such that

$$|c_{i,j}|\cdot p^{-\frac{p}{p-1}\cdot\frac{r'j}{e_K}}\cdot\sigma^{r'i} \to 0$$

as $|i|+|j| \to +\infty$, for all $r' \in [s,r]$ and all $\sigma \in [\sigma_1,\sigma_2]$ (again, one only needs to verify this condition for $r' = s,r$ and $\sigma = \sigma_1,\sigma_2$).

4.7.2. *Closed annuli with punctures.*

We now consider a closed annulus of outer radius 1, with finitely many open disks removed.

Notation 4.22. Let $\delta \in (0,1] \cap p^{\mathbb{Q}}$ and let $f(T) \in \mathcal{O}_{K_0}[T]$ be a monic polynomial that factors as $f(T) = \prod_{i=1}^m (T - [\beta_i])$ for some $\beta_i \in \overline{k}_K^\times$. Let $X_{\delta,f}$ be the rational subset[21] of $\mathbb{D}_K$ defined by

$$X_{\delta,f} := \{|T| \geqslant \delta,\ |f(T)| = 1\} \subset \mathbb{D}_K.$$

When f is understood from the context, we often write X_1 for $X_{1,f}$ and X_δ for $X_{\delta,f}$.

Remark 4.23. Let ξ denote the Gaussian point of $\mathbb{D}_K$ and let ξ denote the type V point corresponding to the open unit circle centered at 0. Then the collection $\Sigma = \{X_{1,f}\}$ forms a basis of open neighborhoods of ξ_{II} inside $\mathbb{D}_K$; while the collection $\Sigma' = \{X_{\delta,f}\}$ forms a basis of open neighborhoods of ξ inside $\mathbb{D}_K$.

Let $\delta \in (0,1] \cap p^{\mathbb{Q}}$ and let $f \in \mathcal{O}_{K_0}[T]$ be a polynomial which takes the form $f(T) = \prod_i (T - [\beta_i]) \in \overline{K}[T]$ with $\beta_i \in \overline{k}_K^\times$. Take the natural inclusion $\iota\colon X_{\delta,f} \to \mathbb{D}_K$ to be the toric chart. To simplify the notation, in the subscripts of period rings we write δ, f instead of $X_{\delta,f}$. From the construction in §4.2, we have perfectoid rings $\widehat{A}_{\delta,f,\infty}$ and $\widetilde{R}_{\delta,f}$ and associated perfect period rings

$$\widetilde{\mathbf{A}}_{\delta,f}, \widetilde{\mathbf{A}}^r_{\delta,f}, \widetilde{\mathbf{A}}^\dagger_{\delta,f}, \widetilde{\mathbf{B}}_{\delta,f}, \widetilde{\mathbf{B}}^r_{\delta,f}, \widetilde{\mathbf{B}}^\dagger_{\delta,f}, \widetilde{\mathbf{C}}^{[s,r]}_{\delta,f}, \widetilde{\mathbf{C}}^r_{\delta,f}, \widetilde{\mathbf{C}}^\dagger_{\delta,f}$$

equipped with commuting actions of φ and Γ. Let $\overline{f}$ denote the reduction of f modulo p. Then $\widetilde{R}_{\delta,f}$ is the rational localization of $\widetilde{R}_{[\delta,1]}$ given by $|\overline{f}(T^\flat)| = 1$.

We have the following explicit imperfect relative period rings for $X_{\delta,f}$ where the corresponding datum is strongly decompleting. Let h denote the degree of the polynomial $f(T)$.

Definition 4.24. (1) Let $R_{\delta,f}$ be the subring of $\widetilde{R}_{\delta,f}$ defined by

$$R_{\delta,f} := \left\{ \sum_{i\in\mathbb{Z}} a_i(T^\flat)^i + \sum_{j=1}^{\infty}\sum_{k=0}^{h-1} \frac{b_{jk}(T^\flat)^k}{\overline{f}(T^\flat)^j} \;\middle|\; \begin{array}{l} a_i, b_{jk} \in \kappa((\overline{\pi}_K)),\ |b_{jk}| \to 0 \text{ as } j \to +\infty, \\ |a_i|\sigma^i \to 0 \text{ as } i \to \pm\infty \text{ for all } \delta \leqslant \sigma \leqslant 1 \end{array} \right\}.$$

(2) For each $0 < r < r_K$, let $\mathbf{A}^r_{\delta,f}$ be a subring of $\widetilde{\mathbf{A}}_{\delta,f}$ given by

$$\mathbf{A}^r_{\delta,f} := \left\{ \sum_{i\in\mathbb{Z}} a_i[T^\flat]^i + \sum_{j=1}^{\infty}\sum_{k=0}^{h-1} \frac{b_{jk}[T^\flat]^k}{f([T^\flat])^j} \;\middle|\; \begin{array}{l} a_i, b_{jk} \in \mathbf{A}^r_K,\ \lambda_r(b_{jk}) \to 0 \text{ as } j \to +\infty, \\ \lambda_r(a_i)\sigma^{ri} \to 0 \text{ as } i \to \pm\infty \text{ for all } \delta \leqslant \sigma \leqslant 1 \end{array} \right\}.$$

[20] In fact, one only needs to verify this condition for $r' = s,r$ and $\sigma = \sigma_1,\sigma_2$.

[21] Roughly speaking, $X_{\delta,f}$ is $\mathbb{D}_K$ minus finitely many open unit disks on the boundary circle $\{|T| = 1\}$ (these disks are centered at $[\beta_i]$'s), then further remove an open disk of radius δ centered at 0.

Consider the norm $\tilde{\lambda}_r$ on $\mathbf{A}^r_{\delta,f}$ given by

$$\tilde{\lambda}_r\left(\sum_{i\in\mathbb{Z}} a_i[T^\flat]^i + \sum_{j=1}^{\infty}\sum_{k=0}^{h-1} b_{jk}\frac{[T^\flat]^k}{f([T^\flat])^j}\right) := \max\left\{\sup_{\delta\leqslant\sigma\leqslant 1}\sup_{i\in\mathbb{Z}}\{\lambda_r(a_i)\sigma^{ri}\}, \sup_{j\geqslant 1, 0\leqslant k\leqslant h-1}\{\lambda_r(b_{jk})\}\right\}.$$

(3) Let

$$\mathbf{A}^\dagger_{\delta,f} := \varinjlim_{0<r<r_K} \mathbf{A}^r_{\delta,f},$$

and let $\mathbf{A}_{\delta,f}$ be the p-adic completion of $\mathbf{A}^\dagger_{\delta,f}$. Note that $\mathbf{A}_{\delta,f}$ is a subring of $\widetilde{\mathbf{A}}_{\delta,f}$.

(4) Let $\mathbf{B}_{\delta,f}$, $\mathbf{B}^r_{\delta,f}$, $\mathbf{B}^\dagger_{\delta,f}$ be the rings $\mathbf{A}_{\delta,f}$, $\mathbf{A}^r_{\delta,f}$, $\mathbf{A}^\dagger_{\delta,f}$ with p inverted.

(5) For $0<s\leqslant r<r_K$, let $\mathbf{C}^{[s,r]}_{\delta,f}$ be the completion of $\mathbf{B}^r_{\delta,f}$ with respect to the norm $\max\{\tilde{\lambda}_r,\tilde{\lambda}_s\}$ (or equivalently, with respect to the family of norms $\{\tilde{\lambda}_t\}_{t\in[s,r]}$). Let

$$\mathbf{C}^r_{\delta,f} := \varprojlim_{0<s\leqslant r} \mathbf{C}^{[s,r]}_{\delta,f}, \qquad \text{and} \qquad \mathbf{C}^\dagger_{\delta,f} = \varinjlim_{0<r<r_K} \mathbf{C}^r_{\delta,f}.$$

Proposition 4.25 ([DMY26, Proposition 11.19, Lemma 11.14])**.** (1) The datum $(R_{\delta,f},\mathbf{A}_{\delta,f})$ is strongly decompleting. The associated imperfect period rings are precisely

$$\mathbf{A}^r_{\delta,f},\mathbf{A}^\dagger_{\delta,f},\mathbf{B}_{\delta,f},\mathbf{B}^r_{\delta,f},\mathbf{B}^\dagger_{\delta,f},\mathbf{C}^{[s,r]}_{\delta,f},\mathbf{C}^r_{\delta,f},\mathbf{C}^\dagger_{\delta,f}.$$

(2) The norm $\tilde{\lambda}_r$ coincides with λ_r, for all $0<r<r_K$.

We also have the following explicit descriptions of $\mathbf{C}^{[s,r]}_{\delta,f}$ and $\mathbf{C}^r_{\delta,f}$.

Lemma 4.26 ([DMY26, Corollary 11.15])**.** (1) For $0<s\leqslant r<r_K$, the ring $\mathbf{C}^{[s,r]}_{\delta,f}$ consists of elements

$$\sum_{i\in\mathbb{Z}} a_i[T^\flat]^i + \sum_{j=1}^{\infty}\sum_{k=0}^{h-1} b_{jk}\frac{[T^\flat]^k}{f([T^\flat])^j}, \quad a_i,b_{jk}\in\mathbf{C}^{[s,r]}_K$$

such that $\lambda_{r'}(a_i)\sigma^{r'i}\to 0$ as $i\to\pm\infty$ for all $r'\in[s,r]$ and $\delta\leqslant\sigma\leqslant 1$, and $\lambda_{r'}(b_{jk})\to 0$ as $j\to+\infty$ for all $r'\in[s,r]$.

(2) For $0<r<r_K$, the ring $\mathbf{C}^r_{\delta,f}$ consists of

$$\sum_{i\in\mathbb{Z}} a_i[T^\flat]^i + \sum_{j=1}^{\infty}\sum_{k=0}^{h-1} b_{jk}\frac{[T^\flat]^k}{f([T^\flat])^j}, \quad a_i,b_{jk}\in\mathbf{C}^r_K$$

such that $\lambda_{r'}(a_i)\sigma^{r'i}\to 0$ as $i\to\pm\infty$ for all $r'\in(0,r]$ and $\delta\leqslant\sigma\leqslant 1$, and $\lambda_{r'}(b_{jk})\to 0$ as $j\to+\infty$ for all $r'\in(0,r]$.

Remark 4.27. We have normalized differential operators d_0 and d_1 on $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$, $\mathbf{C}^r_{[\sigma_1,\sigma_2]}$, and $\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}$ for $0<s\leqslant r$ with r sufficiently small from §4.3.5. A direct computation shows that we have $d_0=(1+\pi)\cdot u_0\cdot\frac{d}{d\pi_K}$ and $d_1=[T^\flat]\cdot\frac{d}{d[T^\flat]}$ on these imperfect period rings, where $u_0=\frac{d\pi_K}{d\pi}$ (see Remark 3.17), which we may assume to be a unit in $\mathbf{C}^r_K$ after shrinking r if necessary. The same statements hold for the imperfect period rings over $X_{\delta,f}$.

Remark 4.28 (From de Rham local systems to p-adic differential equations)**.** Let X be either $X_{[\sigma_1,\sigma_2]}$ or $X_{\delta,f}$, and let $\mathbb{L}$ be a de Rham local system on X. We make a few remarks on the differential operators on the associated p-adic differential equation $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ from [DMY26, §11]. To simply the notation, we mainly explain the case $X=X_{[\sigma_1,\sigma_2]}$; analogous statements hold for $X=X_{\delta,f}$. Given any finite projective module M over $\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}$ (or $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$, $\mathbf{C}^r_{[\sigma_1,\sigma_2]}$) equipped with a continuous Γ-action, the action of $\mathrm{Lie}\,\Gamma$ also induces differential operators ∇_{geom} and ∇_{arith} on M defined by

$$\nabla_{\mathrm{arith}}=\frac{\log(\gamma_0)}{\log\chi(\gamma_0)}, \qquad \text{and} \qquad \nabla_{\mathrm{geom}}=\frac{\log(\tau^{p^m})}{\eta(\tau^{p^m})}$$

as in §4.3.5 (where τ is denoted by τ_1 therein), with $\gamma_0 \in \Gamma_K$ sufficiently close to 1 and m sufficiently large. For our purpose, we again normalize these differential operators by putting

$$D_{[T^\flat]} \coloneqq t^{-1} \cdot [T^\flat]^{-1} \cdot \nabla_{\mathrm{geom}}$$
$$D_{\pi_K} \coloneqq t^{-1} \cdot ((1+\pi) \cdot u_0)^{-1} \cdot \nabla_{\mathrm{arith}}$$

Note that $[T^\flat]$ and $(1+\pi)\cdot u_0$ are units in the corresponding imperfect relative period rings. We caution the reader that, *a priori*, (M, D_{π_K}) (resp. $(M, D_{[T^\flat]})$) could have poles along the vanishing locus of $t = \log(1+\pi)$ when viewed as a differential module over the differential ring $\left(\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}, \frac{d}{d\pi_K}\right)$ (resp. $\left(\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}, \frac{d}{d[T^\flat]}\right)$). Note, however, that $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ is obtained from $D^\dagger_{\mathrm{rig}}(\mathbb{L})$ (cf. Remark 4.13) by modifying along the "$t = 0$ locus". In particular, part of the content of Theorem 4.14 (Theorem 11 in the introduction) asserts that, after this modification procedure, $\left(\mathbf{N}_{\mathrm{dR}}(\mathbb{L}), D_{[T^\flat]}, D_{\pi_K}\right)$ does not have poles, hence it is a genuine differential module (i.e., a relative p-adic differential equation) over the differential ring $\left(\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}, \frac{d}{d[T^\flat]}, \frac{d}{d\pi_K}\right)$. In other words, the modification along the "$t = 0$ locus" removes the singularities from $D^\dagger_{\mathrm{rig}}(\mathbb{L})$ in both the arithmetic and geometric directions! When $X = X_{\delta,f}$, we similarly obtain $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ from $D^\dagger_{\mathrm{rig}}(\mathbb{L})$, and $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ is a differential module over the differential ring $\left(\mathbf{C}^\dagger_{\delta,f}, \frac{d}{d[T^\flat]}, \frac{d}{d\pi_K}\right)$ (see [DMY26, §11.3]). We refer the reader to [DMY26, §11] for more details.

4.8. **A criterion for semistability.**

For later use, we establish a criterion for (pointwise) semistability of p-adic local systems. Throughout §4.8, we assume (X, h) is one of the following:

(1) Let X be a closed annulus $X_{[\sigma_1,\sigma_2]} = \{\sigma_1 \leqslant |T| \leqslant \sigma_2\}$, with $\sigma_1, \sigma_2 \in p^{\mathbb{Q}}$ and $0 < \sigma_1 \leqslant \sigma_2 \leqslant 1$, as in §4.7.1. Let $h\colon X_{[\sigma_1,\sigma_2]} \hookrightarrow \mathbb{D}_K$ be the natural embedding.
(2) Let X be a closed annulus with punctures $X_{\delta,f} = \{\delta \leqslant |T| \leqslant 1, |f(T)| = 1\}$ as in §4.7.2. Let $h\colon X_{\delta,f} \hookrightarrow \mathbb{D}_K$ be the natural embedding.
(3) Let X be a finite étale cover of either $X_{[\sigma_1,\sigma_2]}$ or $X_{\delta,f}$ and let h be the composition of this finite étale map and the natural embedding into $\mathbb{D}_K$.

We have constructed an explicit strongly decompleting datum for (X, h) using §4.7 and Proposition 4.16 (also see [DMY26, §11]). Let $\mathbf{A}^\dagger_X$, $\mathbf{B}^\dagger_X$, $\mathbf{C}^\dagger_X$, etc. denote the corresponding imperfect relative period rings. Let $\mathbb{L}$ be a de Rham $\mathbb{Z}_p$-local system of rank d on X and let $D^\dagger_{\mathrm{rig}}(\mathbb{L})$ be the associated étale (φ, Γ)-module over $\mathbf{C}^\dagger_X$. Let $\mathbf{C}^\dagger_X[\log \pi_K, \log[T^\flat]]$ denote the ring obtained by adjoining free variables $\log \pi_K$ and $\log[T^\flat]$ to $\mathbf{C}^\dagger_X$, equipped with φ and Γ-actions given by

$$\varphi(\log \pi_K) = p \log \pi_K + \log\left(\frac{\varphi(\pi_K)}{\pi_K^p}\right), \quad \varphi(\log[T^\flat]) = p \log[T^\flat] \qquad \text{and}$$
$$\gamma(\log \pi_K) = \log \pi_K + \log\left(\frac{\gamma(\pi_K)}{\pi_K}\right), \quad \gamma(\log[T^\flat]) = \log[T^\flat] + \log\left(\frac{\gamma([T^\flat])}{[T^\flat]}\right)$$

for $\gamma \in \Gamma$.

Remark 4.29. It is computed in [Ber02] that $(\mathbf{C}^\dagger_K[\log \pi_K, \frac{1}{t}])^{\Gamma_K} = K_0$. Using the explicit descriptions of the imperfect period rings in §4.7, one can deduce that $(\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}[\log \pi_K, \log[T^\flat], \frac{1}{t}])^\Gamma = K_0$ and $(\mathbf{C}^\dagger_{\delta,f}[\log \pi_K, \log[T^\flat], \frac{1}{t}])^\Gamma = K_0$.

Proposition 4.30. Suppose that the $\mathbf{C}^\dagger_X[\log[T^\flat], \log \pi_K, \frac{1}{t}]$-module

$$D^\dagger_{\mathrm{rig}}(\mathbb{L}) \otimes_{\mathbf{C}^\dagger_X} \mathbf{C}^\dagger_X[\log[T^\flat], \log \pi_K, \frac{1}{t}]$$

is free and admits a basis $w_1, \ldots, w_d$ such that $\gamma(w_i) = w_i$ for all $\gamma \in \Gamma$ and all $i = 1, \ldots, d$. Then $\mathbb{L}$ is pointwise semistable on X (i.e., semistable at every classical point on X).

We point out that there is no direct "specialization map" relating $D^\dagger_{\mathrm{rig}}(\mathbb{L})$ and $D^\dagger_{\mathrm{rig}}(\mathbb{L}_x)$ for a classical point x. Instead, we directly relate $D^\dagger_{\mathrm{rig}}(\mathbb{L})$ with $D_{\mathrm{st}}(\mathbb{L}_x)$ and use the Γ-invariant basis to produce enough semistable periods at each classical point.

Proof. Let $x \in X$ be a L-rational classical point for some finite extension L/K. Then x is represented by a ring map $f_x\colon A_X \to L$ (actually x is determined by the $\mathrm{Gal}(L/K)$-orbit of such a ring map). Consider the ring map $h^*\colon K\langle T\rangle \to A_X$ induced by the toric chart h. Let $a := f_x \circ h^*(T) \in L^\times$ where T is the coordinate of $\mathbb{D}_K$. Pick a compatible p-power roots $\{a^{1/p^m}\}_{m\geqslant 0}$ of a. This induces a map $\eta\colon \widehat{K}_\infty\langle T^{1/p^\infty}\rangle \to \mathbb{C}_K$ sending $T^{1/p^m} \mapsto a^{1/p^m}$. Since

$$\widehat{A}_{X,\infty} = A_X \widehat{\otimes}_{K\langle T\rangle} \widehat{K}_\infty\langle T^{1/p^\infty}\rangle,$$

the maps f_x and η induces a ring map $e_x\colon \widehat{A}_{X,\infty} \to \mathbb{C}_K$.

In fact, the map e_x can be made Gal_L-equivariant in a suitable sense. First, the action of Gal_L on a^{1/p^m}, given by $g(a^{1/p^m}) = \zeta_{p^m}^{c_m(g)} a^{1/p^m}$ for a unique $c_m(g) \in \mathbb{Z}/p^m$, determines a continuous cocycle $c\colon \mathrm{Gal}_L \to \mathbb{Z}_p$ relative to the cyclotomic character; namely, $c(g_1g_2) = c(g_1) + \chi(g_1)c(g_2)$. Next, recall that the choice of $\{T^{1/p^m}\}_{m\geqslant 0}$ determines a splitting $\Gamma \cong \Gamma_{\mathrm{geom}} \rtimes \Gamma_K$ (cf. §4.1). Then consider a group homomorphism $\rho_x\colon \mathrm{Gal}_L \to \Gamma$ sending $g \mapsto (c(g), g|_{K_\infty})$. Now, consider the action of Gal_L on $\widehat{A}_{X,\infty}$ given by the natural Γ-action and $\rho_x\colon \mathrm{Gal}_L \to \Gamma$. Then $e_x\colon \widehat{A}_{X,\infty} \to \mathbb{C}_K$ is Gal_L-equivariant. Indeed, it suffices to check Gal_L-equivariance for η. On $\widehat{K}_\infty$, the map η is just the identity and Gal_L acts naturally on both the source and target; on the other hand, we have

$$\eta(\rho_x(g)\cdot(T^{1/p^m})) = \eta(\zeta_{p^m}^{c(g)} T^{1/p^m}) = \zeta_{p^m}^{c(g)} a^{1/p^m} = g\cdot(a^{1/p^m}) = g\cdot\eta(T^{1/p^m}),$$

as desired.

The map e_x induces a Gal_L-equivariant map $e_x^\flat\colon \widetilde{R}_X = \widehat{A}^\flat_{X,\infty} \to \mathbb{C}_K^\flat$ which sends $T^\flat \mapsto a^\flat := (a, a^{1/p}, \ldots)$ and thus a Gal_L-equivariant map $\tilde{e}_x\colon \widetilde{\mathbf{A}}_X \to \widetilde{\mathbf{A}}$. It further induces Gal_L-equivariant maps $\tilde{e}_x\colon \widetilde{\mathbf{A}}^\dagger_X \to \widetilde{\mathbf{A}}^\dagger$ and $\tilde{e}_x\colon \widetilde{\mathbf{C}}^\dagger_X \to \widetilde{\mathbf{C}}^\dagger$. Composing with the natural inclusion $\mathbf{C}^\dagger_X \subset \widetilde{\mathbf{C}}^\dagger_X$, we arrive at the "specialization map" $s_x\colon \mathbf{C}^\dagger_X \to \widetilde{\mathbf{C}}^\dagger$. This in turn induces a Gal_L-equivariant specialization map

$$s_x\colon \mathbf{C}^\dagger_X[\log[T^\flat], \log\pi_K, \frac{1}{t}] \to \widetilde{\mathbf{C}}^\dagger_{\log}[\frac{1}{t}]$$

sending $\log[T^\flat] \mapsto \log[a^\flat]$ and $\log\pi_K \mapsto \log\pi_K$.

Observe that if $\mathbb{M}$ is a finite locally free $\mathbb{Z}/p^m$-local system on $\operatorname{Spec}\widetilde{R}_X$ and $D_m(\mathbb{M})$ is the associated φ-module over $W_m(\widetilde{R}_X)$ by [KL15, Proposition 3.2.7], then we have a canonical φ- and Gal_L-equivariant isomorphism

$$D_m(\mathbb{M}) \otimes_{W_m(\widetilde{R}_X), e_x^\flat} W_m(\mathbb{C}_K^\flat) \cong D_m((e_x^\flat)^*\mathbb{M})$$

where $(e_x^\flat)^*\mathbb{M}$ denotes the local system on $\operatorname{Spec}\mathbb{C}_K^\flat$ obtained by base change along $e_x^\flat\colon \widetilde{R}_X \to \mathbb{C}_K^\flat$. Consequently, by unwinding the construction of $\widetilde{D}^{\mathrm{int}}(\mathbb{L})$ (cf. Remark 4.13 and the references therein), we obtain a canonical φ- and Gal_L-equivariant isomorphism $\widetilde{D}^{\mathrm{int}}(\mathbb{L}) \otimes_{\widetilde{\mathbf{A}}_X, \tilde{e}_x} \widetilde{\mathbf{A}} \cong \mathbb{L}_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{A}}$, which further induces an isomorphism $\widetilde{D}^{\mathrm{int},\dagger}(\mathbb{L}) \otimes_{\widetilde{\mathbf{A}}^\dagger_X, \tilde{e}_x} \widetilde{\mathbf{A}}^\dagger \cong \mathbb{L}_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{A}}^\dagger$ by [KL15, Theorem 8.5.3 (d)(e)]. This further induces a φ- and Gal_L-equivariant isomorphism

$$D^\dagger_{\mathrm{rig}}(\mathbb{L}) \otimes_{\mathbf{C}^\dagger_X, s_x} \widetilde{\mathbf{C}}^\dagger = \widetilde{D}^\dagger_{\mathrm{rig}}(\mathbb{L}) \otimes_{\widetilde{\mathbf{C}}^\dagger_X, \tilde{e}_x} \widetilde{\mathbf{C}}^\dagger \cong \mathbb{L}_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{C}}^\dagger.$$

Finally, joining $\log[T^\flat]$, $\log\pi_K$, $\frac{1}{t}$ and using the Γ-invariant basis in the statement of the proposition, we arrive at Gal_L-equivariant isomorphisms

$$\mathbb{L}_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{C}}^\dagger_{\log}[\frac{1}{t}] \cong \left(\bigoplus_{i=1}^d \mathbf{C}^\dagger_X[\log[T^\flat], \log\pi_K, \frac{1}{t}]\cdot w_i\right) \otimes_{\mathbf{C}^\dagger_X[\log[T^\flat], \log\pi_K, \frac{1}{t}], s_x} \widetilde{\mathbf{C}}^\dagger_{\log}[\frac{1}{t}] \cong \bigoplus_{i=1}^d \widetilde{\mathbf{C}}^\dagger_{\log}[\frac{1}{t}]\cdot w_i.$$

Here, Gal_L acts on w_i via $\rho_x\colon \mathrm{Gal}_L \to \Gamma$, thus Gal_L fixes each w_i. By [Ber02, Proposition 3.4] (combined with a Tate twist argument), we have

$$D_{\mathrm{st},L}(\mathbb{L}_x) \cong \left(\mathbb{L}_x \otimes_{\mathbb{Z}_p} \widetilde{\mathbf{C}}^\dagger_{\log}[\frac{1}{t}]\right)^{\mathrm{Gal}_L}.$$

Therefore, $D_{\mathrm{st},L}(\mathbb{L}_x)$ has L_0-dimension d (where L_0 denotes the maximal unramified subfield in L) and thus $\mathbb{L}_x$ is semistable, as desired. □

5. Around points of type I and II

As mentioned in the introduction, to prove the main result, we first prove Theorem 12, which is a local version of the p-adic monodromy theorem. We will prove Theorem 12 by a case-by-case analysis based on the type of the point ξ; in fact, points of type I, II, III, IV, V are treated in §5.1, §5.2, §6, §7, §8–9, respectively.

5.1. Around type I points.

The case of classical type I points is handled in [Shi22]. In fact, we will show that, with some modifications, the proof of [Shi22, Theorem 9.2] remains valid for all non-classical type I points.

Proposition 5.1. Theorem 12 holds for type I points.

Proof. Let X and $\mathbb{L}$ be as in the statement of Theorem 12. Let $x \in X_{\mathbb{C}_K}$ be a type I point. First, by the same argument as in [Shi22, Proposition 9.6], x admits an open neighborhood U_x that is isomorphic to a closed unit disk over $\mathbb{C}_K$. The p-adic Riemann-Hilbert functor of Liu–Zhu [LZ17] yields a vector bundle with connection $(D_{\mathrm{dR}}(\mathbb{L}), \nabla)$ over X. By base change, we obtain a vector bundle with connection $(D_{\mathrm{dR}}(\mathbb{L})_{\mathbb{C}_K}, \nabla)$ over $X_{\mathbb{C}_K}$. By [Shi22, Theorem 9.7], up to shrinking the disk, we may assume that the connection is trivial on U_x.

Notice that U_x contains a K'-rational point for some finite extension K'/K. In particular, the image U'_x of U_x in $X_{K'}$ is isomorphic to the closed unit disk over K'. By construction, the vector bundle with connection $(D_{\mathrm{dR}}(\mathbb{L})_{K'}, \nabla)$ is trivial over U'_x, hence $\mathbb{L}|_{U'_x}$ is horizontally de Rham. By shrinking, we may replace U'_x by some U''_x isomorphic to

$$\mathbb{T}^n_{K'} \cong \mathrm{Spa}\left(K'\langle T^{\pm}\rangle, \mathcal{O}_{K'}\langle T^{\pm}\rangle\right).$$

Finally, by the same argument as in [Shi22, §9.4], there exists a finite extension L/K' such that $\mathbb{L}|_{U''_{x,L}}$ is horizontally de Rham and semistable at every classical point. □

5.2. Around type II points.

Let us turn to type II points. First, note that every type II point is K'-rational for some finite extension K'/K. Therefore, when ξ is a type II point, we may assume that ξ is K-rational after replacing K by a finite extension if necessary (cf. Remark 2.8). In other words, in the case of type II points, it suffices to prove the following.

Proposition 5.2. Theorem 12 holds when ξ is a K-rational type II point on X.

Proof. We first claim that, by enlarging K if necessary, ξ admits a connected affinoid open neighborhood U which admits a smooth formal model over $\mathcal{O}_K$. Indeed, by [Ber07, Proposition 2.2.1] (which corrects the original statements in [Ber93, Proposition 3.6.1]), by enlarging K if necessary, there exist

- a smooth connected projective curve $\mathcal{Y}$ over $\mathcal{O}_K$ with generic fiber Y (viewed as an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$);
- finitely manly closed points $z_1, \ldots, z_n$ on the special fiber Y_s; and
- an open neighborhood U of ξ,

such that U is isomorphic to $\mathrm{sp}^{-1}(Y_s - \{z_1, \ldots, z_n\})$, where $\mathrm{sp}\colon Y \to Y_s$ is the specialization map, and that ξ is the preimage of the generic point of Y_s. It follows from the construction that U is affinoid connected and it admits a smooth affine formal integral model.

Given this, the rest of the proof follows from the arguments in [GY24, §6 & 7]. For the reader's convenience, we recall the proof here (using terminology from *loc. cit.*). Notice that ξ is the unique Shilov point of U. Let $L = \mathscr{H}(\xi)$ be the completed residue field of ξ, which is a complete discrete valuation field. Write $U = \mathcal{U}_\eta$ for some smooth connected formal scheme $\mathcal{U} = \mathrm{Spf}\, R$ over $\mathcal{O}_K$. The restriction $\mathbb{L}|_\xi$ corresponds to a de Rham representation V of Gal_L. By either [Ohk13, Main Theorem] or [Mor14, Corollary 1.2], there exists a

finite Galois extension L'/L such that $V|_{\mathrm{Gal}_{L'}}$ is a semistable representation. According to [GY24, Theorem 6.11, Proposition 6.16, and Theorem 6.17], by shrinking $\mathcal{U}$ and replacing L' by a further finite extension if necessary, there exists a finite Galois extension K'/K and a finite cover $\mathcal{U}' \to \mathcal{U}$ such that

- $\mathcal{U}'$ is affine smooth connected over $\mathcal{O}_{K'}$;
- $\mathcal{U}' \to \mathcal{U}$ factors as $\mathcal{U}' \to \mathcal{U}_{\mathcal{O}_{K'}} \to \mathcal{U}$ where $\mathcal{U}_{\mathcal{O}_{K'}} \to \mathcal{U}$ is the extension of scalars and $\mathcal{U}' \to \mathcal{U}$ induces a finite étale map $\mathcal{U}'_\eta \to \mathcal{U}_{\eta,K'}$ on the generic fibers;
- the inverse image of ξ in $\mathcal{U}'_\eta$ is the unique Shilov point ξ' of $\mathcal{U}'_\eta$;
- the extension $\mathscr{H}(\xi) \to \mathscr{H}(\xi')$ coincides with $L \to L'$.

In particular, the restriction $\mathbb{L}|_{\xi'}$ is a semistable representation of $\mathrm{Gal}_{L'}$. Let $U' = \mathcal{U}'_\eta$. By the purity theorem of Du–Liu–Moon–Shimizu (Theorem 13), we conclude that $\mathbb{L}|_{U'}$ is a semistable local system; in particular, the pullback of $\mathbb{L}$ to U' is pointwise semistable. □

6. Around points of type Ⅲ

The goal of this section is to prove Theorem 12 around type Ⅲ points. Since each type Ⅲ point on $X_{\mathbb{C}_K}$ is K'-rational for some finite extension K'/K, we may again assume that ξ is K-rational after replacing K by a finite extension (cf. Remark 2.8). That is, we shall prove the following

Theorem 6.1. *Theorem 12 holds when ξ is a K-rational type Ⅲ point on X.*

To prove Theorem 6.1, we will carefully study relative p-adic differential equations constructed in [DMY26] (recalled in §4) over suitable open neighborhoods of the type Ⅲ points ξ. As mentioned in the introduction, our strategy is to relate a completed colimit of the imperfect relative period rings over neighborhoods of ξ to the *fake Robba ring* studied by Kedlaya in [Ked07] and [Ked09]. This implies that the differential operators are unipotent "at ξ" after pulling back to a finite étale cover. Our goal is to "propagate" this unipotence to an open neighborhood of ξ. This propagation procedure is the technical core of this section and occupies §6.5–6.6: the special case is treated in §6.5, and the general case in §6.6. Finally, we use the results from §4.8 to conclude the proof of Theorem 6.1.

6.1. The fake Robba ring (à la Kedlaya).

By [Ber07, Proposition 2.2.1], up to replacing K by a finite extension, we may assume that ξ admits an open neighborhood of the form $X_{[\sigma_1,\sigma_2]} = A_K([\sigma_1,\sigma_2])$ (i.e. the closed annulus $\{\sigma_1 \leqslant |T| \leqslant \sigma_2\}$) with $\sigma_1, \sigma_2 \in p^{\mathbb{Q}}$ and ξ is the type Ⅲ point corresponding to the circle of radius σ_ξ where $\sigma_1 < \sigma_\xi < \sigma_2$ and $\sigma_\xi \notin p^{\mathbb{Q}}$. Write $\sigma_\xi = p^{-\frac{p}{p-1}\frac{1}{e_K}\cdot\alpha}$. Then

$$\alpha = -\frac{(p-1)e_K}{p}\cdot \log_p(\sigma_\xi) = -\frac{1}{c_K}\cdot\log_p(\sigma_\xi)$$

is an irrational number, where $c_K = \frac{p}{(p-1)e_K}$. Similarly, consider the rational numbers $\alpha_i = -\frac{1}{c_K}\log_p(\sigma_i)$ for $i = 1, 2$. If $[\sigma_1,\sigma_2]$ runs through closed intervals containing σ_ξ with $\sigma_1, \sigma_2 \in p^{\mathbb{Q}}$, then the closed annuli $X_{[\sigma_1,\sigma_2]}$'s form a basis of open neighborhoods of ξ. Recall the imperfect relative period rings

$$\mathbf{A}^r_{[\sigma_1,\sigma_2]}, \mathbf{A}^\dagger_{[\sigma_1,\sigma_2]}, \mathbf{B}_{[\sigma_1,\sigma_2]}, \mathbf{B}^r_{[\sigma_1,\sigma_2]}, \mathbf{B}^\dagger_{[\sigma_1,\sigma_2]}, \mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}, \mathbf{C}^r_{[\sigma_1,\sigma_2]}, \mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} \tag{6.1}$$

from §4.7. Let us introduce some auxiliary period rings $\mathbf{A}^r_\xi$, $\mathbf{A}^\dagger_\xi$, $\mathbf{B}^r_\xi$, $\mathbf{B}^\dagger_\xi$, $\mathbf{C}^{[s,r]}_\xi$, $\mathbf{C}^r_\xi$, and $\mathbf{C}^\dagger_\xi$, which are certain "completed colimits" of the period rings in (6.1).

Definition 6.2. (1) On each $\mathbf{A}^r_{[\sigma_1,\sigma_2]}$, we define a norm $\lambda_{r,\xi}$ by setting

$$\lambda_{r,\xi}\left(\sum_{i\in\mathbb{Z}} c_i[T^\flat]^i\right) = \sup_{i\in\mathbb{Z}}\left\{\lambda_r(c_i)\cdot\sigma_\xi^{ri}\right\}$$

for each $\sum_{i\in\mathbb{Z}} c_i[T^\flat]^i \in \mathbf{A}^r_{[\sigma_1,\sigma_2]}$ with $c_i \in \mathbf{A}^r_K$. Then we define $\mathbf{A}^r_\xi$ to be the completion of the colimit

$$\mathbf{A}^r_{\xi,\mathrm{colim}} = \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{A}^r_{[\sigma_1,\sigma_2]}$$

with respect to $\lambda_{r,\xi}$. It follows that $\mathbf{A}^r_\xi$ consists of $\sum_{i\in\mathbb{Z}} c_i[T^\flat]^i$ with $c_i \in \mathbf{A}^r_K$ such that $\lambda_r(c_i)\cdot\sigma_\xi^{ri} \to 0$ as $i \to \pm\infty$.

(2) Define $\mathbf{B}^r_\xi = \mathbf{A}^r_\xi[\frac{1}{p}]$.

(3) Define $\mathbf{A}^\dagger_\xi$ (resp. $\mathbf{B}^\dagger_\xi$) as the colimit $\varinjlim_{r\to 0^+} \mathbf{A}^r_\xi$ (resp. $\varinjlim_{r\to 0^+} \mathbf{B}^r_\xi$).

The norm $\lambda_{r,\xi}$ naturally extends to $\mathbf{A}^r_\xi$ and $\mathbf{B}^r_\xi$. If we express an element in $\mathbf{B}^r_\xi$ as $x = \sum_{i,j} c_{i,j}\pi_K^j[T^\flat]^i$ with $c_{i,j} \in K_0'$, then

$$\lambda_{r,\xi}(x) = \sup_{i,j\in\mathbb{Z}} \left\{ |c_{i,j}| \cdot p^{-\frac{p}{p-1}\frac{rj}{e_K}} \cdot \sigma_\xi^{ri} \right\} = \sup_{i,j\in\mathbb{Z}} \left\{ |c_{i,j}| \cdot p^{-\frac{p}{p-1}\frac{1}{e_K}\cdot r\cdot(j+\alpha i)} \right\}.$$

Definition 6.3. (1) Define $\mathbf{C}^{[s,r]}_\xi$ as the completion of $\mathbf{B}^r_\xi$ under the norm $\max\{\lambda_{s,\xi}, \lambda_{r,\xi}\}$ and let $\mathbf{C}^r_\xi$ be the Fréchet completion of $\mathbf{B}^r_\xi$ with respect to the family of norms $\{\lambda_{s,\xi}\}_{s\in(0,r]}$.

(2) Define $\mathbf{C}^\dagger_\xi = \varinjlim_{r\to 0^+} \mathbf{C}^r_\xi$.

Note that $\mathbf{C}^\dagger_\xi$ and $\mathbf{C}^r_\xi$ are equipped with Frobenius structures, as well as geometric and arithmetic differential operators, inherited from those of $\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}$ and $\mathbf{C}^r_{[\sigma_1,\sigma_2]}$.

Remark 6.4. Equivalently, we can define $\mathbf{C}^{[s,r]}_\xi$ as the completion of $K_0'[\pi_K^{\pm 1}, [T^\flat]^{\pm 1}]$ under the norm $\max\{\lambda_{s,\xi}, \lambda_{r,\xi}\}$, that is, it consists of $\sum_{i\in\mathbb{Z}} c_i[T^\flat]^i$ with $c_i \in \mathbf{C}^{[s,r]}_K$ such that $\lambda_t(c_i)\cdot\sigma_\xi^{ti} \to 0$ as $i \to \pm\infty$ for both $t = s$ and $t = r$. This is precisely the *fake annulus* $\mathscr{R}^\lambda_{[p^{-c_K r}, p^{-c_K s}]}$ in the language of [Ked09, Definition 5.1.1][22] where λ stands for the linear functional $\lambda\colon \mathbb{R}^2 \to \mathbb{R}$ sending $(x, y) \mapsto x + \alpha y$. Similarly, $\mathbf{C}^r_\xi$ (resp. $\mathbf{C}^\dagger_\xi$) can be identified with the fake annulus $\mathscr{R}^\lambda_{[p^{-c_K r},1)}$ (resp. the *fake Robba ring* $\mathscr{R}^\lambda$) in *loc. cit.* Moreover, under this identification, the fake Robba ring $\mathscr{R}^\lambda$ is equipped with a Frobenius structure in the sense of [Ked09, Definition 5.1.6].

Remark 6.5. The mod p reduction $R_\xi := \mathbf{A}^\dagger_\xi/p$ can be described as the completion of $\kappa[(\overline{\pi}_K)^{\pm 1}, (T^\flat)^{\pm 1}]$ under the valuation

$$v_\alpha\left(\sum_{i,j} b_{i,j}(\overline{\pi}_K)^j(T^\flat)^i\right) = \frac{p}{p-1}\cdot\frac{1}{e_K}\cdot\inf_{b_{i,j}\neq 0}(j+\alpha i) \tag{6.2}$$

(or equivalently, completion with respect to the norm p^{-v_α}). Note that R_ξ is a field. In fact, if we consider the type Ⅲ point $\overline{\xi}$ on the annulus $A_{\kappa((\overline{\pi}_K))}([\sigma_1,\sigma_2]; T^\flat)$ (in characteristic p) corresponding to the circle of radius σ_ξ, then R_ξ agrees with the completed residue field $\mathscr{H}(\overline{\xi})$.

6.2. Geometric realization of extensions of fake Robba rings.

For any interval $[\sigma_1,\sigma_2]$ as above, the machinery in §4.5 produces a finite projective module $\mathbf{N}_{\mathrm{dR},[\sigma_1,\sigma_2]} = \mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X_{[\sigma_1,\sigma_2]}})$ over $\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}$, equipped with both an arithmetic operator D_{π_K} and a geometric differential operator $D_{[T^\flat]}$, as well as a Frobenius structure. We also consider the integrable and Frobenius compatible connection ("the full connection")

$$\nabla = D_{\pi_K} d\pi_K + D_{[T^\flat]} d[T^\flat],$$

on $\mathbf{N}_{\mathrm{dR},[\sigma_1,\sigma_2]}$. Moreover, for r sufficiently small, we have a finite projective (φ,∇)-module $\mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]}$ over $\mathbf{C}^r_{[\sigma_1,\sigma_2]}$, also equipped with differential operators D_{π_K} and $D_{[T^\flat]}$. The formations of $\mathbf{N}_{\mathrm{dR},[\sigma_1,\sigma_2]}$ and $\mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]}$ are compatible with restriction to a smaller $[\sigma_1,\sigma_2]$ according to [DMY26, Proposition 10.13]. Taking colimits as $\sigma_1,\sigma_2 \to \sigma_\xi$, we obtain finite projective (φ,∇)-modules

$$\mathbf{N}_{\mathrm{dR,colim}} = \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{N}_{\mathrm{dR},[\sigma_1,\sigma_2]}, \qquad \mathbf{N}^r_{\mathrm{dR,colim}} = \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]}$$

[22] Fake annuli and fake Robba rings are first introduced in [Ked07] using a different notation.

over $\mathbf{C}^{\dagger}_{\xi,\mathrm{colim}} = \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]}$ and $\mathbf{C}^{r}_{\xi,\mathrm{colim}} = \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{C}^{r}_{[\sigma_1,\sigma_2]}$, respectively. Note that $\mathbf{C}^{r}_{\xi}$ is the Fréchet completion of $\mathbf{C}^{r}_{\xi,\mathrm{colim}}$ under the family of norms $\{\lambda_{s,\xi}\}_{s\in(0,r]}$. Write

$$\mathbf{N}^{r}_{\mathrm{dR},\xi} \coloneqq \mathbf{N}^{r}_{\mathrm{dR},\mathrm{colim}} \otimes_{\mathbf{C}^{r}_{\xi,\mathrm{colim}}} \mathbf{C}^{r}_{\xi} = \mathbf{N}^{r}_{\mathrm{dR},[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^{r}_{\xi,[\sigma_1,\sigma_2]}} \mathbf{C}^{r}_{\xi}$$

and

$$\mathbf{N}_{\mathrm{dR},\xi} \coloneqq \varinjlim_{r\to 0^+} \mathbf{N}^{r}_{\mathrm{dR},\xi} = \mathbf{N}_{\mathrm{dR},\mathrm{colim}} \otimes_{\mathbf{C}^{\dagger}_{\mathrm{colim}}} \mathbf{C}^{\dagger}_{\xi} = \mathbf{N}_{\mathrm{dR},[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]}} \mathbf{C}^{\dagger}_{\xi}.$$

Recall that $\mathbf{C}^{\dagger}_{\xi}$ is isomorphic to the fake Robba ring $\mathscr{R}^{\lambda}$. We view $(\mathbf{N}_{\mathrm{dR},\xi},\nabla)$ as a finite projective (ϕ,∇)-module over $\mathscr{R}^{\lambda}$, also equipped with differential operators D_{π_K} and $D_{[T^\flat]}$.

The following is Kedlaya's p-adic local monodromy theorem for fake Robba rings.

Theorem 6.6 ([Ked07, Theorem 5.2.4])**.** *There exists a finite separable extension R'/R_ξ with associated fake Robba ring $\mathscr{R}'$ such that $\mathbf{N}_{\mathrm{dR},\xi} \otimes_{\mathscr{R}^{\lambda}} \mathscr{R}'$ becomes unipotent for the full connection ∇; that is, it is a successive extension of trivial connections over $\mathscr{R}'$ of rank 1.*

For the reader's convenience, let us explain the construction of the fake Robba ring $\mathscr{R}'$ associated to R'. This is in fact an analogue of the construction in Remark 3.22. Firstly, the valuation v_α in (6.2) uniquely extends to R', which we still denote as v_a by a slight abuse of notation. The value group of v_α contains $\mathbb{Z}+\alpha\mathbb{Z}$ as a subgroup of finite index. In particular, R' is a *nearly monomial field* in the sense of [Ked07] (see [Ked07, Definition 2.1.1, Remark 2.1.4]). Let $(\mathscr{R}^{\mathrm{int}\prime}, R')$ be the henselian pair that is finite étale over the henselian pair $(\mathbf{A}^{\dagger}_{\xi}, R_\xi)$. Then $\mathscr{R}'$ is defined as

$$\mathscr{R}' \coloneqq \mathbf{C}^{\dagger}_{\xi} \otimes_{\mathbf{A}^{\dagger}_{\xi}} \mathscr{R}^{\mathrm{int}\prime}.$$

Moreover, for sufficiently small r, the finite étale map $\mathbf{A}^{\dagger}_{\xi} \to \mathscr{R}^{\mathrm{int}\prime}$ descends to a finite étale map $\mathbf{A}^{r}_{\xi} \to \mathscr{R}^{\mathrm{int}\prime r}$. Then we put

$$\mathscr{R}'^{r} \coloneqq \mathbf{C}^{r}_{\xi} \otimes_{\mathbf{A}^{r}_{\xi}} \mathscr{R}^{\mathrm{int}\prime r},$$

and we have $\mathscr{R}' = \varinjlim_{r\to 0^+} \mathscr{R}'^{r}$. By [Ked07, §2] (in particular, see Proposition 2.3.4 and Remark 2.3.5), $\mathscr{R}^{\mathrm{int}\prime}$ (resp. $\mathscr{R}'$) is indeed an integral fake Robba ring (resp. a fake Robba ring). Since $\mathbf{C}^{\dagger}_{\xi} \to \mathscr{R}'$ (resp. $\mathbf{C}^{r}_{\xi} \to \mathscr{R}'^{r}$) is finite étale, the differential operators $\frac{d}{d[T^\flat]}$ and $\frac{d}{d\pi_K}$ uniquely extend to $\mathscr{R}'$ (resp. $\mathscr{R}'^{r}$). As a corollary of Theorem 6.6, there exists a sufficiently small r, such that the full connection ∇ on $\mathbf{N}^{r}_{\mathrm{dR},\xi} \otimes_{\mathbf{C}^{r}_{\xi}} \mathscr{R}'^{r}$ is unipotent.

In the remainder of §6.2, let us realize this cover of fake Robba rings $\mathbf{C}^{\dagger}_{\xi} \to \mathscr{R}'$ in geometric terms. First, write $R_{[\sigma_1,\sigma_2]} \coloneqq \mathbf{A}^{\dagger}_{[\sigma_1,\sigma_2]}/p = \mathbf{A}^{r}_{[\sigma_1,\sigma_2]}/p$ for r sufficiently small, as in §4.7 (also see [DMY26, §11]). Recall that $R_{[\sigma_1,\sigma_2]}$ is the ring of analytic functions on the closed annulus over $\kappa((\overline{\pi}_K))$ with radius in $[\sigma_1,\sigma_2]$ and variable $T^\flat$; in particular, it is an affinoid $\kappa((\overline{\pi}_K))$-algebra. Write $\overline{\lambda}_{[\sigma_1,\sigma_2]}$ for the corresponding spectral norm. Equivalently, define v_{α_i} as in (6.2) but use α_i instead of α. Then $R_{[\sigma_1,\sigma_2]}$ is the completion of $\kappa[(\overline{\pi}_K)^{\pm 1},(T^\flat)^{\pm 1}]$ with respect to the valuation $\min\{v_{\alpha_1},v_{\alpha_2}\}$. Moreover, write $R_{\xi,\mathrm{colim}}$ for the colimit $\varinjlim_{\sigma_\xi\in[\sigma_1,\sigma_2]} R_{[\sigma_1,\sigma_2]}$. Then R_ξ agrees with the completion of $R_{\xi,\mathrm{colim}}$ under the valuation v_α from (6.2).

Lemma 6.7. There exist σ_1,σ_2 sufficiently close to σ_ξ, and a finite étale map $R_{[\sigma_1,\sigma_2]} \to R'_{[\sigma_1,\sigma_2]}$ such that

$$R' = R_\xi \otimes_{R_{[\sigma_1,\sigma_2]}} R'_{[\sigma_1,\sigma_2]}.$$

Proof. The assertion is an immediate consequence of Corollary 6.9 below. □

Lemma 6.8. (1) For $\sigma_1 \leqslant \sigma_1' < \sigma_2' \leqslant \sigma_2$, the restriction map $R_{[\sigma_1,\sigma_2]} \to R_{[\sigma_1',\sigma_2']}$ is submetric.

(2) The colimit $R_{\xi,\mathrm{colim}} \coloneqq \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} R_{[\sigma_1,\sigma_2]}$ is a field.

(3) Consider the seminorm $\overline{\lambda}_\alpha$ on $\varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} R_{[\sigma_1,\sigma_2]}$ defined as the infimum of the quotient seminorms induced by $\overline{\lambda}_{[\sigma_1,\sigma_2]}$'s. Then $\overline{\lambda}_\alpha$ agrees with p^{-v_α}. In particular, it is multiplicative and the completion of $\varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} R_{[\sigma_1,\sigma_2]}$ with respect to $\overline{\lambda}_\alpha$ is precisely R_ξ.

Proof. This is standard. In particular, $R_{\xi,\mathrm{colim}}$ is simply the (uncompleted) residue field $\mathbf{k}(\overline{\xi})$ of the type Ⅲ point $\overline{\xi}$ in Remark 6.5. □

Corollary 6.9. The base functor induces an equivalence of categories of finite étale algebras.

$$\left(\varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} R_{[\sigma_1,\sigma_2]}\right)_{\text{fét}} \xrightarrow{\sim} (R_\xi)_{\text{fét}}\,.$$

Proof. This follows from Lemma 6.8 and [KL15, Lemma 2.2.4]. □

Construction 6.10 (Geometric realization of $\mathbf{C}^\dagger_\xi \to \mathscr{R}'$)**.** Write $A_{[\sigma_1,\sigma_2]} = \mathcal{O}(X_{[\sigma_1,\sigma_2]})$ for the ring of analytic functions on the annulus $X_{[\sigma_1,\sigma_2]}$ and recall the perfectoid algebra $\widehat{A}_{[\sigma_1,\sigma_2],\infty}$ from §4.7. By the diagram of equivalences of categories in [DMY26, Proposition 5.28], the finite étale map $R_{[\sigma_1,\sigma_2]} \to R'_{[\sigma_1,\sigma_2]}$ from Lemma 6.7 corresponds to a finite étale map $\widehat{A}_{[\sigma_1,\sigma_2],\infty} \to B$. By [Sch12, §7] (also see [KL15, Proposition 2.6.8]), up to replacing K by $K(\zeta_{p^n})$, we may assume that this finite étale map descends to a finite étale map $A_{[\sigma_1,\sigma_2],n} \to B_0$ where $A_{[\sigma_1,\sigma_2],n}$ is the ring of analytic functions on the p^n-th Kummer cover $X_{[\sigma_1,\sigma_2],n}$ of $X_{[\sigma_1,\sigma_2]}$. Note that, for the purpose of proving Theorem 6.1, we are always allowed to extend the base field and replace $X_{[\sigma_1,\sigma_2]}$ by a finite étale cover. By Remark 4.17, we may further assume that $n = 0$; namely, there exists a finite étale cover $g\colon X'_{[\sigma_1,\sigma_2]} = \mathrm{Spa}(B_0, B_0^\circ) \to X_{[\sigma_1,\sigma_2]}$ so that the corresponding strongly decompleting data given by [DMY26, Proposition 5.28] (also see Proposition 4.16) gives rise to relative imperfect period rings $\mathbf{A}'^r_{[\sigma_1,\sigma_2]}$, $\mathbf{B}'^r_{[\sigma_1,\sigma_2]}$, $\mathbf{C}'^r_{[\sigma_1,\sigma_2]}$, etc., satisfying $\mathbf{A}'^r_{[\sigma_1,\sigma_2]}/p = R'_{[\sigma_1,\sigma_2]}$ for r sufficiently small. Let ξ' denote the pre-image of ξ in $X'_{[\sigma_1,\sigma_2]}$. This is again a point of type Ⅲ. By Lemma 6.7 and the henselianity of $\mathscr{R}^{\mathrm{int}\prime}$, we have

$$\mathscr{R}^{\mathrm{int}\prime} = \mathbf{A}^\dagger_\xi \otimes_{\mathbf{A}^r_{[\sigma_1,\sigma_2]}} \mathbf{A}'^r_{[\sigma_1,\sigma_2]}$$

for r sufficiently small, and thus $\mathscr{R}'^r = \mathbf{C}^r_\xi \otimes_{\mathbf{C}^r_{[\sigma_1,\sigma_2]}} \mathbf{C}'^r_{[\sigma_1,\sigma_2]}$. Let

$$\mathbf{C}^r_{\xi',\mathrm{colim}} \coloneqq \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{C}'^r_{[\sigma_1,\sigma_2]}.$$

Then $\mathscr{R}'^r$ can be identified as the completion $\mathbf{C}^r_{\xi'}$ of $\mathbf{C}^r_{\xi',\mathrm{colim}}$ with respect to the unique extension $\lambda'_{r,\xi}$ of the norm $\lambda_{r,\xi}$ on $\mathbf{C}^r_{\xi,\mathrm{colim}}$, and thus $\mathscr{R}' = \varinjlim_{r\to 0^+} \mathbf{C}^r_{\xi'}$.

Moreover, write $\mathbf{N}'^r_{\mathrm{dR},[\sigma_1,\sigma_2]} = \mathbf{N}^r_{\mathrm{dR}}(\mathbb{L}|_{X'_{[\sigma_1,\sigma_2]}})$ and $\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]} = \mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X'_{[\sigma_1,\sigma_2]}})$. Then for r sufficiently small, we have

$$\begin{aligned}
\mathbf{N}^r_{\mathrm{dR},\xi} \otimes_{\mathbf{C}^r_\xi} \mathscr{R}'^r &= \left(\mathbf{N}^r_{\mathrm{dR},\mathrm{colim}} \otimes_{\mathbf{C}^r_{\xi,\mathrm{colim}}} \mathbf{C}^r_\xi\right) \otimes_{\mathbf{C}^r_\xi} \mathscr{R}'^r \\
&= \left(\varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^r_{[\sigma_1,\sigma_2]}} \mathbf{C}'^r_{[\sigma_1,\sigma_2]}\right) \otimes_{\mathbf{C}^r_{\xi',\mathrm{colim}}} \mathscr{R}'^r \\
&= \left(\varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{N}'^r_{\mathrm{dR},[\sigma_1,\sigma_2]}\right) \otimes_{\mathbf{C}^r_{\xi',\mathrm{colim}}} \mathscr{R}'^r.
\end{aligned}$$

Note that, in the last identification, we make use of the compatibility of the $\mathbf{N}_{\mathrm{dR}}$ functor under finite étale extensions (cf. [DMY26, Proposition 10.13]).

6.3. **Subrings of constants.**

For later use, we need to compute the subrings of $\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}$ and $\mathscr{R}'$ consisting of horizontal elements with respect to $\frac{d}{d[T^\flat]}$. Let κ_1 denote the maximal algebraic extension of $\kappa(\!(\overline{\pi}_K)\!)$ contained in R'. In particular, κ_1 is a finite separable extension of $\kappa(\!(\overline{\pi}_K)\!)$.

Lemma 6.11. There exists σ_1, σ_2 sufficiently close to σ_ξ, such that the inclusion $\kappa_1 \subset R'$ factors as

$$\kappa_1 \hookrightarrow R'_{[\sigma_1,\sigma_2]} \hookrightarrow R'.$$

Proof. Recall that the colimit $R_{\xi,\mathrm{colim}}$ is precisely the uncompleted stalk $\mathbf{k}(\overline{\xi})$ at the type Ⅲ point $\overline{\xi}$ on the annulus $A_{\kappa(\!(\overline{\pi}_K)\!)}([\sigma_1,\sigma_2]; T^\flat)$ corresponding to the circle of radius σ_ξ (cf. Remark 6.5, Lemma 6.8).[23] For simplicity, let Y denote $A_{\kappa(\!(\overline{\pi}_K)\!)}([\sigma_1,\sigma_2]; T^\flat)$ so that $\mathcal{O}(Y) = R_{[\sigma_1,\sigma_2]}$. Let $h\colon Y' \to Y$ be the finite étale cover

[23] Here, the stalk can be taken in either the Berkovich or the adic topology. See §2.4.

given by the finite étale map $R_{[\sigma_1,\sigma_2]} \to R'_{[\sigma_1,\sigma_2]}$ and let $\overline{\xi}' \in Y'$ denote the preimage of $\overline{\xi}$. Since h is finite étale and $h^{-1}(\overline{\xi})$ consists of a singleton, the preimage of a system of open neighborhoods of $\overline{\xi}$ in Y is cofinal among the neighborhoods of $\overline{\xi}'$ in Y'. In particular, the colimit $\varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} R'_{[\sigma_1,\sigma_2]}$ is the uncompleted stalk $\mathbf{k}(\overline{\xi}')$ of the point $\overline{\xi}'$, and $R' = \mathscr{H}(\overline{\xi}')$ is the completed stalk. By [Ber93, Theorem 2.3.3, Proposition 2.4.1], we know that $\mathbf{k}(\overline{\xi}')$ is quasicomplete and is a henselian subfield of $\mathscr{H}(\overline{\xi}')$. By [Tem, Theorem 1.9.5], we conclude that $\mathbf{k}(\overline{\xi}')$ is relatively algebraically closed in $\mathscr{H}(\overline{\xi}')$. This implies that $\kappa_1 \subset \mathbf{k}(\overline{\xi}')$ and thus proves the lemma. □

Remark 6.12. By construction, κ_1 is relatively algebraically closed in R'. Therefore, by elementary field theory, if κ_2/κ_1 is any finite separable extension, then $R' \otimes_{\kappa_1} \kappa_2$ is a field.

By the classical theory of Robba ring (cf. Remark 3.22), the finite separable extension $\kappa_1/\kappa(\!(\overline{\pi}_K)\!)$ corresponds to a finite étale extension of Robba rings $\mathbf{C}^\dagger_K \to \mathscr{R}(\kappa_1)$. In terms of the notation in §1.9, $\mathscr{R}(\kappa_1)$ is abstractly isomorphic to $\mathscr{R}_{K''_0}$ for some finite unramified extension K''_0 of K'_0. Note that the differential operator $\frac{d}{d\pi_K}$ on $\mathbf{C}^\dagger_K$ uniquely extends to $\mathscr{R}(\kappa_1)$. Moreover, $\mathscr{R}(\kappa_1)$ can be naturally identified as a subring of $\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}$ for σ_1, σ_2 sufficiently close to σ_ξ.

Proposition 6.13. Let "$\log[T^\flat]$" be a free variable with $\frac{d}{d[T^\flat]}(\log[T^\flat]) = \frac{1}{[T^\flat]}$ and $\frac{d}{d\pi_K}(\log[T^\flat]) = 0$. Then, after shrinking $[\sigma_1, \sigma_2]$ if necessary, we have

$$\left(\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}\right)^{\frac{d}{d[T^\flat]}=0} = \left(\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}[\log[T^\flat]]\right)^{\frac{d}{d[T^\flat]}=0} = \mathscr{R}(\kappa_1).$$

identifying the differential operators $\frac{d}{d\pi_K}$ on both sides. As an immediate corollary, we have

$$(\mathscr{R}')^{\frac{d}{d[T^\flat]}=0} = \left(\mathscr{R}'[\log[T^\flat]]\right)^{\frac{d}{d[T^\flat]}=0} = \mathscr{R}(\kappa_1).$$

Proof. Unwinding the constructions and applying Lemma 6.11, we know that

$$\mathscr{R}(\kappa_1) \subset \left(\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}\right)^{\frac{d}{d[T^\flat]}=0} \subset \left(\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}[\log[T^\flat]]\right)^{\frac{d}{d[T^\flat]}=0}$$

for σ_1, σ_2 sufficiently close to σ_ξ. It remains to show the reverse inclusions. It suffices to show that $\left(\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}\right)^{\frac{d}{d[T^\flat]}=0} \subset \mathscr{R}(\kappa_1)$. Indeed, using the explicit description of $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$ in Lemma 4.20, we have

$$\left(\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}\right)^{\frac{d}{d[T^\flat]}=0} = \mathbf{C}^{[s,r]}_K$$

for r sufficiently small and for all $0 < s \leqslant r$. This implies $\left(\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}\right)^{\frac{d}{d[T^\flat]}=0} = \mathbf{C}^\dagger_K$, and hence

$$\left(\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^\dagger_K} \mathscr{R}(\kappa_1)\right)^{\frac{d}{d[T^\flat]}=0} = \mathscr{R}(\kappa_1).$$

Consider the finite étale map

$$g\colon \mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^\dagger_K} \mathscr{R}(\kappa_1) \to \mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}.$$

The map g realizes the target as a finite projective module over the source, say of rank m. Take an element $\alpha \in \mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}$ such that $\frac{d}{d[T^\flat]}(\alpha) = 0$. Let $T_\alpha\colon \mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]} \to \mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}$ denote the linear map given by multiplication by α and let $P_\alpha(x) \in (\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^\dagger_K} \mathscr{R}(\kappa_1))[x]$ denote the characteristic polynomial. Since $\frac{d}{d[T^\flat]}(\mathrm{tr}(T^i_\alpha)) = \mathrm{tr}(\frac{d}{d[T^\flat]}(T^i_\alpha)) = 0$ for all $i \geqslant 1$, we know that all coefficients of $P_\alpha(x)$ are $\frac{d}{d[T^\flat]}$-horizontal sections, and thus $P_\alpha(x) \in \mathscr{R}(\kappa_1)[x]$.

Next, we construct a finite separable extension κ_2/κ_1 (with associated extension of Robba rings $\mathscr{R}(\kappa_2)/\mathscr{R}(\kappa_1)$) together with a ring map $\eta\colon \mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^\dagger_K} \mathscr{R}(\kappa_1) \to \mathscr{R}(\kappa_2)$ such that the base change of g along η splits into a direct product $\prod_{i=1}^m \mathscr{R}(\kappa_2)$. For this, we first choose any finite separable extension κ_2/κ_1 such that $\kappa_1 \to \kappa_2$ factors through natural inclusion $\kappa_1 \hookrightarrow R'_{[\sigma_1,\sigma_2]}$. This induces a finite étale map $\delta\colon \mathscr{R}(\kappa_1) \to \mathscr{R}(\kappa_2)$ which factors as $\mathscr{R}(\kappa_1) \hookrightarrow \mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]} \xrightarrow{\mu} \mathscr{R}(\kappa_2)$. Now, the composition $\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} \to \mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]} \xrightarrow{\mu} \mathscr{R}(\kappa_2)$ and the map

δ induce a ring map $\eta\colon \mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^{\dagger}_K} \mathscr{R}(\kappa_1) \to \mathscr{R}(\kappa_2)$. The base change of g along η is a finite étale cover of $\mathscr{R}(\kappa_2)$. Replacing κ_2 by a further separable extension, we can make sure that this finite étale cover splits into direct products $\prod_{i=1}^m \mathscr{R}(\kappa_2)$. To sum up, we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} & \longrightarrow & \mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2) & \longrightarrow & \prod_{i=1}^m \mathscr{R}(\kappa_2) \\
\uparrow{\scriptstyle g} & & \uparrow{\scriptstyle g'} & & \uparrow \\
\mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^{\dagger}_K} \mathscr{R}(\kappa_1) & \longrightarrow & \mathbf{C}^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^{\dagger}_K} \mathscr{R}(\kappa_2) & \longrightarrow & \mathscr{R}(\kappa_2) \\
\uparrow & & \uparrow & \overset{\eta}{\nearrow} & \\
\mathscr{R}(\kappa_1) & \xrightarrow{\delta} & \mathscr{R}(\kappa_2) & &
\end{array}$$

where each square is a push-out square.

Let $(\alpha_1,\dots,\alpha_m) \in \prod_{i=1}^m \mathscr{R}(\kappa_2)$ be the image of α under the composition of the top row of the diagram. Also consider term $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2)$ in the middle column. Since taking base change commutes with taking characteristic polynomials, the multiplication-by-α map on $\mathbf{C}' \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2)$ and the multiplication-by-$(\alpha_1,\dots,\alpha_m)$ map on $\prod_{i=1}^m \mathscr{R}(\kappa_2)$ both have characteristic polynomial $P_\alpha(x) \in \mathscr{R}(\kappa_1)[x]$. In particular, we have

$$P_\alpha(x) = (x-\alpha_1)\cdots(x-\alpha_m) \in \mathscr{R}(\kappa_2)[x]$$

By Cayley–Hamilton, we have an identity $(\alpha-\alpha_1)\cdots(\alpha-\alpha_m)=0$ on $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2)$.

Now, consider $R'' \coloneqq R' \otimes_{\kappa_1} \kappa_2$, which is a field by Remark 6.12. Let $\mathscr{R}' \to \mathscr{R}''$ be the extension of fake Robba rings associated with the finite separable extension R''/R'. Then there is a natural embedding $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2) \to \mathscr{R}''$, and thus $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2)$ is an integral domain. Consequently, $\alpha = \alpha_i$ as an element in $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2)$, for some i. By the injectivity of $\mathscr{R}(\kappa_1) \to \mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]}$ and the faithful flatness of $\mathscr{R}(\kappa_1) \to \mathscr{R}(\kappa_2)$, we conclude that

$$\alpha \otimes 1 \in \big(\{1\} \otimes \mathscr{R}(\kappa_2)\big) \cap \Big(\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]} \otimes \{1\}\Big) = \mathscr{R}(\kappa_1);$$

namely, $\alpha \in \mathscr{R}(\kappa_1)$. This finishes the proof of the proposition. □

From now on, we shrink $[\sigma_1,\sigma_2]$ so that Lemma 6.11 and Proposition 6.13 apply.

6.4. **Unipotence implies semistability.**

Recall from Construction 6.10 that we have realized the finite extension $\mathbf{C}^{\dagger}_{\xi} \to \mathscr{R}'$ of fake Robba rings via a finite étale cover $f\colon X'_{[\sigma_1,\sigma_2]} \to X_{[\sigma_1,\sigma_2]}$. The finite projective module $\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]} = \mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X'_{[\sigma_1,\sigma_2]}})$ over $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]}$ is equipped with an arithmetic differential operator D_{π_K} and a geometric differential operator $D_{[T^\flat]}$, as well as a Frobenius structure. We also consider the full connection $\nabla = D_{\pi_K} d\pi_K + D_{[T^\flat]} d[T^\flat]$ so that $(\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]},\nabla)$ is a finite projective (φ,∇)-module over $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]}$. Similarly, for r sufficiently small, we have a finite projective (φ,∇)-module $\mathbf{N}'^{r}_{\mathrm{dR},[\sigma_1,\sigma_2]} = \mathbf{N}^{r}_{\mathrm{dR}}(\mathbb{L}|_{X'_{[\sigma_1,\sigma_2]}})$ over $\mathbf{C}'^{r}_{[\sigma_1,\sigma_2]}$, equipped with differential operators D_{π_K} and $D_{[T^\flat]}$. Moreover, we know that

$$\mathbf{N}'_{\mathrm{dR},\xi} = \mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]}} \mathscr{R}'$$

is unipotent with respect to ∇. A key ingredient for the proof of Theorem 6.1 is the following proposition, which states that the unipotence of $(\mathbf{N}'_{\mathrm{dR},\xi},\nabla)$ can be spread out to $(\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]},\nabla)$ for some sufficiently small interval $[\sigma_1,\sigma_2]$.

Proposition 6.14. After shrinking $[\sigma_1,\sigma_2]$ if necessary, the differential module $\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]}$ is strongly unipotent for ∇; that is, $\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]}$ is free over $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]}$ and admits a basis $w_1,\dots,w_d$ such that the matrices of $[T^\flat]D_{[T^\flat]}$ and $\pi_K D_{\pi_K}$ relative to this basis are both upper-triangular nilpotent with entries in K''_0. In particular, $\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]}$ is unipotent for the full connection ∇.

The proof of Proposition 6.14 will occupy §6.5–6.6. Granting Proposition 6.14, let us finish the proof of Theorem 6.1.

Proof of Theorem 6.1 assuming Proposition 6.14.
The first step is to compute the geometric horizontal sections after adjoining "$\log[T^\flat]$". Let σ_1, σ_2 be sufficiently close to σ_ξ as in Proposition 6.14. Consider

$$N' \coloneqq \left(\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]}[\log[T^\flat]]\right)^{D_{[T^\flat]}=0}.$$

By Proposition 6.13 and Proposition 6.14, and noting that $D_{[T^\flat]}$ commutes with D_{π_K} and is compatible with Frobenius, we deduce that N' is a finite free $\mathscr{R}(\kappa_1)$-module of rank d (where d is the rank of $\mathbb{L}$) and (N', D_{π_K}) is a differential module over $(\mathscr{R}(\kappa_1), \frac{d}{d\pi_K})$ equipped with Frobenius structure. By the classical p-adic local monodromy theorem of André–Kedlaya–Mebkhout (Theorem 3.23), there exists a finite separable extension κ_2/κ_1 (with the associated Robba ring extension denoted by $\mathscr{R}(\kappa_2)/\mathscr{R}(\kappa_1)$), such that $N' \otimes_{\mathscr{R}(\kappa_1)} \mathscr{R}(\kappa_2)$ becomes unipotent with respect to D_{π_K}. By replacing K by finite extension K' so that $\mathbf{C}^\dagger_{K'}$ contains $\mathscr{R}(\kappa_2)$, we may assume that N' is already unipotent with respect to D_{π_K}. Hence, the $\mathscr{R}(\kappa_1)[\log \pi_K]$-module $N'[\log \pi_K]$ admits a horizontal basis $w_1, \dots, w_d$ with respect to D_{π_K}. Note that $w_1, \dots, w_d$ are also geometrically horizontal as $\frac{d}{d[T^\flat]}(\log \pi_K) = 0$. We arrive at a decomposition

$$\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]}[\log[T^\flat], \log \pi_K] \cong \bigoplus_{i=1}^{d} \mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]}[\log[T^\flat], \log \pi_K] \cdot w_i$$

with each w_i satisfying $D_{[T^\flat]}(w_i) = D_{\pi_K}(w_i) = 0$. Note that all these w_i's are fixed by an open subgroup of Γ. By enlarging K and passing to a p-power Kummer cover of $X'_{[\sigma_1,\sigma_2]}$ (cf. Remark 4.17), we may further assume that all w_i's are fixed by Γ. This allows us to apply Proposition 4.30 (note that the functors $\mathbf{N}_{\mathrm{dR}}$ and $D^\dagger_{\mathrm{rig}}$ coincide after inverting t) and conclude the proof. □

6.5. **Unipotence over annuli: special case.**

In the remainder of this section, we prove Proposition 6.14. Thanks to Theorem 6.6, we already achieve unipotence of ∇ "at ξ" by passing to a finite étale cover. We want to propagate this unipotence to an open neighborhood of ξ. This is akin to the process of extending the region of convergence for solutions of p-adic differential equations over a (fake) poly-annulus.

We first prove Proposition 6.14 in the special case where $\mathscr{R}' = \mathbf{C}^\dagger_\xi$. In this case, $\mathbf{N}^r_{\mathrm{dR},\xi}$ is already unipotent with respect to ∇, for r sufficiently small. Our goal is to show that

$$\mathbf{N}_{\mathrm{dR,colim}} = \varinjlim_{r\to 0^+} \mathbf{N}^r_{\mathrm{dR,colim}} = \varinjlim_{r\to 0^+} \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]}$$

is strongly unipotent. Since

$$\mathbf{N}_{\mathrm{dR},\xi} = \mathbf{N}_{\mathrm{dR,colim}} \otimes_{\mathbf{C}^\dagger_{\xi,\mathrm{colim}}} \mathbf{C}^\dagger_\xi,$$

this can be viewed as a "decompletion question" from $\mathbf{C}^\dagger_\xi$ to $\mathbf{C}^\dagger_{\xi,\mathrm{colim}}$.

6.5.1. *Intrinsic generic radii over log RP subsets.*

We first recall the notion of generic radius of convergence, originally developed by Dwork, Robba, Kedlaya, and others. In the classical setting, suppose (V, D) is a finite nonzero differential module over $(F_\rho, \frac{d}{dt})$ where $\rho > 0$ and F_ρ denotes the completion of $K(t)$ under the ρ-Gauss norm (cf. [Ked22, Definition 9.4.1]). Consider the spectral radius of the operator D defined by $|D|_{\mathrm{sp},V} = \lim_{s\to+\infty} |D^s|_V^{1/s}$, where $|D^s|_V$ denotes the operator norm of D^s. Then the *generic radius of convergence* of (V, D) is defined to be $R(V) = \omega \cdot |D|^{-1}_{\mathrm{sp},V}$, where $\omega = p^{-1/(p-1)}$. The *intrinsic generic radius of convergence* is defined to be

$$IR(V) = \rho^{-1} R(V) = \frac{|\frac{d}{dt}|_{\mathrm{sp},F_\rho}}{|D|_{\mathrm{sp},V}},$$

which is a real number in $(0, 1]$ (cf. [Ked22, Definition 9.4.4, 9.4.7, 6.1.2, 6.1.3]).

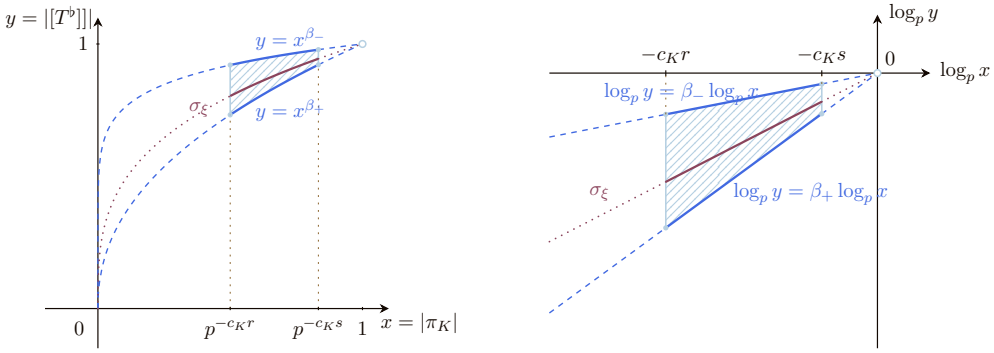


FIGURE 1. A cartoon illustration of $S^{[s,r]}_{[\sigma_1,\sigma_2]}$ (left) and $\log_p S^{[s,r]}_{[\sigma_1,\sigma_2]}$ (right)

To treat the type 𝕀𝕀𝕀 point of interest, we will need generalizations of these notions from [Ked09]. For the setup, let us describe $\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]}$ (resp. $\mathbf{C}^{r}_{[\sigma_1,\sigma_2]}$) in terms of *log-RP subset* (resp. *ind-log-RP subset*) of $(\mathbb{R}_{>0})^2$ in the sense of [Ked09, Definition 4.1.2]. For an interval $I = [\sigma_1, \sigma_2]$, consider the subset

$$\begin{aligned} S^{[s,r]}_{[\sigma_1,\sigma_2]} &:= \left\{(p^{-c_K\cdot u}, \sigma^u) \;\middle|\; u \in [s,r],\, \sigma \in [\sigma_1,\sigma_2]\right\} \\ &= \left\{(x, x^\beta) \;\middle|\; x \in [p^{-c_K\cdot r}, p^{-c_K\cdot s}],\, \beta \in [-\frac{1}{c_K}\log_p \sigma_2, -\frac{1}{c_K}\log_p \sigma_1]\right\} \subset (\mathbb{R}_{>0})^2. \end{aligned}$$

Here $c_K = \frac{p}{(p-1)}\frac{1}{e_K}$ as in Notation 1.9. The subset $S^{(0,r]}_{[\sigma_1,\sigma_2]}$ is defined in a similar way.

Remark 6.15. In the language of [Ked09], $S^{[s,r]}_{[\sigma_1,\sigma_2]}$ (resp. $S^{(0,r]}_{[\sigma_1,\sigma_2]}$) is referred to as a log-RP subset (resp. ind-log-RP subset) because its image under the logarithmic map is a rational polygon (resp. limit of rational polygons); namely,

$$\log_p(S^{[s,r]}_{[\sigma_1,\sigma_2]}) = \left\{(x, \beta x) \middle| x \in [-c_K\cdot r, -c_K\cdot s], \beta \in [-\frac{1}{c_K}\log_p \sigma_2, -\frac{1}{c_K}\log_p \sigma_1]\right\} \subset \mathbb{R}^2.$$

See Figure 1 for an illustration of the log-RP subset. (In the figure, $\beta_- = -\frac{1}{c_K}\log_p \sigma_2$ and $\beta_+ = -\frac{1}{c_K}\log_p \sigma_1$).

We adopt the notation from [Ked09, Notation 4.1.3]. Let $\mathbb{A}^{2,\mathrm{an}}_{K'_0}$ denote the analytic affine plane with coordinates π_K and $[T^\flat]$. For any (ind-)log-RP subset $S \subset \mathbb{R}^2_{>0}$, let $A_{K'_0}(S)$ denote the open subset of $\mathbb{A}^{2,\mathrm{an}}_{K'_0}$ defined by the condition $(|\pi_K|, |[T^\flat]|) \in S$. Then we have

$$\mathbf{C}^{[s,r]}_{[\sigma_1,\sigma_2]} = \Gamma(A_{K'_0}(S^{[s,r]}_{[\sigma_1,\sigma_2]}), \mathcal{O}), \qquad \mathbf{C}^{r}_{[\sigma_1,\sigma_2]} = \Gamma(A_{K'_0}(S^{(0,r]}_{[\sigma_1,\sigma_2]}), \mathcal{O})$$

(note that $\lambda_u(\pi_K) = p^{-c_K u}$). For any $\sigma \in [\sigma_1, \sigma_2]$ and $s \in (0, r]$, consider the point

$$R(\sigma, s) = (p^{-c_K\cdot s}, \sigma^s) = (p^{-c_K\cdot s}, (p^{-c_K\cdot s})^\beta) \in S^{(0,r]}_{[\sigma_1,\sigma_2]}$$

where $\beta = -\frac{1}{c_K}\cdot\log_p \sigma$. This point gives rise to a norm $\lambda_{s,\sigma}$ on $\mathbf{C}^{(0,r]}_{[\sigma_1,\sigma_2]}$ given by[24]

$$\lambda_{s,\sigma}\left(\sum_{i\in\mathbb{Z}} c_i [T^\flat]^i\right) = \sup_{i\in\mathbb{Z}}\left\{\lambda_s(c_i)\cdot\sigma^{si}\right\}.$$

Definition 6.16 ([Ked09, Definition 4.2.1])**.** Let $(M, D_{\pi_K}, D_{[T^\flat]})$ be a finite projective differential module over $(\mathbf{C}^{(0,r]}_{[\sigma_1,\sigma_2]}, \frac{d}{d\pi_K}, \frac{d}{d[T^\flat]})$ and let $R(\sigma, s) \in S^{(0,r]}_{[\sigma_1,\sigma_2]}$ be a point as above. Let $F_{\sigma,s}$ denote the completion of $\mathrm{Frac}(\mathbf{C}^{(0,r]}_{[\sigma_1,\sigma_2]})$ under $\lambda_{\sigma,s}$, and consider the base change $V_{\sigma,s} := M \otimes_{\mathbf{C}^{(0,r]}_{[\sigma_1,\sigma_2]}} F_{\sigma,s}$. Define the *intrinsic generic*

[24] This norm is denoted as $|\cdot|_{R(\sigma,s)}$ in [Ked09, Definition 4.1.6].

radius of M at $R(\sigma, s)$ to be

$$IR(M, R(\sigma,s)) \coloneqq \min\left\{1,\ \frac{|\frac{d}{d[T^\flat]}|_{\mathrm{sp},F_{\sigma,s}}}{|D_{[T^\flat]}|_{\mathrm{sp},V_{\sigma,s}}},\ \frac{|\frac{d}{d\pi_K}|_{\mathrm{sp},F_{\sigma,s}}}{|D_{\pi_K}|_{\mathrm{sp},V_{\sigma,s}}}\right\}.$$

By assumption, we know that $\mathbf{N}^r_{\mathrm{dR},\xi}$ is unipotent for some sufficiently small r; that is, it admits a filtration whose graded pieces are trivial differential modules. In particular, for σ_1, σ_2 sufficiently close to σ_ξ, there exists $r_0 > 0$ such that

$$IR(\mathbf{N}^{r_0}_{\mathrm{dR},[\sigma_1,\sigma_2]}, R(\sigma_\xi, s)) = 1$$

for all $s \in (0, r_0]$ (cf. [Ked09, Remark 4.2.4] and [KX10, Lemma 1.2.9]). Note that $\alpha = -\frac{1}{c_K}\log_p \sigma_\xi$ is irrational by the assumption. Therefore, according to [Ked09, Proposition 6.2.1], by decreasing r_0 and shrinking the interval $[\sigma_1, \sigma_2]$ if necessary, we may assume that

$$IR(\mathbf{N}^{r_0}_{\mathrm{dR},[\sigma_1,\sigma_2]}, R(\sigma, s)) = 1 \tag{6.3}$$

for all $\sigma \in [\sigma_1, \sigma_2]$ and $s \in (0, r_0]$.

6.5.2. *A monomial change of coordinates.*

From the above discussion, we may assume that $\mathbf{N}^{r_0}_{\mathrm{dR},[\sigma_1,\sigma_2]}$ has intrinsic generic radius equal to 1 at all points $R(\sigma, s) \in S^{(0,r_0]}_{[\sigma_1,\sigma_2]}$. To deduce unipotence from this condition, we would like to put ourselves in a setting where we can apply the multivariable p-adic Fuchs theorem[25] of Wang [Wan24, Corollary 3.20] (which slightly generalizes the work of Gachet [Gac99]). However, Wang's theorem only applies to honest poly-annuli rather than the fake ones. The key is to use a change of coordinates to transfer the ind-log-RP subsets into a "standard rectangle".

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix with $a, b, c, d \in \mathbb{Z}$ and $\det A = \pm 1$. Consider the map $\mu_A \colon \mathbb{R}^2_{>0} \to \mathbb{R}^2_{>0}$ sending $(x, y) \mapsto (x^a y^b, x^c y^d)$. We refer to μ_A as a *monomial change of coordinates.*[26] In fact, if we take logarithms on both the source and target, the resulting map

$$\log \mu_A \colon \log_p(\mathbb{R}^2_{>0}) = \mathbb{R}^2 \to \log_p(\mathbb{R}^2_{>0}) = \mathbb{R}^2$$

is simply the affine transformation given by the multiplication of the matrix A. Now, for any $\delta > 0$, let S^δ denote the ind-log-RP subset $[p^{-\delta}, 1) \times [p^{-\delta}, 1) \subset (\mathbb{R}_{>0})^2$. Note that $\log_p S^\delta = [-\delta, 0) \times [-\delta, 0)$. For a suitable choice of A and shrinking $[\sigma_1, \sigma_2]$ if necessary, we can ensure that the image of S^δ under μ_A lies in $S^{(0,r]}_{[\sigma_1,\sigma_2]}$ for sufficiently small δ. Indeed, we first choose A such that the image of the third quadrant (i.e., $(-\infty, 0) \times (-\infty, 0)$) under $\log \mu_A$ lies entirely inside the cone

$$\left\{(x, \beta x) \,\middle|\, x < 0,\ \beta \in [-\frac{\log_p \sigma_2}{c_K}, -\frac{\log_p \sigma_1}{c_K}]\right\}$$

and contains the ray

$$\left\{(x, \beta x) \,\middle|\, x < 0,\ \beta = -\frac{\log_p \sigma_\xi}{c_K}\right\}.$$

Shrinking the interval $[\sigma_1, \sigma_2]$ and multiplying A by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ if necessary, we may assume that the images of the rays $(-\infty, 0) \times \{0\}$ and $\{0\} \times (-\infty, 0)$ under the linear transformation $\log \mu_A$ are precisely the rays $\{(x, \beta x) \,|\, x < 0, \beta = -\frac{1}{c_K}\log_p(\sigma_1)\}$ and $\{(x, \beta x) \,|\, x < 0, \beta = -\frac{1}{c_K}\log_p(\sigma_2)\}$, respectively. For any given $r \in (0, r_0]$, by making δ sufficiently small, we can ensure that the image of $\log_p S^\delta$ under $\log \mu_A$ is entirely contained in $\log_p(S^{(0,r]}_{[\sigma_1,\sigma_2]})$. Consequently, we obtain an injective map

$$\mu_A \colon S^\delta \to S^{(0,r]}_{[\sigma_1,\sigma_2]}.$$

Passing to the ring of functions, we obtain an injective ring homomorphism

$$\mu_A^* \colon \mathbf{C}^r_{[\sigma_1,\sigma_2]} = \Gamma(A_{K'_0}(S^{(0,r]}_{[\sigma_1,\sigma_2]}), \mathcal{O}) \to \Gamma(A_{K'_0}(S^\delta), \mathcal{O})$$

[25] The single variable p-adic Fuchs theorem is due to Christol and Mebkhout (see [Ked22, Theorem 13.6.1]).

[26] This is called a *toric coordinate change* in [Ked09, Definition 4.1.4].

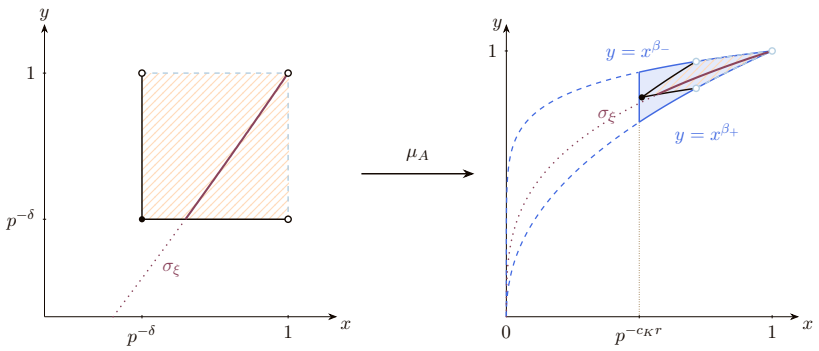


FIGURE 2. A cartoon illustration of $\mu_A \colon S^\delta \to S^{(0,r]}_{[\sigma_1,\sigma_2]}$

sending $\pi_K \mapsto u^a v^b, [T^\flat] \mapsto u^c v^d$. Here, we have identified $A_{K'_0}(S^\delta)$ with the half-open poly-annulus $A_{K'_0}([p^{-\delta}, 1)) \times A_{K'_0}([p^{-\delta}, 1))$ with coordinate functions u, v. The merit here is that $A_{K'_0}(S^\delta)$ is an honest poly-annulus. In what follows, we shall study the p-adic differential equation $\mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]}$ over $\mathbf{C}^r_{[\sigma_1,\sigma_2]}$ using the coordinate u, v by taking base change to $A_{K'_0}([p^{-\delta}, 1)) \times A_{K'_0}([p^{-\delta}, 1))$ along μ_A^*, so that the multivariable p-adic Fuchs theorem applies.

Remark 6.17. The map μ_A sends the rectangle S^δ to a "tilted rectangular region" $\mu_A(S^\delta)$ inside $S^{(0,r]}_{[\sigma_1,\sigma_2]}$. See Figure 2 for an illustration of μ_A, where both the rectangular and the tilted rectangular regions are colored in orange. The curve $y = x^{\beta_-}$ (resp. $y = x^{\beta_+}$) on the right-hand side of the illustration corresponds to the line $y = 1$ (resp. $x = 1$) on the left-hand side. Moreover, the curvy ray (colored in red)

$$\left\{(x, x^\beta) \,\middle|\, p^{-c_K r} \leqslant x < 1,\ \beta = -\frac{1}{c_K} \log_p \sigma_\xi\right\}$$

on the right-hand side corresponds to the fake Robba ring $\mathscr{R}^\lambda$ (as $r \to 0^+$). Its preimage is a line segment (colored in red) with irrational slope on the left-hand side, corresponding to another fake Robba ring $\mathscr{R}^{A^{-1}\lambda}$. The map μ_A^* induces an isomorphism $\mathscr{R}^\lambda \cong \mathbf{C}^\dagger_\xi \xrightarrow{\sim} \mathscr{R}^{A^{-1}\lambda}$ that fits into a commutative diagram

$$\begin{array}{ccc} \Gamma(A_{K'_0}(S^{(0,r]}_{[\sigma_1,\sigma_2]}), \mathcal{O}) & \xrightarrow{\mu_A^*} & \Gamma(A_{K'_0}(S^\delta), \mathcal{O}) \\ \downarrow & & \downarrow \\ \mathscr{R}^\lambda & \xrightarrow{\quad\sim\quad} & \mathscr{R}^{A^{-1}\lambda}. \end{array}$$

The Frobenius structure on $\mathscr{R}^\lambda$ transfers to a Frobenius structure on $\mathscr{R}^{A^{-1}\lambda}$.

Remark 6.18. Let $A^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ be the inverse of A. Then $u \mapsto \pi_K^{a'}[T^\flat]^{b'}$, $v \mapsto \pi_K^{c'}[T^\flat]^{d'}$ induces an injective ring map

$$\nu_A^* \colon \Gamma(A_{K'_0}(S^\delta), \mathcal{O}) \to \Gamma(A_{K'_0}(S^{(0,r_1])}_{[\sigma'_1,\sigma'_2]}), \mathcal{O})$$

for some $0 < r_1 < r$ and for any $\sigma'_1, \sigma'_2 \in p^{\mathbb{Q}}$ satisfying $\sigma_1 < \sigma'_1 < \sigma_\xi < \sigma'_2 < \sigma_2$, such that the composite $\nu_A^* \circ \mu_A^*$ is simply the natural inclusion $\mathbf{C}^r_{[\sigma_1,\sigma_2]} \subset \mathbf{C}^{r_1}_{[\sigma'_1,\sigma'_2]}$. Taking colimit as r and s go to 0, we obtain injective ring maps

$$\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]} \xrightarrow{\mu_A^*} \varinjlim_{\delta \to 0^+} \Gamma(A_{K'_0}(S^\delta), \mathcal{O}) \xrightarrow{\nu_A^*} \mathbf{C}^\dagger_{[\sigma'_1,\sigma'_2]}$$

such that the composite is the natural inclusion. Note that the term in the middle is just the (honest) two-dimensional Robba ring

$$\mathscr{R}_{K'_0} \widehat{\otimes}_{K'_0} \mathscr{R}_{K'_0} = \varinjlim_{r \to 0^+} \varprojlim_{0 < s \leqslant r} \mathscr{R}^{[s,r]}_{K'_0} \widehat{\otimes}_{K'_0} \mathscr{R}^{[s,r]}_{K'_0}$$

with coordinates u, v. Moreover, since both μ_A^* and ν_A^* are given by monomial change of coordinates, it is straightforward to check that the Frobenius structures on $\mathbf{C}^r_{[\sigma_1,\sigma_2]}$ and $\mathbf{C}^\dagger_{[\sigma_1,\sigma_2]}$ induce Frobenius structures on the poly-annulus $\Gamma(A_{K_0'}(S^\delta),\mathcal{O})$ and the two-dimensional Robba ring $\mathscr{R}_{K_0'}\widehat{\otimes}_{K_0'}\mathscr{R}_{K_0'}$.

6.5.3. *Strong unipotence of* ∇.

For $r \in (0, r_0]$ and δ as above, consider

$$N_\delta \coloneqq \mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^r_{[\sigma_1,\sigma_2]},\mu_A^*} \Gamma(A_{K_0'}(S^\delta),\mathcal{O}),$$

equipped with the full connection

$$\nabla = D_u du + D_v dv = \Delta_u \cdot d\log u + \Delta_v \cdot d\log v$$

where $\Delta_u = uD_u$ and $\Delta_v = vD_v$. By Remark 6.17, the map μ_A^* induces an isomorphism

$$\mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}^r_{[\sigma_1,\sigma_2]}} \mathbf{C}^\dagger_\xi = \mathbf{N}_{\mathrm{dR},\xi} \xrightarrow{\sim} N_\delta \otimes_{\Gamma(A_{K_0'}(S^\delta),\mathcal{O})} \mathscr{R}^{A^{-1}\lambda} \tag{6.4}$$

of (φ,∇)-modules, under the identification $\mathscr{R}^\lambda \cong \mathscr{R}^{A^{-1}\lambda}$.

By (6.3), Remark 6.18, and [Ked09, Proposition 4.2.5], we know that N_δ satisfies the Robba condition. Moreover, since N_δ is equipped with a Frobenius structure (cf. Remark 6.18), we know that N_δ admits a set of p-adic exponents $\{a_1,\dots,a_d\}$ with $a_i \in (\mathbb{Q}\cap\mathbb{Z}_p)^2$ such that they have p-adically non-Liouville differences. This is precisely the situation where we can apply the multivariable p-adic Fuchs theorem to N_δ. By [Wan24, Corollary 3.20],[27] we conclude that N_δ is free and admits a basis such that the matrices of Δ_u and Δ_v both have entries in K_0' and their eigenvalues represent (the equivalence class of) the exponents a_i's. With respect to this basis, the full connection can be written as

$$\nabla = d + T_1 \cdot d\log u + T_2 \cdot d\log v,$$

with $T_i \in \mathrm{Mat}_{d\times d}(K_0')$ satisfying $T_1T_2 = T_2T_1$.

We claim that there exists a basis as above so that T_1 and T_2 are upper-triangular nilpotent. Via a change of basis by a matrix in $\mathrm{GL}_d(K_0')$, we may assume that N_δ decomposes into simultaneous generalized eigenspaces of T_1 and T_2. Let M be any one of these direct summands, with corresponding basis $e_1,\dots,e_h$. With respect to this basis, we may assume that $T_i = \lambda_i I + N_i$ with N_i upper-triangular nilpotent, for $i = 1, 2$; namely,

$$\Delta_u = u\frac{d}{du} + \lambda_1 I + N_1, \qquad \Delta_v = v\frac{d}{dv} + \lambda_2 I + N_2.$$

Note that $\lambda_i \in \mathbb{Q}\cap\mathbb{Z}_p$. We claim that λ_1 and λ_2 are actually integers. Granting this claim, we simply modify the basis on M by multiplying $u^{-\lambda_1}v^{-\lambda_2}$ (and repeat this on each direct summand), and we are done.

It remains to prove the claim. Suppose that $(\lambda_1,\lambda_2) \notin \mathbb{Z}^2$. Consider

$$M' \coloneqq M \otimes_{\Gamma(A_{K_0'}(S^\delta),\mathcal{O})} \mathscr{R}^{A^{-1}\lambda}$$

(cf. Remark 6.17), which is a direct summand of $N_\delta \otimes_{\Gamma(A_{K_0'}(S^\delta),\mathcal{O})} \mathscr{R}^{A^{-1}\lambda} \cong \mathbf{N}_{\mathrm{dR},\xi}$ (cf. (6.4)). By assumption, $\mathbf{N}_{\mathrm{dR},\xi}$ is unipotent, and hence so is M'. In particular, M' admits a nonzero horizontal vector for the full connection, say $s = \sum_{i=1}^h s_i e_i \in M'$. We write each $s_i = \sum_{m,n\in\mathbb{Z}} s_i^{mn} u^m v^n$ in terms of power series expansions in the fake Robba ring. Let $\mathbf{s} = (s_1,\cdots,s_h)$ as a column vector and write $\mathbf{s} = \sum_{m,n\in\mathbb{Z}} \mathbf{s}^{mn} u^m v^n$ accordingly. Since $\Delta_u(\mathbf{s}) = \Delta_v(\mathbf{s}) = 0$, we obtain

$$\Big((m+\lambda_1)I_d + N_1\Big)\mathbf{s}^{mn} = 0, \qquad \Big((n+\lambda_2)I_d + N_2\Big)\mathbf{s}^{mn} = 0.$$

If $\lambda_1 \notin \mathbb{Z}$, then $(m+\lambda_1)I + N_1$ is invertible for all m, so $s = 0$. Similarly, if $\lambda_2 \notin \mathbb{Z}$, then $s = 0$. This is a contradiction and justifies the claim.

So far, we have achieved strong unipotence for N^δ. Using ν_A^*, we obtain strong unipotence for $\mathbf{N}^r_{\mathrm{dR},[\sigma_1,\sigma_2]}$ for sufficiently small r and for σ_1,σ_2 sufficiently close to σ_ξ (note that K_0'' is just K_0' here). This settles the special case of Proposition 6.14.

[27] [Wan24, Corollary 3.20] is an enhancement of [Gac99, Théorème pg.216], removing the freeness condition from the latter.

6.6. **Unipotence over annuli: general case.**

Finally, we treat the general case of Proposition 6.14. The idea is to reduce to the special case handled in §6.5.

6.6.1. *Some preparations.*

Recall from §6.2–6.4 the finite étale extension $\mathbf{C}^{\dagger}_{\xi} \to \mathscr{R}'$ of fake Robba rings induced from the finite separable extension R'/R_{ξ}. We have also constructed a finite étale cover $f\colon X'_{[\sigma_1,\sigma_2]} \to X_{[\sigma_1,\sigma_2]}$ that realizes the extension $\mathbf{C}^{\dagger}_{\xi} \to \mathscr{R}'$. For $r > 0$ sufficiently small, the map f induces a finite étale map $\mathbf{C}^{r}_{[\sigma_1,\sigma_2]} \to \mathbf{C}'^{r}_{[\sigma_1,\sigma_2]}$ and a differential module $\mathbf{N}'^{r}_{\mathrm{dR},[\sigma_1,\sigma_2]}$ over $\mathbf{C}'^{r}_{[\sigma_1,\sigma_2]}$, equipped with a Frobenius structure.

To proceed, we first introduce some more notation. Recall that R_{ξ} (resp. R') coincides with the completed stalk $\mathscr{H}(\overline{\xi})$ (resp. $\mathscr{H}(\overline{\xi}')$) (see Remark 6.5 and the proof of Lemma 6.11). In particular, the residue field of R_{ξ} (resp. R') is κ (resp. the algebraic closure κ' of κ in R'). Note that K''_0 in the statement of Proposition 6.14 coincides with $W(\kappa')[1/p]$. The following is an analogue of $\mathbf{A}^{r}_{[\sigma_1,\sigma_2]}$, but with variables x, y instead of π_K and $[T^{\flat}]$.

Definition 6.19. Let $\tau_0 > 0$ such that $\tau_0 \notin p^{\mathbb{Q}}$.

(1) For any $0 < \tau_1 < \tau_0 < \tau_2$ with $\tau_1, \tau_2 \in p^{\mathbb{Q}}$ and any sufficiently small $s > 0$, define

$$\mathbf{A}^{s}_{[\tau_1,\tau_2]}(x,y) \coloneqq \left\{ \sum_{i,j\in\mathbb{Z}} c_{ij}x^i y^j \;\middle|\; c_{ij} \in W(\kappa'), |c_{ij}| \cdot p^{-c_K \cdot si}\tau^{sj} \to 0 \text{ for all } \tau \in [\tau_1,\tau_2], \text{ as } |i|+|j| \to +\infty \right\}.$$

For any $s' \in (0,s]$, we can define a norm $\lambda_{s'}$ on $\mathbf{A}^{s}_{[\tau_1,\tau_2]}(x,y)$ by

$$\lambda_{s'}\left(\sum_{i,j\in\mathbb{Z}} c_{ij}x^i y^j\right) = \sup_{i,j\in\mathbb{Z}} \sup_{\tau\in[\tau_1,\tau_2]} \{|c_{ij}| \cdot p^{-c_K \cdot s'i}\tau^{s'j}\}.$$

(2) Let

$$\mathbf{A}^{s}_{\tau_0,\mathrm{colim}}(x,y) \coloneqq \varinjlim_{\tau_1<\tau_0<\tau_2} \mathbf{A}^{s}_{[\tau_1,\tau_2]}(x,y)$$

and

$$\mathbf{A}^{\dagger}_{\tau_0,\mathrm{colim}}(x,y) \coloneqq \varinjlim_{s\to 0^+} \mathbf{A}^{s}_{\tau_0,\mathrm{colim}}(x,y).$$

Let $\mathbf{A}^{s}_{\tau_0}(x,y)$ be the completion of $\mathbf{A}^{s}_{\tau_0,\mathrm{colim}}(x,y)$ with respect to the norm λ_{s,τ_0} defined by

$$\lambda_{s,\tau_0}\left(\sum_{i,j\in\mathbb{Z}} c_{ij}x^i y^j\right) = \sup_{i,j\in\mathbb{Z}} \{|c_{ij}| \cdot p^{-c_K \cdot si}\tau_0^{sj}\}$$

and let $\mathbf{A}^{\dagger}_{\tau_0}(x,y) = \varinjlim_{s\to 0^+} \mathbf{A}^{s}_{\tau_0}(x,y)$.

(3) Let $\mathbf{B}^{s}_{[\tau_1,\tau_2]}(x,y) = \mathbf{A}^{s}_{[\tau_1,\tau_2]}(x,y)[1/p]$ and $\mathbf{B}^{s}_{\tau_0}(x,y) = \mathbf{A}^{s}_{\tau_0}(x,y)[1/p]$.

(4) Let $\mathbf{C}^{[s',s]}_{[\tau_1,\tau_2]}(x,y)$ (resp. $\mathbf{C}^{s}_{[\tau_1,\tau_2]}(x,y)$) be the completion of $\mathbf{B}^{s}_{[\tau_1,\tau_2]}(x,y)$ with respect to the family of norms $\{\lambda_t\}_{t\in[s',s]}$ (resp. family of norms $\{\lambda_t\}_{t\in(0,s]}$). Let $\mathbf{C}^{\dagger}_{[\tau_1,\tau_2]}(x,y) = \varinjlim_{s>0} \mathbf{C}^{s}_{[\tau_1,\tau_2]}(x,y)$.

(5) Let

$$\mathbf{C}^{s}_{\tau_0,\mathrm{colim}}(x,y) \coloneqq \varinjlim_{\tau_1<\tau_0<\tau_2} \mathbf{C}^{s}_{[\tau_1,\tau_2]}(x,y).$$

and

$$\mathbf{C}^{\dagger}_{\tau_0,\mathrm{colim}}(x,y) \coloneqq \varinjlim_{s\to 0^+} \mathbf{C}^{s}_{\tau_0,\mathrm{colim}}(x,y).$$

Let $\mathbf{C}^{s}_{\tau_0}(x,y)$ be the Fréchet completion of $\mathbf{C}^{s}_{\tau_0,\mathrm{colim}}(x,y)$ with respect to $\{\lambda_{t,\tau_0}\}_{t\in(0,s]}$ and let

$$\mathbf{C}^{\dagger}_{\tau_0}(x,y) = \varinjlim_{s\to 0^+} \mathbf{C}^{s}_{\tau_0}(x,y).$$

The main idea to prove Proposition 6.14 is to construct continuous maps

$$\mathbf{C}'^{r}_{[\sigma_1,\sigma_2]} \xrightarrow{\psi} \mathbf{C}^{s}_{[\tau_1,\tau_2]}(x,y), \qquad \mathbf{C}^{s}_{[\tau_1,\tau_2]}(x,y) \xrightarrow{\phi} \mathbf{C}'^{r'}_{[\sigma_1',\sigma_2']}$$

for suitable $r, s, r', \sigma_1, \sigma_2, \tau_1, \tau_2, \sigma_1', \sigma_2'$ and coordinates x, y, such that ϕ, ψ are compatible with Frobenius and differential structures, and such that $\phi \circ \psi$ and $\psi \circ \phi$ coincide with natural inclusions. Given such (systems of) maps, we can translate the question of p-adic differential equations over $\mathbf{C}'^{r}_{[\sigma_1,\sigma_2]}$ to a question over $\mathbf{C}^{s}_{[\tau_1,\tau_2]}(x,y)$; but the latter is already analyzed in §6.5. As we shall see, to construct these maps, we first construct them on the integral level; that is, we construct continuous maps

$$\mathbf{A}'^{r}_{[\sigma_1,\sigma_2]} \xrightarrow{\psi} \mathbf{A}^{s}_{[\tau_1,\tau_2]}(x,y), \qquad \mathbf{A}^{s}_{[\tau_1,\tau_2]}(x,y) \xrightarrow{\phi} \mathbf{A}'^{r'}_{[\sigma_1',\sigma_2']}$$

with similar properties. The construction of these maps will occupy §6.6.2–6.6.4.

6.6.2. *Algebraization of analytic covers.*

Let $\mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}} \coloneqq \varinjlim_{r\to 0^+} \varinjlim_{\sigma_1<\sigma_\xi<\sigma_2} \mathbf{A}'^{r}_{[\sigma_1,\sigma_2]}$, which is finite étale over $\mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}$. Recall that $\mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}/p = \mathbf{k}(\overline{\xi})$ and $\mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}/p = \mathbf{k}(\overline{\xi}')$. Let $|\cdot|_{\overline{\xi}}$ (resp. $|\cdot|_{\overline{\xi}'}$) denote the corresponding norm on the uncompleted stalk $\mathbf{k}(\overline{\xi})$ (resp. $\mathbf{k}(\overline{\xi}')$). Also note that the finite separable extension $\mathbf{k}(\overline{\xi}')/\mathbf{k}(\overline{\xi})$ induces $\mathbf{k}(\overline{\xi}') \otimes_{\mathbf{k}(\overline{\xi})} R_\xi \cong R'$.

Lemma 6.20. (1) Both $(\mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}, p)$ and $(\mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}, p)$ are henselian pairs.

(2) Let $f \in \mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}$ be an element whose mod p reduction $\overline{f}$ is nonzero. Then f is invertible. If moreover f satisfies $|\overline{f}|_{\overline{\xi}'} < 1$, then there exist a sufficiently small r and σ_1, σ_2 sufficiently close to σ_ξ such that f is topologically nilpotent in $\mathbf{A}'^{r}_{[\sigma_1,\sigma_2]}$.

Proof. (1) Each $(\mathbf{A}^{\dagger}_{[\sigma_1,\sigma_2]}, p)$ is henselian by the proof of [DMY26, Proposition 5.28], so the filtered colimit $(\mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}, p)$ is again henselian. The henselianity of $(\mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}, p)$ follows from finite étaleness.

(2) The first claim on the invertibility of f is a consequence of henselianity.

For the second claim, we first show that the claim holds for $\mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}$ in place of $\mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}$. Suppose $f_0 \in \mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}$ such that $|\overline{f}_0|_{\overline{\xi}} < 1$. Then, up to shrinking $[\sigma_1, \sigma_2]$, we may assume that $\overline{f}_0 \in R_{[\sigma_1,\sigma_2]}$ and $q \coloneqq \overline{\lambda}_{[\sigma_1,\sigma_2]}(\overline{f}_0) < 1$ where $\overline{\lambda}_{[\sigma_1,\sigma_2]}$ is the spectral norm on $R_{[\sigma_1,\sigma_2]}$. By [DMY26, Proposition 5.18(1)], there exists a sufficiently small r_0 and a lift $\widetilde{f}_0 \in \mathbf{A}^{r_0}_{[\sigma_1,\sigma_2]}$ of $\overline{f}_0$ such that $\lambda_r(\widetilde{f}_0) = q^r$ for all $r \in (0, r_0]$. If we write $f_0 = \widetilde{f}_0 + pg$ for $g \in \mathbf{A}^{r_0}_{[\sigma_1,\sigma_2]}$, then $\lambda_r(g) \leqslant \max\{1, \lambda_{r_0}(g)^{r/r_0}\}$. Hence, there exists $r_1 < r_0$ such that $\lambda_r(pg) < 1$ for all $r \in (0, r_1]$. This implies $\lambda_r(f_0) < 1$ for all $r \in (0, r_1]$.

Now we return to the statement for $\mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}$. View $\mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}$ as a finite projective module of rank m over $\mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}$, and write $P_f(T) = T^m + a_{m-1}T^{m-1} + \cdots + a_0 \in \mathbf{A}^{\dagger}_{\xi,\mathrm{colim}}[T]$ for the characteristic polynomial of the multiplication-by-f map. The roots of the mod p reduction $\overline{P}_f(T)$ are Galois conjugates $\tau(\overline{f})$ of $\overline{f} \in \mathbf{k}(\overline{\xi}') = \mathbf{A}'^{\dagger}_{\xi,\mathrm{colim}}/p$ in some fixed algebraic closure $\mathbf{k}(\overline{\xi})^{\mathrm{alg}}$, and each of them satisfies $|\tau(\overline{f})|_{\overline{\xi}'} < 1$. This implies $|\overline{a}_i|_{\overline{\xi}} < 1$ for all i. By the discussion above, we may assume that $a_i \in \mathbf{A}^{r_1}_{[\sigma_1,\sigma_2]}$ and $f \in \mathbf{A}'^{r_1}_{[\sigma_2,\sigma_2]}$ for some r_1, σ_1, σ_2, and that $\lambda_r(a_i) < 1$ for all i and all $r \in (0, r_1]$. Finally, we show that $\lambda_r(f) < 1$ for all $r \in (0, r_1]$. By Cayley–Hamilton, we have $f^m + a_{m-1}f^{m-1} \cdots + a_0 = 0$. If $\lambda_r(f) \geqslant 1$, then

$$\lambda_r(f)^m = \lambda_r(f^m) \leqslant \max_{i<m}\{\lambda_r(a_i)\lambda_r(f)^i\} < \lambda_r(f)^m$$

which is impossible. Here, we have used the fact that λ_r is power-multiplicative on $\mathbf{A}'^{r}_{[\sigma_1,\sigma_2]}$. □

By Krasner's lemma, there is a finite separable extension $F'/\kappa(\overline{\pi}_K, T^\flat)$ such that $R' = F' \otimes_{\kappa(\overline{\pi}_K,T^\flat)} R_\xi$. We view F' as a subfield of R'. Let

$$\overline{G}(X) = X^{d_0} + \sum_{i=0}^{d_0-1} a_i X^i \in \kappa'(\overline{\pi}_K, T^\flat)[X]$$

be the minimal polynomial for a primitive element of the finite separable extension $F'/\kappa'(\overline{\pi}_K, T^\flat)$. Let $\overline{g} \in \kappa'[\overline{\pi}_K, T^\flat]$ be a nonzero polynomial that clears the denominators of the coefficients of $\overline{G}$, that is, $\overline{g} \cdot a_i \in \kappa'[\overline{\pi}_K, T^\flat]$ for all $i = 0, \dots, d_0 - 1$. Pick an (arbitrary) $g \in W(\kappa')[\pi_K, [T^\flat]]$ lifting $\overline{g}$ and pick a lift

$$G[X] \in W(\kappa')[\pi_K, [T^\flat], 1/g][X]$$

of $\overline{G}$. Let $\Delta(G) \in W(\kappa')[\pi_K, [T^\flat], 1/g]$ denote the discriminant of G and consider

$$C_0 \coloneqq W(\kappa')[\pi_K^{\pm}, [T^\flat]^{\pm}, 1/g, 1/\Delta(G)], \qquad D_0 \coloneqq C_0[X]/G(X).$$

By construction, $C_0 \to D_0$ is finite étale. The fraction fields of C_0/p and D_0/p are precisely $\kappa'(\overline{\pi}_K, T^\flat)$ and F', respectively. Moreover, we have $\overline{\Delta}(G) \neq 0$, and is thus a unit in the fraction field $\kappa'(\overline{\pi}_K, T^\flat)$. By Lemma 6.20, there is a natural map $C_0 \to \mathbf{A}^\dagger_{\xi,\mathrm{colim}}$ as both g and $\Delta(G)$ map to units. In particular, we obtain a map $C_0 \to \mathbf{A}^r_{[\sigma_1,\sigma_2]}$ after decreasing r and shrinking $[\sigma_1, \sigma_2]$ if necessary.

Lemma 6.21. For sufficiently small r and for σ_1, σ_2 sufficiently close to σ_ξ, we have

$$D_0 \otimes_{C_0} \mathbf{A}^r_{[\sigma_1,\sigma_2]} \cong \mathbf{A}'^r_{[\sigma_1,\sigma_2]}.$$

Proof. By construction, there is an isomorphism

$$D_0/p \otimes_{C_0/p} R_\xi \cong R'.$$

By quasi-completeness of $\mathbf{k}(\overline{\xi})$ (cf. [Ber93, Theorem 2.3.3, Proposition 2.4.1]), we have an equivalence of categories $\big(\mathbf{k}(\overline{\xi})\big)_{\text{fét}} \xrightarrow{\sim} \big(R_\xi\big)_{\text{fét}}$. This produces an isomorphism

$$D_0/p \otimes_{C_0/p} \mathbf{A}^\dagger_{\xi,\mathrm{colim}}/p \cong \mathbf{A}'^\dagger_{\xi,\mathrm{colim}}/p.$$

Finally, by the henselianity of $(\mathbf{A}^\dagger_{\xi,\mathrm{colim}}, p)$ proved in Lemma 6.20, we obtain an isomorphism

$$D_0 \otimes_{C_0} \mathbf{A}^\dagger_{\xi,\mathrm{colim}} \cong \mathbf{A}'^\dagger_{\xi,\mathrm{colim}},$$

from which the lemma follows. □

6.6.3. *The coordinates x, y.*

To construct ψ and ϕ, we need a further approximation step to relate D_0 with a suitable polynomial ring $W(\kappa')[x, y]$. Recall from §6.2 that the valuation v_α on R_ξ uniquely extends to a valuation (again denoted as v_α) on R'. Since $\kappa'(\overline{\pi}_K, T^\flat) \subset F' \subset R'$, the value group Γ' of F' contains $\mathbb{Z} + \alpha\mathbb{Z}$ as a subgroup of finite index. In particular, the valuation v_α on F' is *monomial* in the sense of [Ked09, Definition 6.1.1]. Pick a $\mathbb{Z}$-basis $\{\beta_1, \beta_2\}$ of Γ' with $\beta_1, \beta_2 > 0$ (note that $\beta_2/\beta_1 \notin \mathbb{Q}$), and then pick elements $t_1, t_2 \in (F')^\times$ such that $v_\alpha(t_i) = \beta_i$.

Proposition 6.22. There exists a smooth algebraic surface S over $\operatorname{Spec} \kappa'$ with function field $F(S) = F'$, a closed point $s_0 \in S$ at which the valuation v_α is centered, and regular parameters $\overline{x}, \overline{y} \in \mathcal{O}_{S,s_0}$ such that $t_i = \overline{u}_i \cdot \overline{x}^{a_i} \overline{y}^{b_i}$ for $i = 1, 2$, with $\overline{u}_i \in \mathcal{O}^\times_{S,s_0}$ and $a_i, b_i \in \mathbb{Z}_{\geqslant 0}$. Here, we say that the valuation is *centered* at s_0 if $\mathfrak{m}_{S,s_0} = \mathcal{O}_{S,s_0} \cap \{a \in F' \mid v_\alpha(a) > 0\}$ where the intersection is taken inside $F' = \operatorname{Frac} \mathcal{O}_{S,s_0}$ (cf. [Ked08, Definition 2.4.1]).

Proof. This follows from the local uniformization theorem of monomial valuations (cf. [KK05, Theorem 1.1], also see [Ked09, Proposition 6.1.3]). □

Write $\gamma_1 = v_\alpha(\overline{x})$ and $\gamma_2 = v_\alpha(\overline{y})$ for the valuations of $\overline{x}$ and $\overline{y}$. Then $\beta_1 = a_1\gamma_1 + b_1\gamma_2$ and $\beta_2 = a_2\gamma_1 + b_2\gamma_2$. In particular, γ_1 and γ_2 are rationally independent. Since S is smooth at s_0 with regular parameters $\overline{x}, \overline{y}$, we have an isomorphism $(\mathcal{O}_{S,s_0})^\wedge_{\mathfrak{m}_{S,s_0}} \cong \kappa'[\![\overline{x}, \overline{y}]\!]$. This implies that $\kappa'[\overline{x}, \overline{y}]$ is dense in $\mathcal{O}_{S,s_0}$ for the topology induced by the valuation v_α. Therefore, $\kappa'(\overline{x}, \overline{y})$ is dense in F', and thus is dense in R'. Also note that, since $\overline{x}, \overline{y}$ are regular parameters at s_0, we have an isomorphism

$$F' \otimes_{\kappa'(\overline{x},\overline{y})} \Omega^1_{\kappa'(\overline{x},\overline{y})/\kappa'} \xrightarrow{\sim} \Omega^1_{F'/\kappa'}$$

as $\Omega^1_{F'/\kappa'} \cong F' \otimes_{\mathcal{O}_{S,s_0}} \Omega^1_{\mathcal{O}_{S,s_0}/\kappa'} \cong F' d\overline{x} \oplus F' d\overline{y}$. This implies $\Omega^1_{F'/\kappa'(\overline{x},\overline{y})} = 0$, so the extension $F'/\kappa'(\overline{x}, \overline{y})$ is finite separable (it is finite because it is finitely generated and has transcendental degree 0).

Remark 6.23. We point out that, the fact that $F'/\kappa'(\overline{x},\overline{y})$ is a finite separable extension and that $\kappa'(\overline{x},\overline{y})$ is dense in F' implies that they have the same completion, but does not imply that they are equal.

Since $F' = \mathrm{Frac}(D_0/p)$, we can write $\overline{x} = \overline{h}_1/\overline{g}_1$ and $\overline{y} = \overline{h}_2/\overline{g}_2$ with $h_1, h_2, g_1, g_2 \in D_0/p$. Pick liftings $h_1, h_2, g_1, g_2 \in D_0$ and consider a ring map $W(\kappa')[x,y] \to D_0[1/(g_1g_2)]$ sending $x \mapsto h_1/g_1$, $y \mapsto h_2/g_2$. This is a lift of $\kappa'[x,y] \longrightarrow \mathcal{O}_{S,s_0} \subset F' = \mathrm{Frac}(D_0/p)$. By further inverting the Jacobian $J \in D_0[1/(g_1g_2)]$ of x, y with respect to $\pi_K, [T^\flat]$, we arrive at an étale ring map ("the framing")

$$\square\colon W(\kappa')[x,y] \longrightarrow D \coloneqq D_0[1/(g_1g_2), 1/J].$$

6.6.4. *Constructions of ψ and ϕ.*

Construction of ψ.

Let $\tau_0 \coloneqq p^{-c_K\cdot\gamma_2/\gamma_1}$ (note that $\tau_0 \notin p^{\mathbb{Q}}$) and let $[\tau_1,\tau_2]$ be an interval containing τ_0, with $\tau_1, \tau_2 \in p^{\mathbb{Q}}$. Consider $\mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x,y)$ and $\mathbf{A}^\dagger_{\tau_0}(x,y)$ defined in §6.6.1, and note that $(\mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x,y), p)$ is a henselian pair. Let $R_{\tau_0,\mathrm{colim}} \coloneqq \mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x,y)/p$ and $R_{\tau_0} \coloneqq \mathbf{A}^\dagger_{\tau_0}(x,y)/p$. Similarly to Remark 6.5 and Lemma 6.8, $R_{\tau_0,\mathrm{colim}}$ (resp. R_{τ_0}) coincides with the uncompleted residue field (resp. completed residue field) of a type Ⅲ point corresponding to a circle of radius τ_0 in characteristic p, and we have an isomorphism $R_{\tau_0} \cong R'$ by sending $x \mapsto \overline{x}$, $y \mapsto \overline{y}$.

By construction, we have inclusions $F' \subset R'$ and $R_{\tau_0,\mathrm{colim}} \subset R_{\tau_0} \cong R'$. We claim that the first inclusion factors through the second via an injection $F' \hookrightarrow R_{\tau_0,\mathrm{colim}}$. To see this, consider $E \coloneqq F' \otimes_{\kappa'(\overline{x},\overline{y})} R_{\tau_0,\mathrm{colim}}$. Note that E is a finite étale $R_{\tau_0,\mathrm{colim}}$-algebra since $F'/\kappa'(\overline{x},\overline{y})$ is finite separable. Consider the composition $m\colon E \otimes_{R_{\tau_0,\mathrm{colim}}} R' \cong F' \otimes_{\kappa'(\overline{x},\overline{y})} R' \to R'$ where the second map is the multiplication map. By the fully faithfulness in [KL15, Lemma 2.2.4(a)], m descends to a unique $R_{\tau_0,\mathrm{colim}}$-algebra map $m_0\colon E \to R_{\tau_0,\mathrm{colim}}$. Finally, composing with $F' \to E$ sending $a \mapsto a \otimes 1$, we get the desired inclusion $F' \hookrightarrow R_{\tau_0,\mathrm{colim}}$.

Given this, we deduce that the natural map $W(\kappa')[x,y] \to \mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x,y)$ factors as the composition

$$W(\kappa')[x,y] \xrightarrow{\square} D \xrightarrow{\eta} \mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x,y)$$

for a unique map $\eta\colon D \longrightarrow \mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x,y)$. Indeed, since $(\mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x,y), p)$ is henselian, it suffices to observe that the modulo p reduction $\kappa'[x,y] \to R_{\tau_0,\mathrm{colim}}$ factors as $\kappa'[x,y] \to D/p \to F' \to R_{\tau_0,\mathrm{colim}}$. Consequently, we obtain a map

$$\eta\colon D \longrightarrow \mathbf{A}^s_{[\tau_1,\tau_2]}(x,y)$$

for sufficiently small $s > 0$ and for τ_1, τ_2 sufficiently close to τ_0.

To construct the desired $\psi\colon \mathbf{A}'^r_{[\sigma_1,\sigma_2]} \to \mathbf{A}^s_{[\tau_1,\tau_2]}(x,y)$, it suffices to produce a map $\psi_1\colon D_0 \to \mathbf{A}^s_{[\tau_1,\tau_2]}(x,y)$ and a map $\psi_2\colon \mathbf{A}^r_{[\sigma_1,\sigma_2]} \to \mathbf{A}^s_{[\tau_1,\tau_2]}(x,y)$ that agrees on C_0 (cf. Lemma 6.21). In fact, we take ψ_1 to be the composition

$$D_0 \to D \xrightarrow{\eta} \mathbf{A}^s_{[\tau_1,\tau_2]}(x,y).$$

To construct ψ_2, we need the following lemma. For the setup, let us write the rational numbers c_Kr and $\alpha_i = -\frac{1}{c_K}\log_p\sigma_i$ (for $i = 1, 2$) in their reduced rational forms

$$c_Kr = \frac{h_1}{k_1}, \qquad -\frac{1}{c_K}\log_p\sigma_2 = \frac{e_1}{m_1}, \qquad -\frac{1}{c_K}\log_p\sigma_1 = \frac{f_1}{n_1}.$$

Note that $e_1/m_1 < \alpha < f_1/n_1$ by assumption.

Lemma 6.24. Let B be a Banach $W(\kappa)$-algebra where $W(\kappa)$ is equipped with the standard p-adic norm. Suppose that $x, y \in B^\times$ are topologically nilpotent units and suppose that $p^{h_1}x^{-k_1}, y^{m_1}x^{-e_1}, x^{f_1}y^{-n_1}$ are power-bounded. Then there exists a unique continuous ring map $\mathbf{A}^r_{[\sigma_1,\sigma_2]} \to B$ sending $\pi_K \mapsto x$, $[T^\flat] \mapsto y$.

Proof. From the description of $\mathbf{A}^r_{[\sigma_1,\sigma_2]}$ in Definition 4.18(2) (cf. Proposition 3.12), we know that

$$\mathbf{A}^r_{[\sigma_1,\sigma_2]} = \left(W(\kappa)[\![\pi_K]\!]\left\langle [T^\flat], \frac{p^{h_1}}{\pi_K^{k_1}}, \frac{[T^\flat]^{m_1}}{\pi_K^{e_1}}, \frac{\pi_K^{f_1}}{[T^\flat]^{n_1}}\right\rangle\right)[\frac{1}{\pi_K}]$$

where completion is taken with respect to the (p, π_K)-adic topology (equivalently, the p-adic topology). In other words, we have

$$\mathbf{A}^r_{[\sigma_1,\sigma_2]} \cong \left(\frac{W(\kappa)[\![\pi_K]\!]\langle [T^\flat], X_0, X_1, X_2\rangle}{\left(p^{h_1} - \pi_K^{k_1} X_0, [T^\flat]^{m_1} - \pi_K^{e_1} X_1, \pi_K^{f_1} - [T^\flat]^{n_1} X_2\right)} \right) [\frac{1}{\pi_K}].$$

The map to B is then given by sending $\pi_K \mapsto x, [T^\flat] \mapsto y, X_0 \mapsto p^{h_1}/x^{k_1}, X_1 \mapsto y^{m_1}/x^{e_1}, X_2 \mapsto x^{f_1}/y^{n_1}$. It is straightforward to check that the map is indeed well-defined and continuous. □

Consider elements $x_1 \coloneqq \eta(\pi_K)$ and $y_1 \coloneqq \eta([T^\flat])$ in $\mathbf{A}^s_{[\tau_1,\tau_2]}(x, y)$. Note that the images of x_1, y_1, $p^{h_1}x_1^{-k_1}$, $y_1^{m_1}x_1^{-e_1}$, $x_1^{f_1}y_1^{-n_1}$ in $\mathbf{A}^\dagger_{\tau_0,\mathrm{colim}}(x, y)$ all have mod p reductions with norm < 1. Hence, by decreasing s and shrinking $[\tau_1, \tau_2]$ if necessary, we may assume that x_1, y_1, $p^{h_1}x_1^{-k_1}$, $y_1^{m_1}x_1^{-e_1}$, $x_1^{f_1}y_1^{-n_1}$ are all topologically nilpotent, by applying Lemma 6.20. Applying Lemma 6.24 to $B = \mathbf{A}^s_{[\tau_1,\tau_2]}(x, y)$, we obtain the desired map $\psi_2 \colon \mathbf{A}^r_{[\sigma_1,\sigma_2]} \to \mathbf{A}^s_{[\tau_1,\tau_2]}(x, y)$. It is routine to check that these two maps agree on C_0.

Construction of ϕ.

Write $c_K s = h_2/k_2$, $-\frac{1}{c_K}\log_p \tau_2 = e_2/m_2$, $-\frac{1}{c_K}\log_p \tau_1 = f_2/n_2$ in their reduced rational forms. Since the images of $1/(g_1g_2)$ and $1/J$ in $\mathrm{Frac}(D_0/p) = F'$ are nonzero, the map $D_0 \to \mathbf{A}'^r_{[\sigma_1,\sigma_2]}$ given by Lemma 6.21 extends to a map

$$\theta \colon D \longrightarrow \mathbf{A}'^r_{[\sigma_1,\sigma_2]}$$

by decreasing r and shrinking $[\sigma_1, \sigma_2]$ if necessary, using the henselianity of $(\mathbf{A}'^\dagger_{\xi,\mathrm{colim}}, p)$ (see Lemma 6.20). Write $x_2 \coloneqq \theta(x)$ and $y_2 \coloneqq \theta(y)$. Since x_2 and y_2 are invertible, and x_2, y_2, $p^{h_2}x_2^{-k_2}$, $y_2^{m_2}x_2^{-e_2}$, $x_2^{f_2}y_2^{-n_2}$ all have reduction of absolute value < 1, (after further decreasing r' and shrinking $[\sigma_1', \sigma_2']$ if necessary), we obtain the desired map $\phi \colon \mathbf{A}^s_{[\tau_1,\tau_2]}(x, y) \to \mathbf{A}'^{r'}_{[\sigma_1',\sigma_2']}$ by applying Lemma 6.24.

Compositions of ψ and ϕ.

For the reader's convenience, let us summarize our constructions. Given any sufficiently small r and sufficiently small interval $[\sigma_1, \sigma_2]$ containing σ_ξ, we have a continuous ring map $\psi \colon \mathbf{A}'^r_{[\sigma_1,\sigma_2]} \to \mathbf{A}^s_{[\tau_1,\tau_2]}(x, y)$ for some s and some sufficiently small interval $[\tau_1, \tau_2]$ containing τ_0. Note that if we decrease r, then ψ remains valid as long as we decrease s accordingly, without changing $[\sigma_1, \sigma_2]$ or $[\tau_1, \tau_2]$.[28] Therefore, we actually obtain a system of maps ψ as r varies. Likewise, we obtain a system of continuous ring maps $\phi \colon \mathbf{A}^s_{[\tau_1,\tau_2]}(x, y) \to \mathbf{A}'^{r'}_{[\sigma_1',\sigma_2']}$ as s varies.

Unwinding the construction, it is straightforward to check that the compositions

$$\mathbf{A}'^r_{[\sigma_1,\sigma_2]} \xrightarrow{\psi} \mathbf{A}^s_{[\tau_1,\tau_2]}(x, y) \xrightarrow{\phi} \mathbf{A}'^{r'}_{[\sigma_1',\sigma_2']}$$

and

$$\mathbf{A}^s_{[\tau_1,\tau_2]}(x, y) \xrightarrow{\phi} \mathbf{A}'^{r'}_{[\sigma_1',\sigma_2']} \xrightarrow{\psi} \mathbf{A}^{s'}_{[\tau_1',\tau_2']}(x, y)$$

are both natural restriction maps (for suitable $r' < r$, $s' < s$, $[\sigma_1', \sigma_2'] \subset [\sigma_1, \sigma_2]$, and $[\tau_1', \tau_2'] \subset [\tau_1, \tau_2]$).

Lastly, by inverting p and passing to Fréchet completions, we obtain systems of continuous ring maps

$$\mathbf{C}'^r_{[\sigma_1,\sigma_2]} \xrightarrow{\psi} \mathbf{C}^s_{[\tau_1,\tau_2]}(x, y), \qquad \mathbf{C}^s_{[\tau_1,\tau_2]}(x, y) \xrightarrow{\phi} \mathbf{C}'^{r'}_{[\sigma_1',\sigma_2']}$$

such that both $\phi \circ \psi$ and $\psi \circ \phi$ are natural restriction maps.

6.6.5. *Proof of Proposition 6.14.*

Passing to colimits, we obtain maps

$$\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]} \xrightarrow{\psi} \mathbf{C}^\dagger_{[\tau_1,\tau_2]}(x, y) \xrightarrow{\phi} \mathbf{C}'^\dagger_{[\sigma_1',\sigma_2']}, \qquad \mathbf{C}^\dagger_{[\tau_1,\tau_2]}(x, y) \xrightarrow{\phi} \mathbf{C}'^\dagger_{[\sigma_1',\sigma_2']} \xrightarrow{\psi} \mathbf{C}^\dagger_{[\tau_1',\tau_2']}(x, y)$$

whose compositions are both natural restriction maps. For τ_1, τ_2 sufficiently close to τ_0, consider

$$N_{[\tau_1,\tau_2]} \coloneqq \mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]} \otimes_{\mathbf{C}'^\dagger_{[\sigma_1,\sigma_2]},\psi} \mathbf{C}^\dagger_{[\tau_1,\tau_2]}(x, y).$$

[28] This follows from the construction. Note that in Lemma 6.20(2), the choices of σ_1 and σ_2 are independent of r.

Using ψ and ϕ, the Frobenius structure on $\mathbf{C}'^{\dagger}_{[\sigma_1,\sigma_2]}$ induces a Frobenius structure on $\mathbf{C}^{\dagger}_{[\tau_1,\tau_2]}(x,y)$. This is essentially due to the construction; indeed, since ϕ and ψ are constructed from the integral level, the induced Frobenius map is indeed a lift of the absolute Frobenius map in characteristic p. Moreover, passing the completed colimits as $\sigma_1,\sigma_2 \to \sigma_\xi$ and $\tau_1,\tau_2 \to \tau_0$, we obtain an isomorphism $\mathscr{R}' \cong \mathbf{C}^{\dagger}_{\tau_0}(x,y)$ of fake Robba rings, compatible with Frobenius structures. Consequently, $N_{[\tau_1,\tau_2]}$ is a finite projective (ϕ,∇)-module over $\mathbf{C}^{\dagger}_{[\tau_1,\tau_2]}(x,y)$ equipped with differential operators D_x and D_y such that

$$N_{[\tau_1,\tau_2]} \otimes_{\mathbf{C}^{\dagger}_{[\tau_1,\tau_2]}(x,y)} \mathbf{C}^{\dagger}_{\tau_0}(x,y)$$

is already unipotent. Finally, applying the arguments in the special case (see §6.5), we deduce that $N_{[\tau_1,\tau_2]}$ is strongly unipotent, by shrinking the interval $[\tau_1,\tau_2]$ if necessary. Using ψ and ϕ, we conclude that $\mathbf{N}'_{\mathrm{dR},[\sigma_1,\sigma_2]}$ is also strongly unipotent, for σ_1,σ_2 sufficiently close to σ_ξ. This finishes the proof of Proposition 6.14.

7. Around points of type Ⅳ

In this section, we prove the following stronger local version of the p-adic monodromy theorem, which immediately implies Theorem 12, for a type Ⅳ point.

Theorem 7.1. *Let X and $\mathbb{L}$ be as in Theorem 12. Let $\xi \in X_{\mathbb{C}_K}$ be a point of type Ⅳ. Then there exist a finite extension L/K, an open neighborhood U of the image of ξ in X_L, and a finite étale cover $f\colon V \to U$, such that $f^*(\mathbb{L}|_U)$ is horizontally semistable in the sense of* [Shi22, Definition 4.13]*; in particular, $f^*(\mathbb{L}|_U)$ is pointwise semistable (i.e. semistable at every classical point in V).*

To prove Theorem 7.1, we first show that every p-adic differential equation is étale locally trivial around any type Ⅳ point.

Theorem 7.2. *Let $\xi \in X_{\mathbb{C}_K}$ be a point of type Ⅳ and let (M,∇) be a vector bundle with connection on a neighborhood of ξ. Then there exists an open neighborhood U of ξ and a finite étale cover $f\colon V \to U$ such that f^*M is a trivial connection.*

Applying Theorem 7.2 to $D_{\mathrm{dR}}(\mathbb{L})$ (together with a descent argument), we reduce to the case where $\mathbb{L}$ is horizontally de Rham. Then Theorem 7.1 follows from the work of Shimizu [Shi22] on potential semistability of horizontally de Rham local systems.

Throughout the section, let $\omega = p^{-\frac{1}{p-1}}$.

7.1. Some preparations.

We first make some elementary observations before proceeding to the proof of Theorem 7.2. By [Ber07, Proposition 2.2.1], a type Ⅳ point on $X_{\mathbb{C}_K}$ admits a basis of open neighborhoods consisting of closed disks. Therefore, without loss of generality we may assume that ξ is a type Ⅳ point on a closed unit disk with coordinate T. Also recall that $\mathbf{k}(\xi)$ denotes the (uncompleted) residue field at ξ, which coincides with the stalk $\mathcal{O}_{X_{\mathbb{C}_K},\xi}$, and $\mathscr{H}(\xi)$ stands for the completed residue field. Moreover, $\mathbf{k}(\xi)$ is a differential field with derivation $\partial = d/dT$. Note that the constant field $\mathbf{k}(\xi)^{\partial=0}$ is $\mathbb{C}_K$, and $\partial\colon \mathbf{k}(\xi) \to \mathbf{k}(\xi)$ is surjective.[29] For later use, we fix an algebraic closure $\mathscr{K}(\xi)$ of $\mathbf{k}(\xi)$. When the type Ⅳ point ξ is understood, we simply write $\mathscr{K}$ for $\mathscr{K}(\xi)$.

Lemma 7.3. Let $\mathbf{k}/\mathbf{k}(\xi)$ be a finite extension. Then there exist an open neighborhood U of ξ and a finite étale cover $f\colon V \to U$ with a unique preimage ξ' lying above ξ such that $\mathbf{k} = \mathcal{O}_{V,\xi'}$; in particular, ξ' has type Ⅳ. Moreover, the derivation $\partial = d/dT$ uniquely extends to a derivation ∂ on $\mathbf{k}$ with constant field $\mathbb{C}_K$, and $\partial\colon \mathbf{k} \to \mathbf{k}$ is surjective.

Proof. The first statement is straightforward. The derivation uniquely extends to $\mathbf{k}$ because $\mathbf{k}/\mathbf{k}(\xi)$ is a finite separable extension. Pick a closed disk $V' \subset V$ containing ξ' with coordinate S. Then we have $\partial = g\frac{d}{dS}$ where $g = \frac{dS}{dT} \in \mathbf{k}^\times$. The desired surjectivity of $\partial\colon \mathbf{k} \to \mathbf{k}$ follows from the surjectivity of $\frac{d}{dS}$ on $\mathbf{k}$. □

[29] To see the surjectivity of ∂, suppose $a \in \mathbf{k}(\xi)$ is defined on some closed disk U containing ξ with coordinate T. Integrating a with respect to T, we obtain an analytic function b that converges on any strictly smaller closed disk containing ξ and such that $\partial b = a$.

As an immediate consequence, $\partial = d/dT$ uniquely extends to a derivation $\partial\colon \mathscr{K} \to \mathscr{K}$. The following corollary is a special case of Theorem 7.2 when the vector bundle has rank 1.

Corollary 7.4. Every rank 1 differential module over $(\mathscr{K}, \partial)$ is trivial.

Proof. Let e be a basis with $\nabla(e) = ae$ for some $a \in \mathscr{K}$. It suffices to find $c \in \mathscr{K}^\times$ such that $\partial c = ac$. Given such c, we have $\nabla(c^{-1}e) = -c^{-2}(ac)e + c^{-1} \cdot ae = 0$ as desired.

Let $\mathbf{k}/\mathbf{k}(\xi)$ be a finite extension that contains a. Applying Lemma 7.3, we obtain a finite étale cover $V \to U$ and a type Ⅳ point $\xi' \in V$ such that ξ' is the unique preimage of ξ and $\mathbf{k} = \mathcal{O}_{V,\xi'}$. By shrinking, we may assume that V is a closed disk. Moreover, there exists $b \in \mathbf{k}$ such that $\partial b = a$, which we may assume to be an analytic function on V. Write $A = \mathcal{O}(V)$, equipped with its spectral norm $|\cdot|_V$. Let m be a sufficiently large integer such that $|p^m b|_V < \omega$. Then the exponential $u = \exp(p^m b)$ converges on V and $u \in A^\times$. Note that the cover of V given by $A[z]/(z^{p^m} - u)$ is finite étale by the Jacobian criterion. Therefore, a root z of $z^{p^m} = u$ in $\mathscr{K}$ satisfies $\partial z = az$ and is precisely the solution we are looking for. □

7.2. Differential modules over $(\mathbf{k}(\xi), \partial)$.

Theorem 7.2 boils down to potential triviality of differential modules over $(\mathbf{k}(\xi), \partial)$. To prove this, we need the following key technical result.

Proposition 7.5. Let $\xi \in X_{\mathbb{C}_K}$ be a point of type Ⅳ and let V be an irreducible differential module over $(\mathbf{k}(\xi), \partial)$ of rank n with trivial determinant (i.e., the determinant $\wedge^n V$ is a trivial differential module). Then either $p|n$ or $n = 1$.

7.2.1. *Radii of optimal convergence.*

To prove this key proposition, let us recall some basic notions and results from the theory of p-adic differenial equations (for example, see [Ked22]). Just for the purpose of proving Proposition 7.5, we temporarily treat all disks and annuli below as Berkovich spaces instead of as adic spaces. This is mainly because the theory of p-adic differential equations are most developed over Berkovich spaces in standard literature. Note that for a type Ⅳ point ξ on a rigid analytic curve X, its Berkovich stalk $\mathcal{O}_{X^{\mathrm{Ber}},\xi}$ coincides with its adic stalk $\mathcal{O}_{X^{\mathrm{ad}},\xi}$ (cf. §2.4). In particular, there is no ambiguity in writing $\mathbf{k}(\xi)$ and $\mathscr{H}(\xi)$.

For the setup, let (M, ∇) be a differential module of rank n over the closed disk $\mathbb{D}_{\mathbb{C}_K}(0, r_0)$ of radius $r_0 \in p^{\mathbb{Q}}$ with coordinate T. For any Berkovich point $z \in \mathbb{D}_{\mathbb{C}_K}(0, r_0)$ (that is, a point of type Ⅰ, Ⅱ, Ⅲ, or Ⅳ), define

$$M_{z,0} \coloneqq M \otimes_{\mathcal{O}(\mathbb{D}_{\mathbb{C}_K}(0,r_0))} \mathscr{H}(z)[\![T - t_z]\!]$$

where t_z denotes the image of T in $\mathscr{H}(z)$ and the map $\mathcal{O}(\mathbb{D}_{\mathbb{C}_K}(0, r_0)) \to \mathscr{H}(z)[\![T - t_z]\!]$ sends $T \mapsto (T - t_z) + t_z$. This is a differential module over $(\mathscr{H}(z)[\![T - t_z]\!], \frac{d}{d(T-t_z)} = \frac{d}{dT})$ with full (formal) horizontal sections; namely,

$$(M_{z,0})^{\nabla=0} \otimes_{\mathscr{H}(z)} \mathscr{H}(z)[\![T - t_z]\!] \xrightarrow{\sim} M_{z,0}.$$

For any $\rho \in (0, r_0)$, let

$$M_{z,\rho} \coloneqq M \otimes_{\mathcal{O}(\mathbb{D}_{\mathbb{C}_K}(0,r_0))} \mathscr{H}(z)\langle \frac{T - t_z}{\rho} \rangle$$

and let $\mathrm{Sol}_z(\rho) \coloneqq (M_{z,0})^{\nabla=0} \cap M_{z,\rho}$ denote the $\mathscr{H}(z)$-vector space of horizontal sections of ∇ that converge on the closed disk of radius ρ.

Definition 7.6. For any Berkovich point $z \in \mathbb{D}_{\mathbb{C}_K}(0, r_0)$, the *radii of optimal convergence*[30] $0 < s_1(M, z) \leqslant \cdots \leqslant s_n(M, z) \leqslant r_0$ are defined by

$$s_i(M, z) \coloneqq \sup\{\rho < r_0 \ : \ \dim_{\mathscr{H}(z)} \mathrm{Sol}_z(\rho) \geqslant n + 1 - i\}.$$

In other words, $s_i(M, z)$ is the radius of the maximal open disk around t_z on which there exist $n - i + 1$ linearly independent horizontal sections of M.

[30] The normalization we use here is slightly different from [Ked22, Definition 11.9.1]. Also note that there is a typo in *loc. cit.*: $n - i$ should be $n + 1 - i$.

Definition 7.7. For any $\mu \in \mathbb{C}_K$ with $|\mu| \leqslant r_0$ and any $0 < \rho \leqslant r_0$, let $x_{\mu,\rho}$ denote the point of type II or III corresponding to the disk $\mathbb{D}_{\mathbb{C}_K}(\mu, \rho)$ of radius ρ centered at μ (see §2.1). We put

$$R_i(M, x_{\mu,\rho}) = \min\{\rho, s_i(M, x_{\mu,\rho})\}, \qquad IR_i(M, x_{\mu,\rho}) = \min\{1, s_i(M, x_{\mu,\rho})/\rho\}$$

for $i = 1, \ldots, n$.

Remark 7.8. The quantities $R_i(M, x_{\mu,\rho})$ and $IR_i(M, x_{\mu,\rho})$ are closely related to the notions of *generic subsidiary radii of convergence* and *intrinsic generic subsidiary radii of convergence* (cf. [Ked22, Definition 9.4.4, 9.4.7, & 9.8.1]). Indeed, let $\mathscr{H}_{\mu,\rho}$ denote the completed residue field at $x_{\mu,\rho}$ and let $M_{\mu,\rho}$ be the base change of (M, ∇) to $\mathscr{H}_{\mu,\rho}$. Then $R_i(M, x_{\mu,\rho})$ (resp. $IR_i(M, x_{\mu,\rho})$) is precisely the generic subsidiary radii of convergence $R_i(M_{\mu,\rho})$ (resp. the intrinsic generic subsidiary radii of convergence $IR_i(M_{\mu,\rho})$) defined in *loc. cit.*

The following local constancy result will be a key ingredient of the proof of Proposition 7.5.

Theorem 7.9. *Let $\xi \in \mathbb{D}_{\mathbb{C}_K}(0, r_0)$ be a type IV point and let (M, ∇) be a differential module of rank n over $\mathbb{D}_{\mathbb{C}_K}(0, r_0)$. Then there exists a closed disk $U \subset \mathbb{D}_{\mathbb{C}_K}(0, r_0)$ containing ξ such that $s_i(M, z)$ is a constant function on U for all $i = 1, \ldots, n$, where the variable z ranges over all Berkovich points on U.*

Proof. This is [Ked15, Lemma 4.5.14] (also see [KS17]). □

7.2.2. *Some reduction steps.*

To prove Proposition 7.5, let us start with some reduction steps.

First, we may assume that the differential module V is represented by a finite free irreducible differential module (M, ∇) over a closed disk $\mathbb{D}_{\mathbb{C}_K}(0, r_0)$ centered at 0 with radius $r_0 \in p^{\mathbb{Q}}$, over which the determinant of M is trivial. By Theorem 7.9, we may further assume that all $s_i(M, z)$'s are constant on $\mathbb{D}_{\mathbb{C}_K}(0, r_0)$, which we simply denote as $s_1, \ldots, s_n$.

Let $r := r(\xi)$ denote the radius of the type IV point ξ. Note that $r < r_0$. We may further assume that $s_1 \leqslant r$. Otherwise, there exists a closed disk $\mathbb{D}_{\mathbb{C}_K}(a, d)$ centered at some point a with radius d satisfying $r < |T - a|_\xi < d < s_1$, where $|\cdot|_\xi$ is the continuous valuation associated with ξ. This disk contains the type IV point ξ and M admits a full set of horizontal solutions over $\mathbb{D}_{\mathbb{C}_K}(a, d)$. The irreducibility then forces $n = 1$ and there is nothing to prove. From now on, we assume that $s_1 \leqslant r$.

Lemma 7.10. We have $s_1 = s_2 = \cdots = s_n$.

Proof. Suppose $s_1 = \cdots = s_j < s_{j+1}$ for some $j < n$. By assumption, we have $R_1(M, x_{0,\rho}) = \cdots = R_j(M, x_{0,\rho}) = s_1$, and $R_{j+1}(M, x_{0,\rho}) = \min\{\rho, s_{j+1}\} > s_1$, for all $\rho \in (r, r_0)$. By [Ked22, Theorem 12.5.2], there exists a decomposition $M = M_1 \oplus M_2$ of differential modules over the open annulus $A_{\mathbb{C}_K}(r, r_0)$, where M_1 has rank j while M_2 has rank $n - j$. This contradicts the irreducibility assumption. □

Let s denote the value $s_1 = \cdots = s_n$. The following is an immediate consequence of Lemma 7.10.

Corollary 7.11. For any $\mu \in \mathbb{C}_K$ with $|\mu| < r_0$ and any $\rho \in [s, r_0)$, we have $R_i(M, x_{\mu,\rho}) = s$ for all $i = 1, \ldots, n$.

Now, we pick $d \in p^{\mathbb{Q}} \cap (r, r_0)$ such that $d \notin \{s \cdot \omega^{-1/p^j} \mid j \in \mathbb{Z}_{\geqslant 0}\}$, and pick $a, c \in \mathbb{C}_K$ with $|c| = d$ and $|T - a|_\xi < d$. Let $b = a + c$. Note that $|T - b|_\xi = d$. In other words, ξ lies on the circle centered at b with radius d. In fact, ξ lies inside the open disk $\mathbb{D}_{\mathbb{C}_K}(a, < d)$. By a change of coordinates, we may assume that $b = 0$. Under this new coordinate, the open disk $\mathbb{D}_{\mathbb{C}_K}(a, < d)$ (and thus ξ) is contained in the circle $\{|T| = d\}$, and hence contained in the closed annulus $\{\alpha \leqslant |T| \leqslant \beta\}$ for any $\alpha, \beta \in p^{\mathbb{Q}}$ satisfying

$$r < \alpha < d < \beta < r_0.$$

In other words, we have

$$\xi \in \mathbb{D}_{\mathbb{C}_K}(a, < d) \subset A_{\mathbb{C}_K}([d, d]) \subset A_{\mathbb{C}_K}([\alpha, \beta]) \subset \mathbb{D}_{\mathbb{C}_K}(0, r_0) \tag{7.1}$$

as illustrated in Figure 7.2.2. By scaling the coordinate T, we may assume that $d = 1$ from now on. Note that the previous assumption on d implies that $s \neq \omega^{1/p^j}$ for all integers $j \geqslant 0$.

Let us make one final reduction before proving Proposition 7.5.

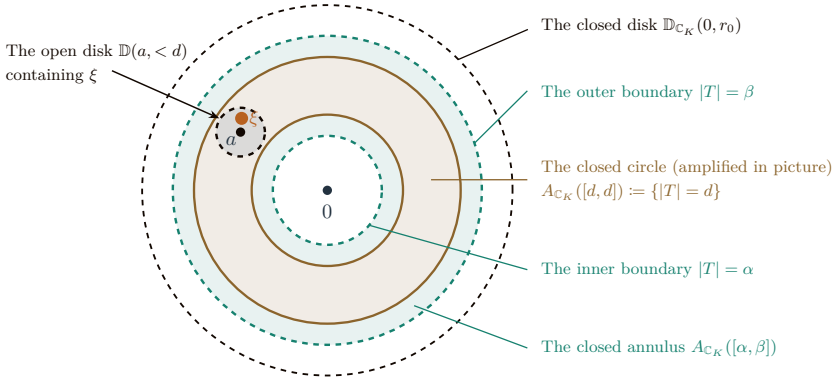


FIGURE 3. A Cartoon of annuli and disks containing ξ

Lemma 7.12. It suffices to prove Proposition 7.5 under the additional assumption that $s<\omega$.

Proof. Let $h\geqslant 0$ be the least integer such that $s^{p^h}<\omega$. If $h=0$, there is nothing to prove, so we assume $h>0$. Note that $s^{p^h}<\omega$ and $s^{p^{h-1}}>\omega$ (here we use $s\neq\omega^{1/p^j}$ for all $j\geqslant 0$). In particular, $s>\omega$. Pick an interval $[\alpha,\beta]$ as above so that

$$(s/\rho)^{p^h}<\omega \text{ and } (s/\rho)^{p^{h-1}}>\omega \quad \text{for all } \rho\in[\alpha,\beta]. \tag{7.2}$$

The idea, which dates back to the works of Christol and Dwork, is to apply Frobenius antecedents to make the generic radii visible. More precisely, consider the "Frobenius map"

$$\phi\colon \{\alpha\leqslant|T|\leqslant\beta\}\to\{\alpha^p\leqslant|T_1|\leqslant\beta^p\}$$

given by $T_1\mapsto T^p$. By [Ked22, Theorem 10.4.4, 10.4.2], there exists a Frobenius antecedent M_1 of M along ϕ; that is, a differential module M_1 over the annulus $\{\alpha^p\leqslant|T_1|\leqslant\beta^p\}$ such that M is the pullback of M_1 along the Frobenius map ϕ. For this new annulus, the open disk that contains the type Ⅳ point $\phi(\xi)$ becomes $\mathbb{D}_{\mathbb{C}_K}(\phi(a),<1)$, while the radii of optimal convergence all become s^p. Hence, for any $\rho_1\in[\alpha^p,\beta^p]$, we have $IR_i(M_1,x_{0,\rho_1})=s^p/\rho_1$ for all $i=1,\dots,n$, and we have $(s^p/\rho_1)^{p^{h-1}}<\omega$ and $(s^p/\rho_1)^{p^{h-2}}>\omega$ by (7.2). Moreover, M_1 is irreducible and has trivial determinant. The latter follows from the uniqueness of Frobenius antecedent. Applying this construction h-times, we obtain M_h (which is irreducible with trivial determinant) over the annulus $\{\alpha^{p^h}\leqslant|T_h|\leqslant\beta^{p^h}\}$ with all radii of optimal convergence equal to $s^{p^h}<\omega$. This finishes the proof of the lemma. □

From now on, we assume $s<\omega$. Pick $r<\alpha<d=1<\beta<r_0$ as above such that $s/\alpha<\omega$. In particular, we have $s/\rho<\omega$ for all $\rho\in[\alpha,\beta]$.

7.2.3. *Proof of Proposition 7.5.*

We are ready to prove Proposition 7.5. Thanks to the reduction steps, we are in the following setup:

- (M,∇) is a finite free differential module over $\mathbb{D}_{\mathbb{C}_K}(0,r_0)$ of rank n;
- For each $i=1,\dots,n$, the radii of optimal convergence $s_i(M,z)$ are constant on $\mathbb{D}_{\mathbb{C}_K}(0,r_0)$, and are all equal to some $s<\omega$;
- For any $\mu\in\mathbb{C}_K$ with $|\mu|<r_0$ and any $\rho\in[s,r_0)$, we have $R_i(M,x_{\mu,\rho})=s$ for all $i=1,\dots,n$;
- We have $r<\alpha<1<\beta<r_0$ with $\alpha,\beta\in p^{\mathbb{Q}}$ such that $s/\rho<\omega$ for all $\rho\in[\alpha,\beta]$, and such that

$$\xi\in\mathbb{D}_{\mathbb{C}_K}(a,<1)\subset A_{\mathbb{C}_K}([1,1])\subset A_{\mathbb{C}_K}([\alpha,\beta])\subset\mathbb{D}_{\mathbb{C}_K}(0,r_0)$$

  as illustrated in Figure (7.2.2).

Let us assume that $n>1$ and $p\nmid n$. Our goal is to find a contradiction. The proof is organized in several steps.

Step 1.

Consider the base change (M_F,∇) of (M,∇) to the fraction field F of the ring of analytic functions on the annulus $\{\alpha\leqslant|T|\leqslant\beta\}$. By [Ked22, Theorem 5.4.2], the differential module (M_F,∇) admits a cyclic vector

v. Write $\nabla^n(v) = \sum_{i=0}^{n-1} c_i \cdot \nabla^i(v)$, where each $c_i = c_i(T)$ is a meromorphic function on the annulus, and let $P(X) \coloneqq X^n - \sum_{i=0}^{n-1} c_i \cdot X^i$ be the characteristic polynomial of v. For any $\mu \in \mathbb{C}_K$ with $|\mu| < r_0$ and any $\rho \in [\alpha, \beta]$, let $\mathscr{H}_{\mu,\rho} = \mathscr{H}(x_{\mu,\rho})$ denote the completed residue field of $x_{\mu,\rho}$ (equipped with the continuous valuation $|\cdot|_{\mu,\rho}$) and let $M_{\mu,\rho}$ denote the base change of M to $\mathscr{H}_{\mu,\rho}$. By construction, v is also a cyclic vector of $M_{\mu,\rho}$, again with characteristic polynomial $P(X)$. Let $\beta_{i,\mu,\rho}$ denote the roots of $P(X)$ in an algebraic closure of $\mathscr{H}_{\mu,\rho}$, for $i = 1, \dots, n$. By the classical result of Christol and Dwork (cf. [Ked22, Theorem 6.5.3, Corollary 6.5.4]), we have $|\beta_{i,\mu,\rho}|_{\mu,\rho} = \omega/s$ for all i (in particular, the $|\cdot|_{\mu,\rho}$-norm of $\beta_{i,\mu,\rho}$ are independent of ρ). Note that the entire spectrum of $M_{\mu,\rho}$ is visible in the sense of [Ked22, Definition 6.5.1] because $\omega/s > 1/\rho$.

Note that $\omega/s \in p^{\mathbb{Q}}$. Pick $\gamma \in \mathbb{C}_K^\times$ with $|\gamma| = \omega/s$ and write $b_i \coloneqq \gamma^{-(n-i)} \cdot c_i \in F$. Consider a new polynomial

$$\widetilde{P}(X) \coloneqq \gamma^{-n} P(\gamma X) = X^n - \sum_{i=0}^{n-1} b_i X^i \in F[X].$$

Over $\mathscr{H}_{\mu,\rho}$, it has roots $\gamma^{-1}\beta_{i,\mu,\rho}$, which all have norm 1. In particular, we have $|b_i|_{\mu,\rho} \leqslant 1$ for all $i = 1, \dots, n-1$, and $|b_0|_{\mu,\rho} = 1$.

Now we consider the type Ⅱ point $x_{0,1}$ and the associated completed residue field $\mathscr{H}_{0,1} = \mathscr{H}(x_{0,1})$. It is a complete non-archimedean field with respect to the 1-Gauss norm $|\cdot|_{0,1}$, with an imperfect residue field $\overline{k}(T)$ (recall that $k = k_K$ is the residue field of K). Then $b_i = b_i(T)$ lies in $\mathcal{O}_{\mathscr{H}_{0,1}}$ for all $0 \leqslant i \leqslant n-1$. For simplicity, let us write $\beta_i = \beta_{i,0,1}$ for the roots of $P(X)$ over an algebraic closure of $\mathscr{H}_{0,1}$, which we fix in the rest of the proof.

Sublemma A. The image of $\widetilde{P}(X)$ under the reduction map $\mathcal{O}_{\mathscr{H}_{0,1}}[X] \to \overline{k}(T)[X]$ lies in $\overline{k}[X]$.

Proof of Sublemma A. Let $\overline{\mu} \in \overline{k}^\times$ be an element and let $\mu \in \mathcal{O}_{\mathbb{C}_K}^\times$ be a lift of $\overline{\mu}$ (for example one may take $\mu = [\overline{\mu}]$). The residue disk corresponding to $\overline{\mu}$ is $\mathbb{D}_{\mathbb{C}_K}(\mu, <1)$. Inside this residue disk, we have type Ⅱ and Ⅲ points $x_{\mu,\rho}$ with $\rho < 1$, corresponding to the closed disk $\mathbb{D}_{\mathbb{C}_K}(\mu, \rho) \subset \mathbb{D}_{\mathbb{C}_K}(\mu, <1)$. Let f be a meromorphic function on the annulus $\{\alpha \leqslant |T| \leqslant \beta\}$. Recall the number $s_{\overline{\mu}}(f)$ defined in [Ked22, Definition 11.1.1]

$$s_{\overline{\mu}}(f) \coloneqq \frac{d}{dr}\Big|_{r \to 0^+} \left(-\log |f|_{x_{\mu,p^{-r}}}\right).$$

It is independent of the lift μ and measures the change in the norm of f when "entering" into the residue disk $\mathbb{D}_{\mathbb{C}_K}(\mu, <1)$. Moreover, if $|f|_{0,1} = 1$, we know that $s_{\overline{\mu}}(f) = \mathrm{ord}_{\overline{\mu}}(\overline{f})$ is simply the order of the zeros of $\overline{f} \in \overline{k}(T)$ at $\overline{\mu}$ (cf. the proof of [Ked22, Proposition 11.1.2]). There are similar definitions and statements for $\overline{\mu} = 0$ and ∞; see *loc. cit.*

We apply the discussion above to the meromorphic functions b_i. Recall that $|b_i|_{0,1} \leqslant 1$. If $|b_i|_{0,1} < 1$ then $\overline{b}_i = 0$ and there is nothing to prove. So we assume that $|b_i|_{0,1} = 1$. If $\overline{\mu} \in \overline{k}^\times$ is a pole of $\overline{b}_i$, then $s_{\overline{\mu}}(b_i) = -m$ for some $m \geqslant 1$ and hence $|b_i|_{\mu,\rho} = \rho^{-m} > 1$ for ρ sufficiently close to 1^-, which is a contradiction. Therefore $\overline{b}_i$ has no poles at any $\overline{\mu} \in \overline{k}^\times$. A similar argument implies that $\overline{b}_i$ has no poles at 0 and ∞ (by considering $x_{0,\rho}$ as $\rho \to 1^-$ and $x_{0,\rho'}$ as $\rho' \to 1^+$, respectively). Therefore, each $\overline{b}_i$ must be a constant, as desired. □

Step 2.

By Sublemma A, any root of the reduction of $\widetilde{P}(X)$ lies in $\overline{k}$. Pick a root $\overline{\eta} \in \overline{k}$ of the reduction of $\widetilde{P}(X)$, and let $\eta = [\overline{\eta}]$ be its Teichmuller lift in $\mathcal{O}_{\mathbb{C}_K}^\times$. Let $\lambda = \eta\gamma$, which has norm $|\lambda| = |\gamma| = \omega/s$. By the construction, we know that $|\beta_i - \lambda|_{0,1} \leqslant \omega/s$ for all $i = 1, \dots, n$. Moreover, we have $|\beta_{i_0} - \lambda|_{0,1} < \omega/s$ for at least one of the indices i_0.

Sublemma B. In fact, we have $|\beta_i - \lambda|_{0,1} < \omega/s$ for all $i = 1, \dots, n$.

Proof of Sublemma B. Suppose $|\beta_i - \lambda|_{0,1} = \omega/s$ for some indices i. Let l be the number of such indices; in particular, $1 \leqslant l \leqslant n-1$. Consider the free rank-1 differential module $L_{-\lambda}$ over the annulus $A_{\mathbb{C}_K}([\alpha, \beta])$ with basis $e_{-\lambda}$ defined by $D(e_{-\lambda}) = -\lambda e_{-\lambda}$. Then consider the differential module

$$M_{-\lambda} \coloneqq M \otimes_{\Gamma_{\mathbb{C}_K}([\alpha,\beta])} L_{-\lambda}$$

where $\Gamma_{\mathbb{C}_K}([\alpha,\beta])$ denotes analytic functions on the annulus $A_{\mathbb{C}_K}([\alpha,\beta])$ (cf. §1.9). If we write $\nabla_{-\lambda}$ for the resulting differential on $M_{-\lambda}$, then $\nabla_{-\lambda}=\nabla-\lambda$. That is, for all $x\in M$, we have

$$\nabla_{-\lambda}(x\otimes e_{-\lambda})=(\nabla(x)-\lambda x)\otimes e_{-\lambda}.$$

From now on, we shall identify an element $x\in M$ with its image $x\otimes e_{-\lambda}\in M_{-\lambda}$. In particular, for the cyclic vector v chosen previously, we have $P(\nabla_{-\lambda}+\lambda)v=0$. Moreover, $\{v,\nabla_{-\lambda}(v),\cdots,\nabla_{-\lambda}^{n-1}(v)\}$ form a cyclic basis of $M_{-\lambda}$ with characteristic polynomial $P(X+\lambda)$, whose roots in the algebraic closure of $\mathscr{H}_{0,1}$ are $\beta_i-\lambda$.

Since $|\lambda|=\omega/s$, $\exp(\lambda T)$ has convergence radius s and thus $R_1(L_{-\lambda},x_{0,\rho})=\min\{\rho,\frac{\omega}{|\lambda|}\}=s$ for all $\rho\in[\alpha,\beta]$. This in turn implies that $R_1(M_{-\lambda},x_{0,\rho})\geqslant s$ for all $\rho\in[\alpha,\beta]$ by [Ked22, Lemma 9.4.6]. We claim that $R_n(M_{-\lambda},x_{0,1})>s$. Otherwise, we must have

$$R_1(M_{-\lambda},x_{0,1})=\cdots=R_n(M_{-\lambda},x_{0,1})=s.$$

Since the entire spectrum of $M_{-\lambda}$ is visible at $x_{0,1}$, by the result of Christol and Dwork again (cf. [Ked22, Theorem 6.5.3, Corollary 6.5.4]), the roots $\beta_i-\lambda$ of $P(X+\lambda)$ all have norms $|\beta_i-\lambda|_{0,1}=\omega/s$. This contradicts the fact that $|\beta_{i_0}-\lambda|_{0,1}<\omega/s$ for at least one of the indices i_0.

Consequently, up to shrinking the interval $[\alpha,\beta]$ if necessary, we have $R_n(M_{-\lambda},x_{0,\rho})>s$ for all $\rho\in[\alpha,\beta]$. But this would imply that $M_{-\lambda}$ (and thus M) is not irreducible, by the same splitting argument in the proof of Lemma 7.10 (using [Ked22, Theorem 12.5.2]). This finishes the proof of the lemma. □

Step 3.

Now we finish the proof of Proposition 7.5. By Sublemma B, we have

$$\big|c_{n-1}-n\lambda\big|_{0,1}=\big|\sum_{i=1}^{n}\beta_i-n\lambda\big|_{0,1}<\omega/s.$$

Since $|n\lambda|=|\lambda|=\omega/s$ (this is where we use $p\nmid n$), we deduce that $|c_{n-1}|_{0,1}=\omega/s>1$. Now, consider the element $w^{\det}:=v\wedge\nabla(v)\wedge\cdots\wedge\nabla^{n-1}(v)$ in the determinant of $M_{0,1}=M\otimes_{\Gamma_{\mathbb{C}_K}([\alpha,\beta])}\mathscr{H}_{0,1}$. Differentiating $w^{\det}$ yields $\nabla(w^{\det})=c_{n-1}w^{\det}$. By assumption, $M_{0,1}$ has trivial determinant, so there exists some element $\delta\in\mathscr{H}_{0,1}^{\times}$ such that $\delta w^{\det}$ is horizontal; namely, $\nabla(\delta w^{\det})=(\frac{d}{dT}\delta)w^{\det}+\delta\cdot c_{n-1}w^{\det}=0$. This implies $c_{n-1}=-(\frac{d}{dT}\delta)\cdot\delta^{-1}$. However, $|\frac{d}{dT}\delta|_{0,1}\leqslant|\delta|_{0,1}$ for any $\delta\in\mathscr{H}_{0,1}^{\times}$. This forces $|c_{n-1}|_{0,1}\leqslant 1$, and we find the desired contradiction.

7.3. **Proof of Theorem 7.2.**

We will reduce Theorem 7.2 to Proposition 7.5 using a similar strategy as in [Ked22, §21.5]. Let us start with the following observation on differential modules over $\mathscr{K}$. Recall that $\mathscr{K}=\mathscr{K}(\xi)$ is a fixed algebraic closure of $\mathbf{k}(\xi)$.

Lemma 7.13. Let V be a differential module over $(\mathscr{K},\partial)$ that admits a subquotient of dimension 1. Then V admits a nontrivial horizontal section.

Proof. By assumption, there exist differential submodules $V''\subset V'\subset V$ such that V'/V'' is one dimensional. By Corollary 7.4, V'/V'' is necessarily a trivial differential module. As in §7.1, there exist an open neighborhood U of ξ and a finite étale cover $f\colon U'\to U$ with a unique preimage ξ' lying above ξ such that $V''\subset V'\subset V$ descend to differential modules $M''\subset M'\subset M$ on U'. We may assume that $U'=\mathbb{D}_{\mathbb{C}_K}(0,r_0)$ is a closed disk, with coordinate T. Let $r=r(\xi')$ be the radius of the type Ⅳ point ξ'. Replacing U' by a smaller disk if necessary, we may further assume that all radii of optimal convergence are constant on U'.

Pick $a\in\mathbb{C}_K$ and $d\in p^{\mathbb{Q}}$ such that $r<|T-a|_{\xi'}<d<r_0$. Then we have $\xi'\in\mathbb{D}_{\mathbb{C}_K}(a,<d)\subset U'=\mathbb{D}_{\mathbb{C}_K}(0,r_0)$. At the type Ⅱ point $x_{a,d}$ (namely, the Gauss point of $\mathbb{D}_{\mathbb{C}_K}(a,d)$), the existence of the trivial subquotient M'/M'' implies $IR_n(M,x_{a,d})=1$ by [KX10, Lemma 1.2.9]. This implies $s_n(M,x_{a,d})\geqslant d$. By the constancy of radii of optimal convergence, we must have $s_n(M,z)\geqslant d$ for all Berkovich points $z\in U'$. In particular, we have $s_n(M,a)\geqslant d$. Now, we pick $d'\in p^{\mathbb{Q}}$ such that $|T-a|_{\xi'}<d'<d$ so that $\mathbb{D}_{\mathbb{C}_K}(a,d')$ is strictly contained inside $\mathbb{D}_{\mathbb{C}_K}(a,<d)$ and $\xi'\in\mathbb{D}_{\mathbb{C}_K}(a,d')$. Then $s_n(M,a)\geqslant d$ implies that at least one horizontal section of M converges on the entire disk $\mathbb{D}_{\mathbb{C}_K}(a,d')$. This provides the desired horizontal section. □

Given this, we can prove the following upgraded version of Proposition 7.5.

Proposition 7.14. Every irreducible differential module over $(\mathscr{K}, \partial)$ is trivial (and in particular has dimension 1).

Proof. Let (V, ∇) be an irreducible differential module over $(\mathscr{K}, \partial)$ of rank n. First, suppose that $p \nmid n$. Note that the determinant of V is trivial by Corollary 7.4. Then we can descend V to a differential module over $\mathbf{k} = \mathbf{k}(\xi')$ with trivial determinant, where ξ' is a type IV point lying above ξ as before. Applying Proposition 7.5, we conclude that $n = 1$ and V is trivial.

For the rest of the proof, we assume $p|n$. Our goal is to find a contradiction. Consider $W \coloneqq \mathrm{End}_{\mathscr{K}}(V)$ equipped with the differential $\nabla_W(f) = \nabla \circ f - f \circ \nabla$. By choosing a basis of V, we can write $\nabla = \partial + H$ for some $H \in \mathrm{Mat}_{n\times n}(\mathscr{K}) = \mathrm{End}_{\mathscr{K}}(V)$. For any matrix $B \in \mathrm{Mat}_{n\times n}(\mathscr{K}) = \mathrm{End}_{\mathscr{K}}(V)$, we have $\nabla_W(B) = \partial B + HB - BH$. Taking trace, we obtain $\mathrm{tr}(\nabla_W(B)) = \partial(\mathrm{tr}B)$. In particular, the trace-zero subspace $W_0 \coloneqq \ker(\mathrm{tr}\colon \mathrm{End}_{\mathscr{K}}(V) \to \mathscr{K})$ is a differential submodule of W of dimension $n^2 - 1$. Since p is coprime to $n^2 - 1$, there exists an irreducible subquotient of W_0 with rank coprime to p. By the previous paragraph, this subquotient is trivial of dimension 1. By Lemma 7.13, W_0 has a nontrivial horizontal section under ∇_W. In other words, we have found a (nonzero) traceless endomorphism $B\colon V \to V$ that commutes with ∇. In the matrix form (under the basis chosen above), we have $\partial B = BH - HB$. By induction, we deduce $\partial(B^i) = B^iH - HB^i$ for all $i \geqslant 1$, and hence $\partial(\mathrm{tr}(B^i)) = \mathrm{tr}(\partial(B^i)) = 0$. Therefore, the characteristic polynomial of B has coefficients in $(\mathscr{K})^{\partial=0} = \mathbb{C}_K$ by Lemma 7.3. In particular, all eigenvalues of B lie in $\mathbb{C}_K$. Let $\lambda \in \mathbb{C}_K$ be one of the eigenvalues of B and consider $V_\lambda \coloneqq \ker(B - \lambda I)$. It is clear that $B - \lambda I \neq 0$ as B is traceless. Since B commutes with ∇, we know that $V_\lambda \subset V$ is a proper differential submodule. This contradicts the irreducibility of V and finishes the proof of the proposition. □

Now we are ready to prove Theorem 7.2.

Proof of Theorem 7.2. Let $M_{\mathbf{k}(\xi)}$ denote the restriction of (M, ∇) to $\mathbf{k}(\xi)$ and let $M_{\mathscr{K}}$ denote its base change to $\mathscr{K} = \mathscr{K}(\xi)$. It suffices to show that $(M_{\mathscr{K}}, \nabla)$ is a trivial differential module over $(\mathscr{K}, \partial)$ (then the desired statement follows from Lemma 7.3). By Proposition 7.14, $M_{\mathscr{K}}$ admits a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M_{\mathscr{K}}$ where each quotient M_i/M_{i-1} is a trivial differential module of dimension 1. We need to show that this series splits.

We prove this by induction. Suppose that M_{i-1} has horizontal basis $e_1, \ldots, e_{i-1}$. Let $\overline{e}_i \in M_i/M_{i-1}$ be a horizontal section under ∇, and let $\widetilde{e}_i \in M_i$ be any lift of $\overline{e}_i$. Then we have $\nabla(\widetilde{e}_i) = \sum_{j=1}^{i-1} a_j e_j$ with $a_j \in \mathscr{K}$. By Lemma 7.3, for each $j = 1, \ldots, i-1$, there exists $c_j \in \mathscr{K}$ with $\partial c_j = a_j$. Then $e_i \coloneqq \widetilde{e}_i - \sum_{j=1}^{i-1} c_j e_j$ satisfies $\nabla(e_i) = 0$, as desired. This finishes the proof. □

7.4. **Proof of Theorem 7.1.**

To prove Theorem 7.1, we need a version of Theorem 7.2 over some finite extension of K rather than $\mathbb{C}_K$.

Lemma 7.15. Let $X = \mathrm{Sp}(A)$ be a smooth geometrically connected affinoid rigid analytic curve over K and let (M, ∇) be a vector bundle with connection on X whose global section is a finite free A-module. If $M_{\mathbb{C}_K}$ is a trivial connection, then so is M.

Proof. Let $A_{\mathbb{C}_K} = \mathcal{O}(X_{\mathbb{C}_K}) = A\widehat{\otimes}_K\mathbb{C}_K$. By picking a basis of M over A, we can write $\nabla = d + \Omega$ for some $\Omega \in \mathrm{Mat}_{n\times n}(A)$. Finding a horizontal basis of $M_{\mathbb{C}_K}$ over $A_{\mathbb{C}_K}$ is equivalent to finding a matrix $F \in \mathrm{GL}_n(A_{\mathbb{C}_K})$ such that $dF + \Omega F = 0$. Note that any two such matrices differ by a matrix in $\mathrm{GL}_n(\mathbb{C}_K)$.

Choose any classical point $x_0 \in X$ defined over a finite Galois extension L/K. Write $A_L = \mathcal{O}(X_L) = A \otimes_K L$. The evaluation map $A_{\mathbb{C}_K} \to \mathbb{C}_K$ at the point x_0 yields a matrix $F(x_0) \in \mathrm{GL}_n(\mathbb{C}_K)$. Let $F_0 \coloneqq F \cdot F(x_0)^{-1}$. Then we have $dF_0 + \Omega F_0 = 0$ and $F_0(x_0) = I_n$ is the identity matrix. These two conditions uniquely characterize F_0. For any $g \in \mathrm{Gal}_L$, since g fixes Ω and commutes with the differential d, we again have $dg(F_0) + \Omega g(F_0) = 0$ and $g(F_0)(x_0) = I_n$. The uniqueness then implies $g(F_0) = F_0$ for all $g \in \mathrm{Gal}_L$. By Corollary 2.3, we have $(A_{\mathbb{C}_K})^{\mathrm{Gal}_L} = A_L$ and hence $F_0 \in \mathrm{GL}_n(A_L)$. In other words, the base change M_L is a trivial connection. Finally, the horizontal sections $(M_L)^{\nabla=0}$ of M_L descend to those of M by Galois descent. This finishes the proof of the lemma. □

Remark 7.16. The freeness assumption on M in Lemma 7.15 is redundant. This simplified version is already sufficient for our purpose, and we leave the more general situation to the interested reader.

Corollary 7.17. Let X be a smooth rigid analytic curve over K and let $\xi \in X_{\mathbb{C}_K}$ be a point of type Ⅳ. Let (M, ∇) be a vector bundle with connection defined on a neighborhood of the image of ξ in X. Then there exist a finite extension L/K, an open neighborhood U of the image of ξ in X_L, and a finite étale cover $f\colon V \to U$ such that f^*M is a trivial connection.

Proof. This follows immediately from Theorem 7.2 and Lemma 7.15. □

Finally, we finish the proof of Theorem 7.1.

Proof of Theorem 7.1. Applying Corollary 7.17 to $D_{\mathrm{dR}}(\mathbb{L})$, we can find a finite extension L/K, an open neighborhood U of the image of ξ in X_L, and a finite étale cover $f\colon V \to U$ such that $D_{\mathrm{dR}}(f^*\mathbb{L}) = f^*(D_{\mathrm{dR}}(\mathbb{L}))$ is a trivial connection. In particular, $f^*\mathbb{L}$ is horizontally de Rham in the sense of [Shi22, Definition 9.1]. After shrinking U and V and enlarging L if necessary, we may assume that V is isomorphic to a circle over L. By [Shi22, Theorem 1.9], after a further finite extension of L, $f^*\mathbb{L}$ becomes horizontally semistable, thus semistable at every classical point in V. □

8. Around points of type V: special case

The goal of this section is to prove Theorem 12 for a special class of type V points — those type V points lying on closed unit disks. Later in §9, we will deduce the general case of type V points from this.

For the setup, let ξ_{II} be the (type Ⅱ) Gaussian point of the closed unit disk $\mathbb{D}_K = \mathbb{D}_K(0,1)$ over K and let ξ denote the type V point corresponding to the open unit disk $\mathbb{D}_K(0,<1) \subset \mathbb{D}_K$ (cf. §2.1). Both ξ_{II} and ξ are K-rational points, and ξ lives in the closure of ξ_{II}.

Let $\delta \in p^{\mathbb{Q}} \cap (0,1)$ be a real number and $f(T) = \prod_{i=1}^{n}(T - [\beta_i]) \in \mathcal{O}_{K_0}[T]$ be a polynomial with $\beta_i \in \overline{k}_K^{\times}$. Consider the closed annulus $A_K([\delta,1]) = \{\delta \leqslant |T| \leqslant 1\}$ and let $X_{\delta,f} \subset A_K([\delta,1])$ be the rational subset obtained by removing finitely many residue disks centered at $[\beta_i]$ on the boundary circle $\{|T| = 1\}$ (cf. Notation 4.22). We equip $X_{\delta,f}$ with the toric chart $X_{\delta,f} \hookrightarrow A_K([\delta,1]) \hookrightarrow \mathbb{D}_K$. Let $X_{1,f} \coloneqq \{|T| = 1\} \cap X_{\delta,f}$ denote the "outer boundary", which admits a smooth integral model over $W(k_K)$. Note that $X_{\delta,f}$'s form a basis of open neighborhoods of ξ while $X_{1,f}$'s form a basis of open neighborhoods of ξ_{II}. When f is clear from the context, we simply write X_δ for $X_{\delta,f}$ and write X_1 for $X_{1,f}$. Moreover, recall the relative period rings $\widetilde{\mathbf{A}}^\dagger_{\delta,f}$, $\widetilde{\mathbf{B}}^\dagger_{\delta,f}$, $\widetilde{\mathbf{C}}^\dagger_{\delta,f}$, $\mathbf{A}^\dagger_{\delta,f}$, $\mathbf{B}^\dagger_{\delta,f}$, $\mathbf{C}^\dagger_{\delta,f}$ etc. constructed in §4.7.2.

The main result of this section is the following.

Theorem 8.1. *Let $\mathbb{L}$ be a de Rham $\mathbb{Z}_p$-local system on $X_\delta = X_{\delta,f}$ for some δ and f. Suppose that $\mathbb{L}$ is semistable on the "outer boundary" $X_{1,f}$. Then, up to replacing K by a finite extension and shrinking X_δ (i.e. increasing δ and removing finitely many more residue disks on the outer boundary), there exists a finite étale cover $g\colon X' \to X_\delta$ such that $g^*\mathbb{L}$ is pointwise semistable.*

Since $\mathbb{L}$ is already semistable on the outer boundary, we are mainly interested in the half-open annulus $A_K([\delta,1);T)$. To prove Theorem 8.1, we consider the p-adic differential equation $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ over $\mathbf{C}^\dagger_{\delta,f}$, equipped with both arithmetic and geometric differential operators (cf. §4.5 & 4.7.2). As we shall see, the semistability of $\mathbb{L}$ on the boundary $X_{1,f}$ will force $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ to be "strongly unipotent" with respect to the arithmetic differential operator D_{π_K}. For a more precise statement, consider the rational subset

$$V_{\delta,f} \coloneqq \{\delta \leqslant |[T^\flat]| \leqslant 1,\ |f([T^\flat])| = 1\} \subset A_{K'_0}([\delta,1];[T^\flat])$$

obtained by removing finitely many residue disks on the circle $\{|[T^\flat]| = 1\}$, and let $A_{\delta,f}$ denote the ring of analytic functions on $V_{\delta,f}$. From the explicit description of $\mathbf{C}^r_{\delta,f}$ (Lemma 4.26), we have $\left(\mathbf{C}^r_{\delta,f}\right)^{\frac{d}{d\pi_K}=0} = A_{\delta^r,f}$. The key technical result is the following.

Theorem 8.2. *Let $\mathbb{L}$ be a de Rham $\mathbb{Z}_p$-local system of rank d on $X_\delta = X_{\delta,f}$ for some δ and f. Suppose that $\mathbb{L}$ is semistable on $X_{1,f}$. Then, up to enlarging K and shrinking X_δ, there exists $r > 0$ such that the $\mathbf{C}^r_{\delta,f}$-module $\mathbf{N}^r_{\mathrm{dR}}(\mathbb{L})$ is free and admits a basis $e_1, \ldots, e_d$ such that the matrix of $\pi_K D_{\pi_K}$ relative to this*

basis is upper-triangular nilpotent with entries in $A_{\delta^r,f}$. In other words, $\mathbf{N}^r_{\mathrm{dR}}(\mathbb{L})$ is strongly unipotent with respect to the arithmetic differential operator D_{π_K}.

Given this, we may assume that the restriction of $\mathbf{N}_{\mathrm{dR}}(\mathbb{L})$ on the half-open annulus $A_K([\delta,1);T)$ is strongly unipotent with respect to D_{π_K}. Then, similarly to our treatment in §6, we can produce enough semistable periods on $A_K([\delta,1);T)$ after pulling back to some finite étale cover. The proof of Theorem 8.2 will occupy §8.1 – §8.4, from which we deduce Theorem 8.1 in §8.5 – §8.6.

8.1. **Completed perfection at the type Ⅱ point.**

For the rest of the section, let $\mathbb{L}$ be a de Rham $\mathbb{Z}_p$-local system of rank d on $X_\delta = X_{\delta,f}$ as in the statement of Theorem 8.1 and assume that $\mathbb{L}$ is semistable on $X_1 = X_{1,f}$.

The first step is to consider the completed stalk $\mathscr{H}(\xi_{\mathrm{Ⅱ}})$ at the type Ⅱ point, which is given by the completion of $K(T)$ under the 1-Gauss norm. Write $\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{(\mathrm{pf})} \coloneqq \cup_{m\geqslant 0}\mathscr{H}(\xi_{\mathrm{Ⅱ}})(T^{1/p^m})$ and let $\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}$ denote its completion under the p-adic topology. Then $\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}$ a complete discrete valuation field, with residue field being the perfection $k(T)^{\mathrm{perf}}$ of $k(T)$. Write

$$\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}_\infty \coloneqq \bigcup_{n\geqslant 1}\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}(\zeta_{p^n}),$$

and let $\mathscr{H}'$ denote the maximal unramified extension of $W(k(T)^{\mathrm{perf}})[p^{-1}]$ inside $\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}_\infty$. Let $\mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$ denote the imperfect period ring associated with the p-adic field $\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}$ in the classical theory. Using the notation in §1.9, $\mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$ is simply the Robba ring $\mathscr{R}_{\mathscr{H}'}$ over $\mathscr{H}'$, with coordinate π_K. Consider the restriction of $\mathbb{L}$ to the type Ⅱ point $\xi_{\mathrm{Ⅱ}}$, and then to $\widetilde{\xi}_{\mathrm{Ⅱ}} = \mathrm{Spa}(\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}, \mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf},\circ})$. This gives rise to a Galois representation over $\mathrm{Gal}_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$, which we shall denote by $\mathbb{L}|_{\widetilde{\xi}_{\mathrm{Ⅱ}}}$.

Lemma 8.3. There is a natural isomorphism

$$\mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X_1}) \otimes_{\mathbf{C}^\dagger_{1,f}} \mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}} \xrightarrow{\sim} \mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{\widetilde{\xi}_{\mathrm{Ⅱ}}})$$

compatible with arithmetic differential operators D_{π_K}, where $\mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X_1})$ is the relative p-adic differential equation constructed in [DMY26] (cf. §4.5), while $\mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{\widetilde{\xi}_{\mathrm{Ⅱ}}})$ is the classical p-adic differential equation of Berger [Ber02] (cf. §3.7).

Proof. Let A_{X_1} denote the ring of analytic functions on X_1. The map $A_{X_1} \to \mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}$ together with a choice of a common geometric point (i.e. a choice of a map $\overline{A}_{X_1} \to \overline{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$, cf. [Sta26, 0BND]) yields a continuous map of Galois groups $\mathrm{Gal}_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}} \to \pi_1^{\text{ét}}(X_1)$, which induces a map $\Gamma_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}} \to \Gamma$. Furthermore, the map $A_{X_1} \to \mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}$ induces natural maps $\widetilde{\mathbf{A}}_{1,f} \to \widetilde{\mathbf{A}}_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$ and $\mathbf{C}^\dagger_{1,f} \to \mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$ compatible with φ- and $\Gamma_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$-actions. Now, consider the natural isomorphism

$$(\mathbb{L}|_{X_1} \otimes_{\mathbb{Z}_p} W(\overline{A}^{\wedge,\flat}_{X_1})) \otimes_{W(\overline{A}^{\wedge,\flat}_{X_1}))} W(\overline{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}^{\wedge,\flat}) \cong \mathbb{L}|_{\widetilde{\xi}_{\mathrm{Ⅱ}}} \otimes_{\mathbb{Z}_p} W(\overline{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}^{\wedge,\flat})$$

compatible with φ- and $\mathrm{Gal}_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$-actions. Unwinding the construction of $D^\dagger_{\mathrm{rig}}(\cdot)$, we deduce a natural isomorphism

$$D^\dagger_{\mathrm{rig}}(\mathbb{L}|_{X_1}) \otimes_{\mathbf{C}^\dagger_{1,f}} \mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}} \cong D^\dagger_{\mathrm{rig}}(\mathbb{L}|_{\widetilde{\xi}_{\mathrm{Ⅱ}}})$$

compatible with φ- and $\Gamma_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$-actions. The desired statement then follows from [Ber02, Théorème 5.10]. □

Since $\mathbb{L}|_{\widetilde{\xi}_{\mathrm{Ⅱ}}}$ is semistable, as an immediate consequence of Lemma 8.3, we know that

$$\mathbf{N}_{\mathrm{dR},\widetilde{\xi}_{\mathrm{Ⅱ}}} \coloneqq \mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X_1}) \otimes_{\mathbf{C}^\dagger_{1,f}} \mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$$

is strongly unipotent with respect to D_{π_K};[31] that is, there exists a basis of $\mathbf{N}_{\mathrm{dR},\widetilde{\xi}_{\mathrm{Ⅱ}}}$ over $\mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}$, with respect to which the differential $\Delta_{\pi_K} \coloneqq \pi_K D_{\pi_K}$ acts through an upper-triangular nilpotent matrix with coefficients in $\mathscr{H}' = \left(\mathbf{C}^\dagger_{\mathscr{H}(\xi_{\mathrm{Ⅱ}})^{\mathrm{pf}}}\right)^{\frac{d}{d\pi_K}=0}$.

[31] According to [Ked06, Proposition 5.2.6], there is no difference between unipotence and strongly unipotence.

8.2. **Arithmetic unipotence at the type Ⅱ point.**

Let L denote the completion of $K_0'([T^\flat])$ with respect to the 1-Gauss norm and let $|\cdot|_L$ denote the resulting norm on L. The Robba ring $\mathscr{R}_L$ with coordinate π_K (see §1.9) can be naturally viewed as a subring of $\mathbf{C}^\dagger_{\mathscr{H}(\xi_{\text{Ⅱ}})^{\text{pf}}}$, and the map $\mathbf{C}^\dagger_{1,f} \to \mathbf{C}^\dagger_{\mathscr{H}(\xi_{\text{Ⅱ}})^{\text{pf}}}$ factors as $\mathbf{C}^\dagger_{1,f} \to \mathscr{R}_L \to \mathbf{C}^\dagger_{\mathscr{H}(\xi_{\text{Ⅱ}})^{\text{pf}}}$, compatibly with differential operators $\frac{d}{d\pi_K}$.

Lemma 8.4. Let N_0 be a finite projective differential module over $(\mathscr{R}_L, \frac{d}{d\pi_K})$. Let $\mathbb{C}_L$ be a completed algebraic closure of L containing $\mathscr{H}'$ and write $N_{\mathbb{C}_L} \coloneqq N_0 \otimes_{\mathscr{R}_L} \mathscr{R}_{\mathbb{C}_L}$. If $N_{\mathbb{C}_L}$ is strongly unipotent with respect to D_{π_K}, then so is N_0.

Proof. By assumption, $N_{\mathbb{C}_L}$ admits a basis $e_1, \dots, e_d$ over $\mathscr{R}_{\mathbb{C}_L}$, with respect to which $\Delta_{\pi_K} = \pi_K D_{\pi_K}$ takes the form

$$\Delta_{\pi_K} = \pi_K \cdot \frac{d}{d\pi_K} + Z$$

where $Z \in \mathrm{Mat}_{d\times d}(\mathbb{C}_L)$ is upper-triangular nilpotent. Consider the $\mathbb{C}_L$-vector space $\ker(\Delta^d_{\pi_K}) \subset N_{\mathbb{C}_L}$. We claim that $\ker(\Delta^d_{\pi_K})$ is precisely the $\mathbb{C}_L$-span of $e_1, \dots, e_d$. It suffices to show that, if $x = \sum_{i=1,\dots,d} a_i e_i$ (with $a_i \in \mathscr{R}_{\mathbb{C}_L}$) lies in $\ker(\Delta^d_{\pi_K})$, then $a_i \in \mathbb{C}_L$ for all $1 \leqslant i \leqslant d$. We first show $a_d \in \mathbb{C}_L$. Write $a_d = \sum_{j\in\mathbb{Z}} b_j \pi_K^j$ with $b_j \in \mathbb{C}_L$, and write $\Delta^d_{\pi_K}(x) = \sum_i \widetilde{a}_i e_i$. Note that $\widetilde{a}_i = 0$ for all i since $x \in \ker(\Delta^d_{\pi_K})$. To proceed, we compute $\widetilde{a}_d$ in terms of the a_i's. Since $Z(e_i)$ lies in the span of $\{e_1, \dots, e_{i-1}\}$, we have

$$\widetilde{a}_d = (\pi_K \frac{d}{d\pi_K})^d(a_d) = \sum_{j\neq 0} j^d b_j \pi_K^j = 0.$$

Thus, $b_j = 0$ whenever $j \neq 0$, and $a_d \in \mathbb{C}_L$. Next, since $a_d \in \mathbb{C}_L$, we have $x - a_d e_d \in \ker(\Delta^d_{\pi_K})$. Applying the same argument as above inductively, we conclude that $a_i \in \mathbb{C}_L$ for all i, as desired.

The rest of the proof is essentially Galois descent. First note that $(\mathbb{C}_L)^{\mathrm{Gal}_L} = L$ à la Ax–Tate–Sen. Consider the Gal_L-action on the Robba ring $\mathscr{R}_{\mathbb{C}_L}$ given by the usual Gal_L-action on the coefficients in $\mathbb{C}_L$ and the trivial action on π_K. We equip $\mathscr{R}_L$ and N_0 with the trivial Gal_L-actions and equip $N_{\mathbb{C}_L} = N_0 \otimes_{\mathscr{R}_L} \mathscr{R}_{\mathbb{C}_L}$ with the induced action. Note that Gal_L commutes with $\Delta_{\pi_K} = \pi_K D_{\pi_K}$. In particular, $\ker(\Delta^d_{\pi_K})$ is Gal_L-stable.

Choose a basis $f_1, \dots, f_d$ of N_0. For each $1 \leqslant i \leqslant d$ and $j \in \mathbb{Z}$, consider the $\mathbb{C}_L$-linear Gal_L-equivariant projections

$$\lambda_{i,j} \colon N_{\mathbb{C}_L} \to \mathbb{C}_L$$

given by sending $\sum_{i=1}^d (\sum_{j\in\mathbb{Z}} b_{i,j}\pi_K^j) \cdot f_i$ (with $b_{i,j} \in \mathbb{C}_L$) to $b_{i,j}$. Since $\ker(\Delta^d_{\pi_K})$ is d-dimensional Gal_L-stable $\mathbb{C}_L$-sub-vector space in $N_{\mathbb{C}_L}$, there exist d such linear maps among $\lambda_{i,j}$'s such that their restrictions to $\ker(\Delta^d_{\pi_K})$ give rise to a Gal_L-equivariant isomorphism

$$\lambda \coloneqq \ker(\Delta^d_{\pi_K}) \xrightarrow{\sim} \mathbb{C}_L^d.$$

For each $1 \leqslant i \leqslant d$, let $\widetilde{f}_i$ be the pre-image of the i-th standard basis of $\mathbb{C}_L^d$ under λ. Then Gal_L fixes $\widetilde{f}_i$, so $\widetilde{f}_i \in N_0$ for all $1 \leqslant i \leqslant d$. Note that $\widetilde{f}_1, \dots, \widetilde{f}_d$ form another basis of N_0. Write $W_0 \subset \ker(\Delta^d_{\pi_K})$ for the L-span of $\widetilde{f}_1, \dots, \widetilde{f}_d$, which is a d-dimensional L-subspace. Note that Δ_{π_K} preserves W_0 and acts nilpotently on W_0. So we can pick an L-basis $g_1, \dots, g_d$ with respect to which Δ_{π_K} acts through an upper-triangular nilpotent matrix Z_0. This then yields an $\mathscr{R}_L$-basis of N_0 such that $\Delta_{\pi_K} = \pi_K \frac{d}{d\pi_K} + Z_0$, as desired. □

As an immediate consequence of Lemma 8.4, we know that

$$\mathbf{N}_{\mathrm{dR},\xi_{\text{Ⅱ}}} \coloneqq \mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X_1}) \otimes_{\mathbf{C}^\dagger_{1,f}} \mathscr{R}_L = \mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X_\delta}) \otimes_{\mathbf{C}^\dagger_{\delta,f}} \mathscr{R}_L$$

is strongly unipotent with respect to D_{π_K}. Therefore, for r sufficiently small, we have

$$\mathbf{N}^r_{\mathrm{dR},\xi_{\text{Ⅱ}}} \coloneqq \mathbf{N}^r_{\mathrm{dR}}(\mathbb{L}|_{X_1}) \otimes_{\mathbf{C}^r_{1,f}} \mathscr{R}^r_L = \mathbf{N}^r_{\mathrm{dR}}(\mathbb{L}|_{X_\delta}) \otimes_{\mathbf{C}^r_{\delta,f}} \mathscr{R}^r_L$$

is strongly unipotent with respect to D_{π_K}.

8.3. **Propagation of arithmetic unipotence.**

To prove Theorem 8.2, it remains to propagate the unipotence along the map $\mathbf{C}^r_{\delta,f} \to \mathscr{R}^r_L$, for some sufficiently small r. To this end, we first prove a propagation result on the level of relative Robba rings.

Let us introduce some notation. For any $\delta \in (0,1) \cap p^{\mathbb{Q}}$ and any monic polynomial $g = g(T) \in \mathcal{O}_{K_0'}[T]$ of the form $g(T) = \prod_i (T - [\gamma_i])$ with $\gamma_i \in \overline{k}_K^{\times}$ (so that $|g| = 1$ for the 1-Gauss norm), consider the rational subset

$$V_{\delta,g} := \{\delta \leqslant |[T^\flat]| \leqslant 1,\ |g([T^\flat])| = 1\} \subset A_{K_0'}([\delta,1];[T^\flat]),$$

by removing finitely many residue disks from the closed annulus $A_{K_0'}([\delta,1];[T^\flat])$. Let $A_{\delta,g}$ be the ring of analytic functions on $V_{\delta,g}$, equipped with the spectral norm $|\cdot|_{\delta,g}$. Note that $K_0'([T^\flat]) \subset \mathrm{Frac}(A_{\delta,g}) \subset L$, and L coincides with the completion of $\mathrm{Frac}(A_{\delta,g})$ under the 1-Gauss norm.

Remark 8.5. Our strategy is inspired by [Ked06, Proposition 5.3.3], where a similar result is proved for certain over-convergent algebras (which are called *dagger algebras* in *loc. cit.*). If we ignore the removed residue disks on X_1 for the moment, then a tempting choice of such a dagger algebra (in order to apply [Ked06, Proposition 5.3.3]) could be

$$A^{\mathrm{naive},\dagger} := K_0'\langle [T^\flat]^{\pm 1}\rangle^\dagger,$$

where the over-convergence condition is both in the inward direction (i.e. towards radius δ) and the outward direction (i.e. towards radius $1+\epsilon$ for some $\epsilon > 0$). However, our p-adic differential equation is only defined over $\mathbf{C}^r_{\delta,f}$, whose coefficients in $[T^\flat]$ are not over-convergent in the outward direction. Nevertheless, we will show that the argument of [Ked06, Proposition 5.3.3] can be adapted to our setting by considering a "one-sided" dagger algebra $\varinjlim_{\delta \to 1^-} A_{\delta,g}$ instead. For the precise statement, see Proposition 8.7 below.

Notation 8.6. Let $\delta \in (0,1) \cap p^{\mathbb{Q}}$ and $g \in \mathcal{O}_{K_0'}[T]$ as above.

(1) For each $0 < s \leqslant r$, consider the ring $\mathscr{R}^{[s,r]}_{A_{\delta,g}}$ with coordinate π_K as defined in §1.9. In particular, $\mathscr{R}^{[s,r]}_{A_{\delta,g}}$ is the affinoid K_0'-algebra of analytic functions on the affinoid space $V_{\delta,g} \times A_{K_0'}([p^{-c_K \cdot r}, p^{-c_K \cdot s}])$.

(2) Consider the following norms on $\mathscr{R}^{[s,r]}_{A_{\delta,g}}$:

(i) The spectral norm $\|\cdot\|_{\delta,g,[s,r]}$ on the product $V_{\delta,g} \times A_{K_0'}([p^{-c_K \cdot r}, p^{-c_K \cdot s}])$ is given by

$$\|\sum_{i \in \mathbb{Z}} a_i \pi_K^i\|_{\delta,g,[s,r]} := \sup_{i \in \mathbb{Z}} \sup_{t \in [s,r]} \{|a_i|_{\delta,g} \cdot p^{-c_K \cdot ti}\}.$$

with $a_i \in A_{\delta,g}$.

(ii) The "relative 1-Gauss norm" $\|\cdot\|_{L,[s,r]}$ is defined to be

$$\|\sum_{i \in \mathbb{Z}} a_i \pi_K^i\|_{L,[s,r]} := \sup_{i \in \mathbb{Z}} \sup_{t \in [s,r]} \{|a_i|_L \cdot p^{-c_K \cdot ti}\}.$$

where we view each a_i as an element of L via the inclusion $A_{\delta,g} \subset L$ and $|\cdot|_L$ is the 1-Gauss norm on L.

Proposition 8.7. Let $\delta \in (0,1) \cap p^{\mathbb{Q}}$ and $g \in \mathcal{O}_{K_0'}[T]$ as above. Let (M, Δ) be a finite projective differential module over $(\mathscr{R}^{[s_0,r_0]}_{A_{\delta,g}}, \pi_K \frac{d}{d\pi_K})$ of rank d for some s_0, r_0 satisfying $s_0 < r_0/p$. Suppose that the base change

$$M_L := M \otimes_{\mathscr{R}^{[s_0,r_0]}_{A_{\delta,g}}} \mathscr{R}^{[s_0,r_0]}_L$$

admits a strongly unipotent basis (i.e. with respect to this basis, Δ takes the form $\pi_K \frac{d}{d\pi_K} + Z_L$ for some upper-triangular nilpotent matrix $Z_L \in \mathrm{Mat}_{d \times d}(L)$). Then M becomes strongly unipotent after shrinking $V_{\delta,g}$. More precisely, given any interval $[s,r] \subset (s_0, r_0)$ with $s < r/p$, up to increasing δ, there exists a monic polynomial $g' \in \mathcal{O}_{K_0'}[T]$ of the form $g'(T) = \prod_{i=1}^m (T - [\gamma_i'])$ with $\gamma_i' \in \overline{k}_K^{\times}$, such that the base change

$$M_{\delta,g'g} := M \otimes_{\mathscr{R}^{[s_0,r_0]}_{A_{\delta,g}}} \mathscr{R}^{[s,r]}_{A_{\delta,g'g}}$$

admits a strongly unipotent basis with respect to which $\Delta = \pi_K D_{\pi_K}$ acts through an upper-triangular nilpotent matrix with entries in $A_{\delta,g'g}$.

In the remainder of this subsection, we explain how the arguments in [Ked06] carry over to our setting, pointing out differences along the way. We start with the following analogue of [Ked06, Proposition 2.4.2], which is an essential ingredient in the proof of Proposition 8.7.

Lemma 8.8. For any $x \in A_{\delta,g}$ and any $\epsilon \in (0,1]$, we have

$$|x|_{\delta^\epsilon,g} \leqslant |x|_L^{1-\epsilon} \cdot |x|_{\delta,g}^{\epsilon}$$

Proof. We first claim that it suffices to prove the inequality for elements of the form $x = \frac{h}{g^N}$ for some $h \in K_0'[[T^\flat]^{\pm 1}]$ and $N \in \mathbb{Z}_{\geqslant 0}$. Indeed, these rational functions are dense in $A_{\delta,g}$ under the spectral norm. Suppose that $x \in A_{\delta,g}$ and that x_n is a sequence of such rational functions with $|x_n - x|_{\delta,g} \to 0$. Then, since $V_{\delta^\epsilon,g} \subset V_{\delta,g}$, we have $|x_n - x|_{\delta^\epsilon,g} \leqslant |x_n - x|_{\delta,g} \to 0$ and $|x_n - x|_L \leqslant |x_n - x|_{\delta,g} \to 0$, and the claim follows.

For the rational function $x = \frac{h}{g^N}$, we have $|x|_{\delta,g} = \max\{|h|_\delta, |h|_1\}$ and $|x|_L = |h|_1$. Here, $|\cdot|_\rho$ stands for the ρ-Gauss norm; that is, for $h = \sum_{j\in\mathbb{Z}} a_j[T^\flat]^j$ with $a_j \in K_0'$, we put $|h|_\rho = \sup_j\{|a_j| \cdot \rho^j\}$. For each monomial $a_j[T^\flat]^j$, we have $|a_j|\delta^{\epsilon\cdot i} = |a_j|^{1-\epsilon}(|a_j|\delta^i)^\epsilon$. Taking the supremum over j yields the desired inequality $|x|_{\delta^\epsilon,g} \leqslant |x|_L^{1-\epsilon} \cdot |x|_{\delta,g}^{\epsilon}$. □

Corollary 8.9. For every element $x \in \mathscr{R}_{A_{\delta,g}}^{[s,r]}$ and every $\epsilon \in (0,1]$, we have

$$\|x\|_{\delta^\epsilon,g,[s,r]} \leqslant \|x\|_{L,[s,r]}^{1-\epsilon} \cdot \|x\|_{\delta,g,[s,r]}^{\epsilon}.$$

Proof. This follows by applying Lemma 8.8 coefficient-wise to $x = \sum_i x_i \cdot \pi_K^i$. □

Lemma 8.10. Keep the assumption of Proposition 8.7. Then, after shrinking $V_{\delta,g}$ (i.e., increasing δ and replacing g by $g'g$ as in the statement of Proposition 8.7), M becomes finite free over $\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}$.

Proof. Note that M_L is free by assumption. Choosing a basis $e_1, \ldots, e_d$ of M_L and consider the isomorphism $S_L\colon (\mathscr{R}_L^{[s_0,r_0]})^d \xrightarrow{\sim} M_L$ sending $(a_1, \ldots, a_d) \mapsto \sum_{i=1}^d a_i e_i$. Pick a finite set of generators $\{m_j\}_{j\in J}$ of the finite projective $\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}$-module M and write $e_i = \sum_{j\in J} c_j m_j$ for each i, with $c_j \in \mathscr{R}_L^{[s_0,r_0]}$. Such an expression is not unique in general.

Observe that every element in $\mathscr{R}_L^{[s_0,r_0]}$ can be approximated by elements in $\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}$ (up to shrinking $V_{\delta,g}$). To see this, it suffices to notice that $K_0'([T^\flat])[\pi_K^{\pm 1}] \subset \mathscr{R}_L^{[s_0,r_0]}$ is dense under the relative 1-Gauss norm $\|\cdot\|_{L,[s_0,r_0]}$, and then note that each element in $K_0'([T^\flat])$ is defined on some $V_{\delta,g'g}$ (by increasing δ if necessary).

Now, we approximate each c_j by $c_j' \in \mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}$ as above, and let $e_i' = \sum_{j\in J} c_j' m_j \in M$. We obtain a linear map $S\colon \left(\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}\right)^d \to M$ sending $(a_1, \ldots, a_d) \mapsto \sum_{i=1}^d a_i e_i'$, such that S approximates S_L. Similarly, let us consider the dual basis $e_1^\vee, \ldots, e_d^\vee$ of $M_L^\vee$ and consider $S_L'\colon (\mathscr{R}_L^{[s_0,r_0]})^d \xrightarrow{\sim} M_L^\vee$ sending $(a_1, \ldots, a_d) \mapsto \sum_{i=1}^d a_i e_i^\vee$. By choosing a finite set of generators of $M^\vee$ and repeating the approximation process above, we arrive at a linear map $S'\colon \left(\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}\right)^d \to M^\vee$ that approximates S_L'. Taking duals, we obtain $S'^\vee\colon M \to \left(\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}\right)^d$.

Choosing sufficiently accurate approximates, we can ensure that $P := S'^\vee \circ S - \mathrm{id}$ satisfies $\|P\|_{L,[s_0,r_0]} < 1$. Applying Corollary 8.9 entry-wise to P, we have

$$\|P\|_{\delta^\epsilon,g,[s_0,r_0]} \leqslant \|P\|_{L,[s_0,r_0]}^{1-\epsilon} \cdot \|P\|_{\delta,g,[s_0,r_0]}^{\epsilon}.$$

The right hand side is $\leqslant 1$ for ϵ sufficiently close to 0^+. Hence, by choosing such an ϵ and replacing δ by δ^ϵ, we obtain $\|P\|_{\delta,g,[s_0,r_0]} < 1$ and so $S'^\vee \circ S$ is invertible. Therefore, S is a split injection and $M \cong (\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]})^d \oplus M'$ for a finite projective module M' over $\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}$. Since M has rank d, we must have $M' = 0$. This finishes the proof. □

Given the preparations above, Proposition 8.7 follows from the same argument as in the proof of [Ked06, Proposition 5.3.3], using Lemma 8.8 and Corollary 8.9 instead of Proposition 2.4.2 in *loc. cit.* Let us briefly sketch out the argument below.

Proof of Proposition 8.7. Fix $[s,r] \subset (s_0, r_0)$ with $s < r/p$. By assumption, there exists a strongly unipotent basis of M_L over $\mathscr{R}_L^{[s_0,r_0]}$ with respect to which Δ acts as $\Delta = \pi_K \frac{d}{d\pi_K} + Z_L$, where Z_L is an upper-triangular nilpotent matrix with coefficients in L. Let e be the nilpotency index of Z_L. By Lemma 8.10, we may assume that M is free over $\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}$. Consider the operators $h_l \colon M \to M$ defined by

$$h_0 = \Delta^{e-1}, \qquad h_l = (1 - \frac{\Delta^2}{l^2})^e h_{l-1} \quad \text{for } l \geqslant 1$$

as in [Ked06, Lemma 5.3.1]. Then by the same argument as in the proof of [Ked06, Lemma 5.3.1], we have that for any $w \in M_L$, the sequence $\{h_l(w)\}_{\ell=0}^{\infty}$ converges to an element $h_\infty(w)$ in M_L under $\| \cdot \|_{L,[s_0,r_0]}$, satisfying $\Delta(h_\infty(w)) = 0$. The key is then to prove the following analogue of [Ked06, Lemma 5.3.2].

Claim. Let $w \in M$ and $[s', r'] \subset (s_0, r_0)$. There exists $\delta' \geqslant \delta$ such that $h_\infty(w)$ lies in

$$M \otimes_{\mathscr{R}_{A_{\delta,g}}^{[s_0,r_0]}} \mathscr{R}_{A_{\delta',g}}^{[s',r']}.$$

Proof of Claim. We follow a similar idea as in the proof of [Ked06, Lemma 5.3.2]. Write $w_l := h_{l+1}(w) - h_l(w)$. Then by the proof of *loc. cit.*, we have $\|w_l\|_{L,[s',r']} \leqslant c \cdot \alpha^l$ and $\|w_l\|_{\delta,g,[s',r']} \leqslant c' \cdot (\alpha')^l$ for some $c, c', \alpha, \alpha' > 0$ with $\alpha < 1$ and $\alpha' \geqslant 1$ (here $[s', r'] \subset (s_0, r_0)$ is used to ensure that we can find such $\alpha < 1$). Now by Corollary 8.9 (this replaces the role of [Ked06, Proposition 2.4.2] in the proof of [Ked06, Lemma 5.3.2]), we have

$$\|w_l\|_{\delta^\epsilon,g,[s',r']} \leqslant c^{1-\epsilon}(c')^\epsilon(\alpha^{1-\epsilon}(\alpha')^\epsilon)^l$$

for all l. Choosing ϵ such that $\alpha^{1-\epsilon}(\alpha')^\epsilon < 1$ and taking $\delta' = \delta^\epsilon$, we obtain the desired statement in the claim.

Thanks to the claim, for a given $w \in M$, we may assume that $h_\infty(w) \in M$ (after increasing δ if necessary), which again satisfies $\Delta(h_\infty(w)) = 0$. This puts us in a situation where we can apply the induction argument as in the proof of [Ked06, Proposition 5.3.3]. Now the argument in the proof of [Ked06, Proposition 5.2.6] carries over *verbatim* and produces the desired strongly unipotent basis. □

8.4. **Proof of Theorem 8.2.**

Using Proposition 8.7, we now prove Theorem 8.2. The first step is to achieve unipotence over $\mathbf{C}_{\delta,f}^{[s,r]}$ for some suitable choice of $[s,r]$. For any $[s,r]$ with r sufficiently small, let

$$\mathbf{N}_{\mathrm{dR}}^{[s,r]}(\mathbb{L}|_{X_\delta}) := \mathbf{N}_{\mathrm{dR}}^{r}(\mathbb{L}|_{X_\delta}) \otimes_{\mathbf{C}_{\delta,f}^{r}} \mathbf{C}_{\delta,f}^{[s,r]}.$$

Recall that we already know that $\mathbf{N}_{\mathrm{dR},\xi_{\mathrm{II}}}^{r_0} = \mathbf{N}_{\mathrm{dR}}^{r_0}(\mathbb{L}|_{X_\delta}) \otimes_{\mathbf{C}_{\delta,f}^{r_0}} \mathscr{R}_L^{r_0}$ is strongly unipotent with respect to D_{π_K}, for some sufficiently small r_0. We further assume that $\lambda_r(u_{\pi_K} - 1) < p^{-\frac{1}{p-1}}$ for all $0 < r \leqslant r_0$, where $u_{\pi_K} = \varphi(\pi_K)/\pi_K^p$. In particular, we have a well-defined element $\log(u_{\pi_K})$ in $\mathbf{C}_K^{r_0}$.

Lemma 8.11. Up to replacing K by an unramified extension and shrinking $X_\delta = X_{\delta,f}$, there exists $0 < s < r < r_0$ with $s < r/p$, such that $\mathbf{N}_{\mathrm{dR}}^{[s,r]}(\mathbb{L}|_{X_\delta})$ is free over $\mathbf{C}_{\delta,f}^{[s,r]}$ and admits a basis with respect to which $\Delta_{\pi_K} = \pi_K D_{\pi_K}$ acts through an upper-triangular nilpotent matrix with entries in $A_{\delta^r,f}$.

Proof. Pick $s_0 \in (0, r_0/p]$. Then $\mathbf{N}_{\mathrm{dR}}^{[s_0,r_0]}(\mathbb{L}|_{X_\delta}) \otimes_{\mathbf{C}_{\delta,f}^{[s_0,r_0]}} \mathscr{R}_L^{[s_0,r_0]}$ is strongly unipotent with respect to D_{π_K}.

By Lemma 4.26, $\mathbf{C}_{\delta,f}^{[s_0,r_0]}$ is precisely the ring of analytic functions on the rational subdomain

$$W_{\delta,f}^{[s_0,r_0]} = \{|\pi_K| = p^{-c_K \cdot t},\ \delta^t \leqslant |[T^\flat]| \leqslant 1,\ |f([T^\flat])| = 1,\ t \in [s_0, r_0]\}$$

inside the two-dimensional poly-annulus

$$A_{K_0'}([\delta^{r_0}, 1]; [T^\flat]) \times A_{K_0'}([p^{-c_K r_0}, p^{-c_K s_0}]; \pi_K).$$

Therefore, we have containment of rational subsets

$$V_{\delta^{s_0},f} \times A_{K_0'}([p^{-c_K \cdot r_0}, p^{-c_K \cdot s_0}]) \subset W_{\delta,f}^{[s_0,r_0]} \subset V_{\delta^{r_0},f} \times A_{K_0'}([p^{-c_K \cdot r_0}, p^{-c_K \cdot s_0}]).$$

In particular, for any choice of $\delta' \in [\delta^{s_0}, 1)$, we have

$$V_{\delta',f} \times A_{K_0'}([p^{-c_K \cdot r_0}, p^{-c_K \cdot s_0}]) \subset W_{\delta,f}^{[s_0,r_0]},$$

which induces an injection

$$\iota_1 \colon \mathbf{C}_{\delta,f}^{[s_0,r_0]} \to \mathscr{R}_{A_{\delta',f}}^{[s_0,r_0]}$$

compatible with the arithmetic differential operator D_{π_K}. The base change of $\mathbf{N}_{\mathrm{dR}}^{[s_0,r_0]}(\mathbb{L}|_{X_\delta})$ along ι_1 then produces a differential module satisfying the assumption of Proposition 8.7. Consequently, by choosing $[s,r] \subset (s_0,r_0)$ with $s < r/p$ and increasing δ', we know that $\mathbf{N}_{\mathrm{dR}}^{[s_0,r_0]}(\mathbb{L}|_{X_\delta})$ becomes strongly unipotent with respect to D_{π_K} after further base change to $\mathscr{R}_{A_{\delta',g'f}}^{[s,r]}$ for some $g' \in \mathcal{O}_{K_0'}[T]$. Up to replacing K by an unramified extension[32] (so that the coefficients of g' lie in K_0) and shrinking X_δ, we may assume that

$$\mathbf{N}_{\mathrm{dR}}^{[s_0,r_0]}(\mathbb{L}|_{X_\delta}) \otimes_{\mathbf{C}_{\delta,f}^{[s_0,r_0]}} \mathscr{R}_{A_{\delta',f}}^{[s,r]}$$

admits a basis with respect to which Δ_{π_K} acts through an upper-triangular matrix with entries in $A_{\delta',f}$.

Finally, we pick $\delta'' \in ((\delta')^{1/r}, 1)$ so that

$$W_{\delta'',f}^{[s,r]} \subset V_{\delta',f} \times A_{K_0'}([p^{-c_K \cdot r}, p^{-c_K \cdot s}]),$$

which induces an injection

$$\iota_2 \colon \mathscr{R}_{A_{\delta',f}}^{[s,r]} \to \mathbf{C}_{\delta'',f}^{[s,r]}$$

compatible with D_{π_K}. Since the composition $\iota_2 \circ \iota_1$ is the natural inclusion and we have

$$\left(\mathbf{C}_{\delta'',f}^{[s,r]}\right)^{\frac{d}{d\pi_K}=0} = A_{\delta''^r,f}$$

from the explicit description of $\mathbf{C}_{\delta,f}^{[s,r]}$ (cf. Lemma 4.26), we obtained the desired strong unipotence. □

We are now ready to prove Theorem 8.2. The idea is to use Frobenius to translate the arithmetic unipotence from $\mathbf{C}_{\delta,f}^{[s,r]}$ to $\mathbf{C}_{\delta,f}^{r}$.

Proof of Theorem 8.2. To simplify notations, write $N^{[s,r]} \coloneqq \mathbf{N}_{\mathrm{dR}}^{[s,r]}(\mathbb{L}|_{X_\delta})$. By Lemma 8.11, we may assume that $0 < s < r/p$ and $N^{[s,r]}$ is free over $\mathbf{C}_{\delta,f}^{[s,r]}$ and admits a basis with respect to which Δ_{π_K} acts as

$$\Delta_{\pi_K} = \pi_K \frac{d}{d\pi_K} + Z, \tag{8.1}$$

for some upper-triangular nilpotent matrix Z with coefficients in $A_{\delta^r,f}$.

Write $s_n \coloneqq s/p^n$ and $r_n \coloneqq r/p^n$. By [DMY26, Construction 10.7], there are compatible isomorphisms

$$\phi_n \colon N^{[s_n,r_n]} \otimes_{\mathbf{C}_{\delta,f}^{[s_n,r_n]},\varphi} \mathbf{C}_{\delta,f}^{[s_{n+1},r_{n+1}]} \xrightarrow{\sim} N^{[s_{n+1},r_{n+1}]}.$$

Our goal is to construct a basis $\mathscr{B}_n$ of $N^{[s_n,r_n]}$ for each $n \geqslant 1$, such that $\mathscr{B}_n$'s agree on the overlap and $\Delta_{\pi_K} = \pi_K \frac{d}{d\pi_K} + Z$ relative to these bases for the same Z as in (8.1). We will construct $\mathscr{B}_n$ inductively.

Recall that we have a well-defined element $\log(u_{\pi_K}) \in \mathbf{C}_K^r$. Assuming $\mathscr{B}_n$ is given, we construct $\mathscr{B}_{n+1}$ as follows. Write $\mathscr{B}_n = \{e_1, \ldots, e_d\}$, and consider $\mathscr{B}'_{n+1} = \{\phi_n(e_1), \ldots, \phi_n(e_d)\}$ which is a basis of $N^{[s_{n+1},r_{n+1}]}$. Since Δ_{π_K} is compatible with Frobenius (cf. [DMY26, Construction 10.7]), we have

$$D_{\pi_K}(\phi(x)) = \frac{d\varphi(\pi_K)}{d\pi_K} \cdot \phi(D_{\pi_K}(x))$$

for all $x \in N^{[s_n,r_n]}$. Hence

$$\Delta_{\pi_K}(\phi(x)) = \frac{\pi_K}{\varphi(\pi_K)} \cdot \frac{d\varphi(\pi_K)}{d\pi_K} \cdot \phi(\Delta_{\pi_K}(x)).$$

This implies

$$\Delta_{\pi_K}(\mathscr{B}'_{n+1}) = \mathscr{B}'_{n+1} \cdot \varphi(Z) \cdot \big(p + \Delta_{\pi_K}(\log u_{\pi_K})\big).$$

We modify $\mathscr{B}'_{n+1}$ into a new basis

$$\mathscr{B}''_{n+1} \coloneqq \mathscr{B}'_{n+1} \cdot \exp\big(-\log u_{\pi_K} \cdot \varphi(Z)\big),$$

which is well-defined as Z is nilpotent. It is straightforward to check that $\Delta_{\pi_K}(\mathscr{B}''_{n+1}) = p \cdot \mathscr{B}''_{n+1} \cdot \varphi(Z)$. In particular, $\mathscr{B}''_{n+1}$ is a strongly unipotent basis for $N^{[s_{n+1},r_{n+1}]}$. We will further modify it so that it agrees

[32] Note that replacing K by an unramified extension does not change π_K.

with $\mathscr{B}_n$ on the overlap $\mathbf{C}_{\delta,f}^{[s_n,r_{n+1}]}$. To this end, let $C_n \in \mathrm{GL}_d(\mathbf{C}_{\delta,f}^{[s_n,r_{n+1}]})$ be the change of basis matrix so that

$$\mathscr{B}''_{n+1} = \mathscr{B}_n \cdot C_n \tag{8.2}$$

on the overlap. Differentiating both sides of (8.2) gives $\Delta_{\pi_K}(C_n) = p \cdot C_n \cdot \varphi(Z) - Z \cdot C_n$. Since $\Delta_{\pi_K}(\varphi(Z)) = \Delta_{\pi_K}(Z) = 0$ and Z is nilpotent with $Z^d = 0$, we deduce that $\Delta_{\pi_K}^{2d-1}(C_n) = 0$. On the other hand, using the explicit description of $\mathbf{C}_{\delta,f}^{[s,r]}$ (cf. Lemma 4.26), one computes that

$$\ker(\Delta_{\pi_K}^m \colon \mathbf{C}_{\delta,f}^{[s,r]} \to \mathbf{C}_{\delta,f}^{[s,r]}) = A_{\delta^r,f}$$

for all $m \geqslant 1$. Hence, the coefficients of C_n all lie in $A_{\delta^{r_{n+1}},f} \subset \mathbf{C}_{\delta,f}^{[s_{n+1},r_{n+1}]}$. Applying the same argument to C_n^{-1}, we deduce that the coefficients of C_n^{-1} also lie in $A_{\delta^{r_{n+1}},f} \subset \mathbf{C}_{\delta,f}^{[s_{n+1},r_{n+1}]}$. Therefore,

$$\mathscr{B}_{n+1} = \mathscr{B}''_{n+1} \cdot C_n^{-1},$$

is a well-defined basis of $N^{[s_{n+1},r_{n+1}]}$ that coincides with $\mathscr{B}_n$ on $\mathbf{C}_{\delta,f}^{[s_n,r_{n+1}]}$, as desired. This finishes the proof of the theorem. □

8.5. **Arithmetic horizontal sections.**

In the remainder of the section, we prove Theorem 8.1. We want to show that, up to enlarging K and shrinking X_δ, there exists a finite étale cover over which $\mathbb{L}$ becomes pointwise semistable. By Theorem 8.2, we may assume that $\mathbf{N}_{\mathrm{dR}}^r(\mathbb{L})$ is strongly unipotent with respect to the arithmetic differential operator D_{π_K} over $\mathbf{C}_{\delta,f}^r$ for some sufficiently small r. For any $\sigma \in [\delta,1) \cap p^{\mathbb{Q}}$, consider the embedding $A_{K_0'}([\delta,\sigma];T) \subset X_\delta$ which induces a natural map $\mathbf{C}_{\delta,f}^r \to \mathbf{C}_{[\delta,\sigma]}^r$. Here $\mathbf{C}_{[\delta,\sigma]}^r$ is the imperfect relative period ring for the annulus $A_{K_0'}([\delta,\sigma];T)$ constructed in §4.7.1. Then the base change

$$N_{[\delta,\sigma]}^r \coloneqq \mathbf{N}_{\mathrm{dR}}^r(\mathbb{L}) \otimes_{\mathbf{C}_{\delta,f}^r} \mathbf{C}_{[\delta,\sigma]}^r \cong \mathbf{N}_{\mathrm{dR}}^r(\mathbb{L}|_{A_{K_0'}([\delta,\sigma])})$$

is a finite free $\mathbf{C}_{[\delta,\sigma]}^r$-module equipped with commuting differential operators $D_{[T^\flat]}$ and D_{π_K}, as well as a Frobenius structure, where the last isomorphism follows from [DMY26, Proposition 10.13]. Moreover, $N_{[\delta,\sigma]}^r$ is strongly unipotent with respect to D_{π_K} for all $\sigma \in [\delta,1) \cap p^{\mathbb{Q}}$.

To prove Theorem 8.1, we will first study the D_{π_K}-horizontal sections after adjoining $\log \pi_K$. Then we study the geometric differential operator on these arithmetic horizontal sections using the classical p-adic local monodromy theorem of André–Kedlaya–Mebkhout. As a preparation, we first compute the arithmetic horizontal sections in relevant relative period rings.

Lemma 8.12. For any $0 < \sigma_1 \leqslant \sigma_2 \leqslant 1$ and $0 < s \leqslant r$, we have

$$\left(\mathbf{C}_{[\sigma_1,\sigma_2]}^{[s,r]}\right)^{\frac{d}{d\pi_K}=0} = \left(\mathbf{C}_{[\sigma_1,\sigma_2]}^{[s,r]}[\log \pi_K]\right)^{\frac{d}{d\pi_K}=0} = \Gamma_{K_0'}([\sigma_1^r,\sigma_2^s];[T^\flat])$$

where the right-hand side is the ring of analytic functions on the annulus $A_{K_0'}([\sigma_1^r,\sigma_2^s];[T^\flat])$ (cf. §1.9). Taking limit as $s \to 0^+$, we obtain

$$\left(\mathbf{C}_{[\sigma_1,\sigma_2]}^{r}\right)^{\frac{d}{d\pi_K}=0} = \left(\mathbf{C}_{[\sigma_1,\sigma_2]}^{r}[\log \pi_K]\right)^{\frac{d}{d\pi_K}=0} = \begin{cases} \Gamma_{K_0'}([\sigma_1^r,1);[T^\flat]) & \sigma_2 < 1 \\ \Gamma_{K_0'}([\sigma_1^r,1];[T^\flat]) & \sigma_2 = 1 \end{cases}.$$

Proof. Since $\log \pi_K$ is transcendental over $\mathbf{C}_{[\sigma_1,\sigma_2]}^{[s,r]}$, it suffices to show

$$\left(\mathbf{C}_{[\sigma_1,\sigma_2]}^{[s,r]}\right)^{\frac{d}{d\pi_K}=0} = \Gamma_{K_0'}([\sigma_1^r,\sigma_2^s];[T^\flat]).$$

But this follows from the explicit description of the period ring $\mathbf{C}_{[\sigma_1,\sigma_2]}^{[s,r]}$ in Lemma 4.20. □

Remark 8.13. Let us emphasize that the arithmetic horizontal sections of $\mathbf{C}_{[\sigma_1,\sigma_2]}^r$ and $\mathbf{C}_{[\sigma_1,\sigma_2]}^r[\log \pi_K]$ are actually independent of the outer radius σ_2 (when $\sigma_2 < 1$).

In what follows, we will mostly use "$\mathscr{R}^?_?$" (cf. §1.9) to denote the corresponding rings in coordinate "$[T^\flat]$". More precisely, by a slight abuse of notation, we write

$$\mathscr{R}^{[s,r]}_{K'_0} \coloneqq \Gamma_{K'_0}([p^{-r\cdot c_K}, p^{-s\cdot c_K}]; [T^\flat]) \text{ and } \mathscr{R}^{r}_{K'_0} \coloneqq \Gamma_{K'_0}([p^{-r\cdot c_K}, 1); [T^\flat]).$$

We also write $w_\delta \coloneqq -\frac{1}{c_K}\log_p \delta$ so that

$$\mathscr{R}^{w_\delta r}_{K'_0} = \Gamma_{K'_0}([\delta^r, 1); [T^\flat]).$$

Now, we return to the situation at the beginning of §8.5. In particular, $N^r_{[\delta,\sigma]}$ is strongly unipotent with respect to D_{π_K}.

Corollary 8.14. After joining $\log \pi_K$, the arithmetic horizontal sections

$$N^{r,\nabla_{\mathrm{arith}}} \coloneqq \left(N^r_{[\delta,\sigma]}[\log \pi_K]\right)^{D_{\pi_K}=0}$$

(equipped with the geometric differential operator $D_{[T^\flat]}$) is a finite free differential module over $(\mathscr{R}^{w_\delta r}_{K'_0}, \frac{d}{d[T^\flat]})$, equipped with a Frobenius structure. Moreover, it is independent of σ.

Proof. Note that $D_{[T^\flat]}$ commutes with D_{π_K}. The statements (except those regarding Frobenius structures) follow from the strong unipotence assumption and the Lemma 8.12.

For the Frobenius structure, first note that Frobenius preserves the arithmetic horizontal sections. Indeed, we have an identity

$$D_{\pi_K}\circ\phi = \frac{d\varphi(\pi_K)}{d\pi_K}\cdot\phi\circ D_{\pi_K} \tag{8.3}$$

on $N^r_{[\delta,\sigma]}$, by the compatibility of the differential operators and Frobenius (cf. [DMY26, Construction 10.7]). Since $\frac{d}{d\pi_K}\circ\varphi(\log\pi_K) = \frac{d\varphi(\pi_K)}{d\pi_K}\cdot\varphi(\pi_K)^{-1}$, the identity (8.3) remains valid on $N^r_{[\delta,\sigma]}[\log\pi_K]$. Therefore, for $x\in N^r_{[\delta,\sigma]}[\log\pi_K]$, if $D_{\pi_K}(x)=0$, then $D_{\pi_K}(\phi(x))=0$.

Finally, the original Frobenius structure ϕ on $N^r_{[\delta,\sigma]}$ induces a Frobenius isomorphism

$$\phi\colon N^{r,\nabla_{\mathrm{arith}}}\otimes_{\mathscr{R}^{w_\delta r}_{K'_0},\varphi}\mathscr{R}^{w_\delta r/p}_{K'_0} \xrightarrow{\sim} N^{\frac{r}{p},\nabla_{\mathrm{arith}}}$$

over $\mathscr{R}^{w_\delta r/p}_{K'_0}$, as desired. □

8.6. **Achieving semistability.**

We are ready to prove Theorem 8.1. By Corollary 8.14,

$$N^{\dagger,\nabla_{\mathrm{arith}}} \coloneqq \varinjlim_{r\to 0^+} N^{r,\nabla_{\mathrm{arith}}}$$

is a finite free differential module (of rank d) over $(\mathscr{R}^\dagger_{K'_0}, \frac{d}{d[T^\flat]})$ equipped with a Frobenius structure, where $\mathscr{R}^\dagger_{K'_0} \coloneqq \varinjlim_{r\to 0^+}\mathscr{R}^{w_\delta r}_{K'_0}$ is the *geometric* Robba ring in coordinate $[T^\flat]$. Note that the mod p reduction of the integral Robba ring inside $\mathscr{R}^\dagger_{K'_0}$ is precisely $\kappa(\!(T^\flat)\!)$. By the classical p-adic local monodromy theorem (Theorem 3.23), there exists a finite separable extension $E'/\kappa(\!(T^\flat)\!)$ with associated Robba ring extension $\mathscr{R}^\dagger_{K'_0}\to\mathscr{R}'$, such that $N^{\dagger,\nabla_{\mathrm{arith}}}$ becomes (strongly) unipotent for the geometric differential operator $D_{[T^\flat]}$ after tensoring with $\mathscr{R}'$. Our goal is to find a finite étale cover of X_δ (after enlarging K and shrinking X_δ if necessary) that realizes this extension of Robba rings.

By [Kat86, Theorem 1.4.1], there exists a finite étale $\kappa[(T^\flat)^{\pm 1}]$-algebra Q[33] together with an isomorphism

$$\alpha\colon Q\otimes_{\kappa[(T^\flat)^{\pm1}]}\kappa(\!(T^\flat)\!) \xrightarrow{\sim} E'.$$

Recall from §4.7.2 the perfectoid algebra $\widehat{A}_{\delta,f,\infty}$ with its tilt $\widetilde{R}_{\delta,f}$. Consider the finite étale map

$$\widetilde{R}_{\delta,f}\to Q\otimes_{\kappa[(T^\flat)^{\pm1}]}\widetilde{R}_{\delta,f}.$$

[33] This corresponds to what Katz calls a *special* finite étale cover of $\mathbb{G}_{m,\kappa}$

By [DMY26, Proposition 5.28], this corresponds to some finite étale maps $R_{\delta,f} \to R'$ and $\widehat{A}_{\delta,f,\infty} \to B$. By the same argument as in Construction 6.10, after replacing K by $K(\zeta_{p^n})$ and X_δ by its p^n-th Kummer cover $X_{\delta,n}$ for some $n \geqslant 0$ if necessary, we may assume that the finite étale map $\widehat{A}_{\delta,f,\infty} \to B$ descends to a finite étale map $A_{\delta,f} \to B_0$. Let $\mathrm{pr}\colon X'_\delta \to X_\delta$ be the corresponding finite étale cover, and let $\mathbf{A}^r_{X'_\delta}$, $\mathbf{B}^r_{X'_\delta}$, $\mathbf{C}^r_{X'_\delta}$, etc. denote the corresponding imperfect relative period rings (cf. §4.6). In particular, we have $\mathbf{A}^r_{X'_\delta}/p = R'$ for r sufficiently small.

Theorem 8.1 is an immediate consequence of the following proposition.

Proposition 8.15. After enlarging K if necessary, there exists a finite étale cover $X''_\delta \to X'_\delta$ such that $\mathbb{L}$ becomes pointwise semistable after pulling back along the composition $X''_\delta \to X'_\delta \xrightarrow{\mathrm{pr}} X_\delta$.

Proof. By assumption, $\mathbb{L}$ is already pointwise semistable on the outer boundary $X_1 \subset X_\delta$. So, it suffices to construct a finite étale cover $X''_\delta \to X'_\delta$ (after enlarging K if necessary) such that for any closed annulus $A_K([\delta,\sigma]) \subset X_\delta$ with $\sigma \in [\delta,1) \cap p^{\mathbb{Q}}$, the pullback of $\mathbb{L}|_{A_K([\delta,\sigma])}$ over $X''_{[\delta,\sigma]} \coloneqq A_K([\delta,\sigma]) \times_{X_\delta} X''_\delta$ is pointwise semistable.

Consider the finite étale cover $X'_{[\delta,\sigma]} \coloneqq A_K([\delta,\sigma]) \times_{X_\delta} X'_\delta$ of the annulus $A_K[\delta,\sigma]$ and let $\mathbf{C}^\dagger_{X'_{[\delta,\sigma]}}$ denote the corresponding imperfect relative period rings (cf. §4.6). By [DMY26, Proposition 5.28, Corollary 5.31], the isomorphism

$$\widetilde{R}_{X'_{[\delta,\sigma]}} = \widetilde{R}_{[\delta,\sigma]} \otimes_{\kappa[(T^\flat)^{\pm 1}]} Q \cong \widetilde{R}_{[\delta,\sigma]} \otimes_{\kappa((T^\flat))} E'$$

induces an isomorphism

$$\mathbf{C}^\dagger_{X'_{[\delta,\sigma]}} \cong \mathbf{C}^\dagger_{[\delta,\sigma]} \otimes_{\mathscr{R}_{K'_0}} \mathscr{R}',$$

which is compatible with both differential operators D_{π_K} and $D_{[T^\flat]}$. Thus, by Lemma 8.12, we have

$$\left(\mathbf{C}^\dagger_{X'_{[\delta,\sigma]}}[\log \pi_K]\right)^{\frac{d}{d\pi_K}=0} = \mathscr{R}'.$$

Now, since $\mathbf{N}_{\mathrm{dR}}(\cdot)$ is compatible with finite étale covers by [DMY26, Proposition 10.13], we have

$$\begin{aligned}\left(\mathbf{N}_{\mathrm{dR}}(\mathrm{pr}^*\mathbb{L}|_{X'_{[\delta,\sigma]}})[\log \pi_K]\right)^{D_{\pi_K}=0} &= \left(\mathbf{N}_{\mathrm{dR}}(\mathbb{L}|_{X_{[\delta,\sigma]}})[\log \pi_K] \otimes_{\mathbf{C}^\dagger_{[\delta,\sigma]}} \mathbf{C}^\dagger_{X'_{[\delta,\sigma]}}\right)^{D_{\pi_K}=0} \\ &= \left(N^\dagger_{[\delta,\sigma]}[\log \pi_K] \otimes_{\mathscr{R}_{K'_0}} \mathscr{R}'\right)^{D_{\pi_K}=0} = N^{\dagger,\nabla_{\mathrm{arith}}} \otimes_{\mathscr{R}_{K'_0}} \mathscr{R}'.\end{aligned}$$

which is (strongly) unipotent with respect to the geometric differential $D_{[T^\flat]}$. Therefore, we obtain a decomposition

$$\mathbf{N}_{\mathrm{dR}}(\mathrm{pr}^*\mathbb{L}|_{X'_{[\delta,\sigma]}})[\log \pi_K, \log[T^\flat]] \cong \bigoplus_{i=1}^{d} \mathbf{C}^\dagger_{X'_{[\delta,\sigma]}}[\log \pi_K, \log[T^\flat]] \cdot w_i$$

compatible with D_{π_K} and $D_{[T^\flat]}$ on both sides, where each w_i satisfies $D_{[T^\flat]}(w_i) = D_{\pi_K}(w_i) = 0$. Finally, note that all these w_i's are fixed by an open subgroup of Γ. By enlarging K and passing to a p-power Kummer cover of X'_δ (cf. Remark 4.17), we may further assume that all w_i's are fixed by Γ. Hence, we can apply Proposition 4.30 (note that the functors $\mathbf{N}_{\mathrm{dR}}$ and $D^\dagger_{\mathrm{rig}}$ coincide after inverting t) and conclude the proof. □

9. Around points of type V: general case

In this section, we prove Theorem 12 for general type V points by reducing to the special case treated in §8. Since every type V point on $X_{\mathbb{C}_K}$ is K'-rational for some finite extension K'/K, it suffices to assume that ξ is K-rational and prove the following.

Theorem 9.1. *Theorem 12 holds when ξ is a K-rational type* V *point on X.*

A key ingredient in the proof is a spread-out argument (cf. Theorem 9.2), which will also play an essential role in the proof of the main theorem in §10.

9.1. **A spread-out theorem.**

The following spread-out theorem will be used in both §9.3 and in the proof of Theorem 1 in §10.

Theorem 9.2. *Let X be a smooth projective curve over K and let X^{ad} denote the associated adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. Let ξ be a non-type I point on X^{ad} and let $U \subset X^{\mathrm{ad}}$ be an affinoid open neighborhood of ξ. Let $f\colon V \to U$ be a finite étale cover. Then, up to shrinking U, there exists a finite cover $\tilde{f}\colon Y \to X$ from another smooth projective curve Y over K such that $f\colon V \to U$ is precisely the pullback of $\tilde{f}^{\mathrm{ad}}\colon Y^{\mathrm{ad}} \to X^{\mathrm{ad}}$ along $U \subset X^{\mathrm{ad}}$.*

Theorem 9.2 relies on the following approximation result (Proposition 9.3). This is closely related to a result of Raynaud [Ray94, Corollaire 3.5.2] (in the setting of Berkovich curves), which he refers to as the *p-adic Runge's Theorem*, as it can be viewed as an analogue of Runge's theorem in the theory of complex Riemann surfaces. Although we only require the case of curves, we present the result in full generality.

Proposition 9.3. Let X be a smooth projective variety over K and let $K(X)$ denote the field of rational functions on X. Let ξ be a point in X^{ad} and let $U \subset X^{\mathrm{ad}}$ be a connected affinoid open neighborhood of ξ. Then, after shrinking U if necessary, $K(X) \cap \Gamma(U, \mathcal{O}_{X^{\mathrm{ad}}})$ is dense in $\Gamma(U, \mathcal{O}_{X^{\mathrm{ad}}})$ with respect to the topology induced from the spectral norm on U. Here the intersection is taken in the fractional field of $\Gamma(U, \mathcal{O}_{X^{\mathrm{ad}}})$.

Proof. The idea is to reduce to the case of the projective spaces $\mathbb{P}^n_K$. Let n be the dimension of X over K. Let us pick a finite surjective cover $f\colon X \to \mathbb{P}^n_K$ such that f is smooth (thus finite étale) on an affinoid neighborhood containing ξ. To see such a cover exists, one can embed $X \subset \mathbb{P}^N_K$ as a closed subscheme where $N > n$, and consider a linear subspace $L \subset \mathbb{P}^N_K$ of complementary dimension (so L is isomorphic to a copy of $\mathbb{P}^{N-n-1}_K$) such that L is disjoint from X. The projection map from $\mathbb{P}^N_K$ that projects "away from L" determines a finite map $f\colon X \to \mathbb{P}^n_K$. We will arrange L such that the singular locus does not contain ξ. To see that this is possible, let ξ be a point of (cohomological) dimension d_0 and let Z be a d_0-dimensional irreducible closed subscheme of X that contains ξ (i.e., such that $\xi \in Z^{\mathrm{ad}}$). We need to arrange f so that it is generically smooth on Z. For this, we pick a closed point $z \in Z$ and arrange the linear subspace L so that it does not intersect the tangent space $T_z X$ of X inside $\mathbb{P}^N_K$. The resulting finite map $f\colon X \to \mathbb{P}^n_K$ is then smooth at the point $z \in Z$, so it is generically smooth on Z, as desired.

Now let U be an affinoid open neighborhood of ξ in X^{ad}. By shrinking U, we may assume that f is finite étale on U. Let $\eta = f(\xi)$ and choose a connected affinoid open neighborhood W of η. We may assume that W is contained the analytification of an affine open subscheme of $\mathbb{P}^n_K$. Let $V \coloneqq f^{-1}(W)$. Then V is affinoid as f is finite. Let us first consider the following commutative diagram

$$\begin{array}{ccc} K(\mathbb{P}^n_K) \cap \Gamma(W, \mathcal{O}_{\mathbb{P}^{n,\mathrm{ad}}_K}) & \xrightarrow{\alpha} & \Gamma(W, \mathcal{O}_{\mathbb{P}^{n,\mathrm{ad}}_K}) \\ \downarrow & & \downarrow \\ K(X) \cap \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}}) & \xrightarrow{\beta} & \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}}). \end{array}$$

We caution the reader that V might be disconnected, in which case $K(X) \cap \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}})$ stands for the subring of analytic functions on V obtained by restricting rational functions on X. We will first prove the density result on V (after shrinking W around η if necessary), and then pass to a smaller neighborhood of ξ contained in the neighborhood U that we start with.

Claim. If the map α has dense image, then β also has dense image.

To prove the claim, let us consider an affine open subscheme $W_0 = \operatorname{Spec} A_0 \subset \mathbb{P}^n_K$ such that $W \subset W_0$, and let $U_0 = W_0 \times_{\mathbb{P}^n_K} X$ be the pullback scheme. We may assume that W_0 is small enough so that $U_0 = \operatorname{Spec} B_0$ for a finite étale algebra B_0 over A_0 (and we identify A_0 as a sub-algebra of B_0). In particular, there exist elements $x_1, \ldots, x_m \in B_0$ which generate B_0 as a module over A_0. Let $B = \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}})$ and $A = \Gamma(W, \mathcal{O}_{\mathbb{P}^{n,\mathrm{ad}}_K})$. Then B is generated by $x_1, \ldots, x_m$ as an A-module. By scaling, we may assume that $x_1, \ldots, x_m$ all have spectral norm $\leqslant 1$ on V. Now, for any $b \in B$, we can write $b = \sum_{i=1}^m a_i x_i$ with $a_i \in A$. By assumption, for any $\epsilon > 0$,

there exists $a_i' \in K(\mathbb{P}_K^n) \cap \Gamma(W, \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad}}})$ such that $|a_i - a_i'|_W < \epsilon$. Take

$$b' = \sum_{i=1}^{m} a_i' x_i \in K(X) \cap \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}}).$$

Then $|b - b'|_V < \epsilon$ (notice that $|a_i - a_i'|_V \leqslant |a_i - a_i'|_W$). This finishes the proof of the claim.

By Lemma 9.4 below, we may further shrink the neighborhood W of η so that α has dense image. Therefore, for this choice of W, the map β in the diagram above also has dense image. Next, let $U_0 \subset U \cap V$ be a rational subset of V containing ξ, given by

$$U_0 = \{x \in V \mid |h_j(x)| \leqslant |h(x)|, 1 \leqslant j \leqslant r\,;\, h(x) \neq 0\};$$

namely,

$$\Gamma(U_0, \mathcal{O}_{X^{\mathrm{ad}}}) = B\langle \frac{h_1, \ldots, h_r}{h} \rangle$$

where $B = \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}})$ as above. Pick $a \in K^\times$ such that $|a| < |h(\xi)|$. Taking the intersection of U_0 with $\{x \in V \mid |h(x)| \geqslant |a|\}$, we may assume that U_0 has the form

$$U_0 = \{x \in V \mid |h_j(x)| \leqslant |h(x)|, 1 \leqslant j \leqslant r, |h(x)| \geqslant |a|\}\,.$$

By the density result for V proved above, there exist elements $q, q_1, \ldots, q_r \in K(X) \cap \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}})$ such that $|h - q|_V < |a|$ and $|h_j - q_j|_V < |a|$ for all $j = 1, \ldots, r$. Then we can identify U_0 as

$$U_0 = \{x \in V \mid |q_j(x)| \leqslant |q(x)|, 1 \leqslant j \leqslant r, |q(x)| \geqslant |a|\}\,;$$

namely,

$$\Gamma(U_0, \mathcal{O}_{X^{\mathrm{ad}}}) = B\langle \frac{q_1, \ldots, q_r, a}{q} \rangle \tag{9.1}$$

Consider the inclusions of rings

$$\Big(K(X) \cap \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}})\Big)[\frac{1}{q}] \;\subset\; K(X) \cap \Gamma(U_0, \mathcal{O}_{X^{\mathrm{ad}}}) \;\subset\; \Gamma(U_0, \mathcal{O}_{X^{\mathrm{ad}}}).$$

By the description in (9.1), we know that $(K(X) \cap \Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}}))[\frac{1}{q}]$ is dense in $\Gamma(U_0, \mathcal{O}_{X^{\mathrm{ad}}})$. Therefore, $K(X) \cap \Gamma(U_0, \mathcal{O}_{X^{\mathrm{ad}}})$ is dense in $\Gamma(U_0, \mathcal{O}_{X^{\mathrm{ad}}})$. Finally, by considering the connected component of U_0 that contains ξ, we arrive at the desired proposition. □

The following lemma is used in the proof of the proposition above.

Lemma 9.4. Let U be a connected affinoid open neighborhood of some point $\xi \in \mathbb{P}_K^{n,\mathrm{ad}}$. Then, up to shrinking U, the intersection $K(\mathbb{P}_K^n) \cap \Gamma(U, \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad}}})$ is dense in $\Gamma(U, \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{ad}}})$ with respect to the spectral norm on U.

Proof. We first place U inside a standard n-dimensional unit polydisk $\mathbb{D}_K^n$ such that the coordinates $t_1, \ldots, t_n \in K(\mathbb{P}_K^n)$ are global rational functions on $\mathbb{P}_K^n$. Without loss of generality, we may assume that U is a Laurent domain of the form

$$U = \{z \in \mathbb{D}_K^n \mid |f_i(z)| \leqslant a_i, |g_j(z)| \geqslant b_j\}$$

for some $f_i, g_j \in \mathcal{O}(\mathbb{D}_K^n)$ and $a_i, b_j \in p^{\mathbb{Q}}$. Now we may replace f_i, g_j by f_i', g_j' which are polynomials in $t_1, \ldots, t_n$ while keeping

$$U = \{z \in \mathbb{D}_K^n \mid |f_i'(z)| \leqslant a_i, |g_j'(z)| \geqslant b_j\}.$$

In particular, these f_i''s and g_j''s are rational functions on $\mathbb{P}_K^n$, in which case, the claim becomes evident. □

To proceed, we also need the following perturbation lemma.

Lemma 9.5 (p-adic perturbation of finite covers of affinoids)**.** Let Y be an affinoid smooth rigid analytic curve over K, viewed as an adic space. Let ξ be a non-type I point on Y and let $V = \mathrm{Spa}(A, A^+)$ be an open neighborhood of ξ that is a rational subset of Y. Let V' be a finite cover of V defined by a monic irreducible polynomial

$$F(Z) = Z^d + a_{d-1}Z^{d-1} + \cdots + a_0$$

where $a_i \in A$, which remains irreducible as a polynomial over $\mathcal{O}_{Y,\xi}$. Then there exists sufficiently small $\epsilon \in \mathbb{R}_{>0}$ such that the following holds: for any polynomial

$$G(Z) = Z^d + b_{d-1}Z^{d-1} + \cdots + b_0$$

with $b_i \in A$ such that $|a_i - b_i|_V < \epsilon$ for the spectral norm $|\cdot|_V$ on V, up to replacing V by a smaller rational subset containing ξ if necessary, $G(Z)$ defines the same finite cover $V' \to V$.

Proof. It suffices to show that, as long as ϵ is sufficiently small, $F(Z)$ and $G(Z)$ define the same extension of the stalk $\mathcal{O}_{Y,\xi}$. First suppose that ξ has type Ⅱ, Ⅲ, or Ⅳ. Then the natural map $\mathcal{O}_{Y,\xi} \to \mathcal{O}^\wedge_{Y,\xi} = \mathscr{H}(\xi)$ induces an equivalence between the categories of finite extensions of $\mathcal{O}_{Y,\xi}$ and that of $\mathscr{H}(\xi)$. This follows, for example, from considering the Berkovich stalk $\mathcal{O}_{Y^{\mathrm{Ber}},\xi}$ and the fact that it is *quasicomplete* in the sense of [Ber93, Theorem 2.3.3, Proposition 2.4.1], and using the fact that we have an identification $\mathcal{O}^\wedge_{Y^{\mathrm{Ber}},\xi} = \mathcal{O}^\wedge_{Y,\xi} = \mathscr{H}(\xi)$ (see §2.4). Therefore, it suffices to show that for sufficiently small ϵ, $F(Z)$ and $G(Z)$ define the same extension of $\mathscr{H}(\xi)$. Note that the condition $|a_i - b_i|_V < \epsilon$ in particular implies that $|a_i - b_i|_\xi < \epsilon$. Thus, the claim follows from Krasner's lemma since $\mathscr{H}(\xi)$ is a complete non-archimedean field.

Now suppose that ξ has type V and let $\xi_{\mathrm{Ⅱ}}$ denote the unique type Ⅱ point that specializes to ξ. Then $\xi_{\mathrm{Ⅱ}} \in V$. By Lemma 2.12, the inclusion $\mathcal{O}_{Y,\xi} \hookrightarrow \mathcal{O}_{Y,\xi_{\mathrm{Ⅱ}}}$ induces an equivalence between finite extensions of $\mathcal{O}_{Y,\xi}$ and that of $\mathcal{O}_{Y,\xi_{\mathrm{Ⅱ}}}$. Given this claim, it suffices to check that $F(Z)$ and $G(Z)$ define the same extension over $\mathcal{O}_{Y,\xi_{\mathrm{Ⅱ}}}$. Notice that the condition $|a_i - b_i|_V < \epsilon$ implies that $|a_i - b_i|_{\xi_{\mathrm{Ⅱ}}} < \epsilon$. So we are reduced to the case of type Ⅱ points, and we are done. □

Proof of Theorem 9.2. By shrinking U (and hence V) and passing to connected components of V, we may assume that

$$\Gamma(V, \mathcal{O}_{X^{\mathrm{ad}}}) = \Gamma(U, \mathcal{O}_{X^{\mathrm{ad}}})[T]/F(T)$$

for a monic polynomial $F(T) = \sum_{i=0}^{l} \beta_i T^i$ with coefficients $\beta_i \in \Gamma(U, \mathcal{O}_{X^{\mathrm{ad}}})$ (in particular, $\beta_l = 1$) such that $F(T)$ is irreducible in $\mathcal{O}_{X^{\mathrm{ad}},\xi}[T]$. By Proposition 9.3, upon further shrinking U if necessary, for any $\epsilon > 0$, there exists $\widetilde{\beta}_i \in K(X) \cap \Gamma(U, \mathcal{O}_{X^{\mathrm{ad}}})$ such that $|\beta_i - \widetilde{\beta}_i|_U < \epsilon$ for all $0 \leqslant i \leqslant l-1$ (set $\widetilde{\beta}_l = 1$). Here $|\cdot|_U$ stands for the spectral norm on U. Applying Lemma 9.5, we deduce that for ϵ sufficiently small, up to further shrinking U, the polynomials $F(T)$ and $\widetilde{F}(T) = \sum_{i=0}^{l} \widetilde{\beta}_i T^i$ define the same finite étale cover $V \to U$. Now, since $\widetilde{F}(T) \in K(X)[T]$, it defines a finite separable extension $L/K(X)$. Let Y be the normalization of X in L, so that we have $V \cong Y^{\mathrm{ad}} \times_{X^{\mathrm{ad}}} U$. □

9.2. **Open neighborhoods of type V points.**

Our proof of Theorem 9.1 will be based on the existence of a nice toric chart on a neighborhood of ξ, which enables us to reduce to the case where X is a suitable open subset of a disk (cf. §9.3). To proceed, let us first introduce some convenient open neighborhoods of type V points.

Let $\delta \in \mathbb{D}_K$ be the type V point corresponding to the open disk $\mathbb{D}_K(0, <1)$ and let $\delta_{\mathrm{Ⅱ}}$ be its associated type Ⅱ point, which is the Gauss point of the boundary circle. Let $X_{\sigma,f}$ be a neighborhood of δ as described in Notation 4.22, where $\sigma \in p^{\mathbb{Q}} \cap (0,1)$. Let us introduce the following notation.

Definition 9.6. Let $\mathcal{Y}$ be a smooth connected projective curve over $\mathcal{O}_K$. Let Y be the adic generic fiber over K and let $\mathcal{Y}_s$ be the special fiber over k. Let $\mathrm{sp}\colon Y \to \mathcal{Y}_s$ be the specialization map. Then the preimage of the generic point of $\mathcal{Y}_s$ under sp consists of a single type Ⅱ point $\xi_{\mathcal{Y}}$ on Y.

(1) For every closed point $z \in \mathcal{Y}_s$, let D_z denote the preimage $\mathrm{sp}^{-1}(z)$. Note that the interior $\mathring{D}_z$ of D_z is isomorphic to an open unit disk[34] and D_z is the disjoint union of $\mathring{D}_z$ with a type V point ξ_z.[35] We identify $\mathring{D}_z$ with an open unit disk under a choice of coordinate T_z.
(2) For $\sigma \in (0,1) \cap p^{\mathbb{Q}}$, let $\mathring{D}_{z;\sigma}$ denote the half-open annulus $\{\sigma \leqslant |T_z| < 1\}$ inside $\mathring{D}_z$, and let

$$D_{z;\sigma} \coloneqq \mathring{D}_{z;\sigma} \cup \{\xi_z\}.$$

[34] In fact, $\mathring{D}_z$ is a *virtual open unit disk* (over K); namely, the disk does not necessarily contain a K-rational point. However, when z is a k-point, the disk $\mathring{D}_z$ indeed contains a K-rational point, and hence it is a "genuine" open unit disk over K.

[35] Note that the closure of the type Ⅱ point $\xi_{\mathcal{Y}} \in Y$ precisely consists of $\xi_{\mathcal{Y}}$ and all of such ξ_z's.

(3) Let $m \geqslant 1$. Let $z, z_1, \ldots, z_m$ be $m+1$ (ordered) distinct closed points in Y_s, and let $\sigma \in (0,1] \cap p^{\mathbb{Q}}$. We define $Y_\sigma(z, z_1, \ldots, z_m)$ to be the open subset of Y given by the conditions $|T_z| \geqslant \sigma$ and $|T_{z_i}| = 1$ for $i = 1, \ldots, m$.[36] When the points $z, z_1, \ldots, z_m$ are understood from the context, we shall simply write Y_σ for $Y_\sigma(z, z_1, \ldots, z_m)$.

Now, let X be as in Theorem 9.1, and let ξ be a K-rational type V point on X. Let ξ_{II} be the associated type II point, which is necessarily also K-rational.

Lemma 9.7. (1) After enlarging K if necessary, there exists an open neighborhood V of ξ such that
- $V = Y_{\sigma_0}(z, z_1, \ldots, z_m)$ for a smooth connected projective curve $\mathcal{Y}$ over $\mathcal{O}_K$, distinct closed points $z, z_1, \ldots, z_m$ on $\mathcal{Y}_s$, and a real number $\sigma_0 \in (0,1) \cap p^{\mathbb{Q}}$ as in Definition 9.6;
- $\xi_{\mathrm{II}} = \xi_{\mathcal{Y}}$;
- $\xi = \xi_z$ (in particular, z is a k-point of $\mathcal{Y}_s$);[37]

Note that the special fiber $\mathcal{Y}_s$ of $\mathcal{Y}$ is uniquely determined by the imperfect residue field of the completed stalk $\mathscr{H}(\xi_{\mathrm{II}})$.

(2) In the situation of (1), the collection $\Psi = \{Y_1(z, z_1, \ldots, z_m, z'_1, \ldots, z'_l)\}$ forms a basis of open neighborhoods of ξ_{II} in X, where $z'_1, \ldots, z'_l$ are closed points of $\mathcal{Y}_s$ distinct from $z, z_1, \ldots, z_m$. Note that $Y_1(z, z_1, \ldots, z_m, z'_1, \ldots, z'_l)$ does not contain the type V point ξ.

(3) In the situation of (1), the collection $\Psi' = \left\{Y_\sigma(z, z_1, \ldots, z_m, z'_1, \ldots, z'_l);\ \sigma \in [\sigma_0, 1) \cap p^{\mathbb{Q}}\right\}$ forms a basis of open neighborhoods of ξ in X, where $z'_1, \ldots, z'_l$ are closed points of $\mathcal{Y}_s$ distinct from $z, z_1, \ldots, z_m$.

Proof. This follows from [Ber07, Proposition 2.2.1]. □

9.3. **The reduction step: a nice toric chart.**

To prove Theorem 9.1, our strategy is to reduce the general situation to the case where $X = X_\sigma$ as in Theorem 8.1.

Let ξ be a K-rational type V point on X and let ξ_{II} be the associated type II point. Up to enlarging K, we may assume that ξ admits an open neighborhood of the form $V \cong Y_{\sigma_0} = Y_{\sigma_0}(z, z_1, \ldots, z_m)$ as in Definition 9.6. By further enlarging K and shrinking V (that is, enlarging the set of closed points $\{z, z_1, \ldots, z_m\}$ and increasing σ_0) if necessary, there exists a toric chart

$$h \colon V = Y_{\sigma_0} \to \mathbb{D}_K$$

such that the following holds.

(1) There exist $c_1, \ldots, c_n \in \overline{k}_K^\times$ and $\sigma \in (0,1) \cap p^{\mathbb{Q}}$, such that h factors as

$$h \colon V \to X_{\sigma,f} \hookrightarrow \mathbb{D}_K \tag{9.2}$$

for some $f = \prod_{i=1}^n (T - [c_i]) \in \mathcal{O}_{K_0}[T]$, where the first map is finite étale;

(2) $\delta_{\mathrm{II}} = h(\xi_{\mathrm{II}})$ is the Gaussian point of $\mathbb{D}_K$, and $\delta = h(\xi)$ is the type V point in $\mathbb{D}_K$ corresponding to the open unit disk centered at 0. Moreover, we have $h^{-1}(\delta_{\mathrm{II}}) = \{\xi_{\mathrm{II}}\}$ and $h^{-1}(\delta) = \{\xi\}$.

(3) $h^{-1}\left(A_K[\sigma, 1)\right) = \mathring{D}_{z;\sigma_0} \cong A_K([\sigma_0, 1); T_z)$;

(4) $h^{-1}\left(X_{1,f}\right) = Y_1 \subset Y_{\sigma_0}$;

We remark that, by enlarging K if necessary, the affinoid neighborhood

$$U \coloneqq Y_1 = Y \backslash (D_z \sqcup \cdots \sqcup D_{z_m})$$

of ξ_{II} can be realized as the adic generic fiber of a smooth affine formal scheme $\mathfrak{U}$, where the latter is the completion of $\mathcal{Y}$ along the locally closed subscheme $\mathcal{Y}_s - z, z_1, \ldots, z_m\}$. Likewise, $X_{1,f}$ is the adic generic fiber of a smooth affine formal scheme $\mathfrak{D}_f$, obtained as the completion of the p-adic formal scheme $\widehat{\mathbb{A}}^1_{\mathcal{O}_K}$ along $\mathbb{A}^1_k - \{0, c_1, \ldots, c_n\}$. By shrinking $X_{1,f}$ (and hence also shrinking V) if necessary, we may further assume that $\mathfrak{D}_f$ admits a formal model $\mathfrak{D}_f^\circ$ over $W(k)$.

Using the toric chart h, we construct another toric chart that behaves well at ξ_{II}.

[36] Roughly speaking, the condition "$|T_z| \geqslant \sigma$" means "removing the open disk of radius σ_i" inside D_{z_i}. (Here, if $\sigma_i = 1$ for some i, the condition $|T_{z_i}| \geqslant 1$ simply means removing the entire D_{z_i}.)

[37] Since z is a k-point of Y_s, the disk $\mathring{D}_z$ is a "genuine" open unit disk over K; namely, $\mathring{D}_z \cong \mathbb{D}_K(0, 1^-)$, with coordinate T_z. Hence, $\mathring{D}_{z;\sigma} \cong A_K([\sigma, 1); T_z)$ is a "genuine" half-open annulus over K with radius $[\sigma, 1)$.

Proposition 9.8. By enlarging K and shrinking both U and V (that is, enlarging the set of closed points $\{z, z_1, \dots, z_m\}$ and increasing σ_0) if necessary, there exists a finite étale map

$$g\colon V = Y_{\sigma_0} \to X_{\sigma', f'}$$

where $f' = \prod_{i=1}^{l}(T - [c_i'])) \in \mathcal{O}_{K_0}[T]$ for some $c_1', \dots, c_l' \in \overline{k}_K^{\times}$, and $\sigma' \in (0, 1] \cap p^{\mathbb{Q}}$, such that

(1) $g(\xi_{\mathrm{II}})$ is the Gaussian point of $\mathbb{D}_K$, and $g(\xi)$ is the type V point in $\mathbb{D}_K$ corresponding to the open unit disk centered at 0. Moreover, we have $g^{-1}(\delta_{\mathrm{II}}) = \{\xi_{\mathrm{II}}\}$ and $g^{-1}(\delta) = \{\xi\}$;
(2) $g^{-1}\big(X_{1,f'}\big) = U = Y_1$;
(3) the restriction $g|_U\colon U \to X_{1,f'}$ is the adic generic fiber of a finite étale map $\widehat{g}\colon \mathfrak{U} \to \mathfrak{D}_{f'}$ between smooth affine formal schemes.

Proof. We will first construct such a map on a neighborhood U of ξ_{II} satisfying condition (3), then show that it extends to a neighborhood V of ξ, after possibly enlarging the base field K.

Step 1 (Factor h on the integral models).

We start with the finite étale map $h\colon V = Y_{\sigma_0} \to X_{\sigma,f}$ from (9.2), which restricts to a finite étale map $h\colon U = Y_1 \to X_{1,f}$. Up to shrinking U and $X_{1,f}$ if necessary, there exists a finite map $\widehat{h}\colon \mathfrak{U} \to \mathfrak{D}_f$ that gives rise to h upon passing to the adic generic fibers (*a priori*, we only have such a map up to taking admissible blowing-ups, but this can be avoided once we are allowed to shrink the source and target; cf. [Ray74]). Passing to special fibers, we obtain

$$\widehat{h}_s\colon \mathfrak{U}_s = \mathcal{Y}_s \backslash \{z, z_1, \dots, z_m\} \to \mathfrak{D}_{f;s} = \mathbb{A}^1_k \backslash \{0, c_1, \dots, c_n\},$$

which extends to a finite map $\mathcal{Y}_s \to \mathbb{P}^1_k$ between smooth projective curves over k. This map factors as

$$\mathcal{Y}_s \xrightarrow{\phi_s} \mathcal{Y}_s^{(N)} \xrightarrow{\psi_s} \mathbb{P}^1_k,$$

where ϕ_s denotes the p^N-th power relative Frobenius map and ψ_s is finite separable. After further shrinking U and $X_{1,f}$, this restricts to a factorization

$$\mathfrak{U}_s \xrightarrow{\phi_s} \mathfrak{U}_s^{(N)} \xrightarrow{\psi_s} \mathfrak{D}_{f;s}$$

of $\widehat{h}_s$, where the second map ψ_s is finite étale.

Step 2 (Lift the finite étale map ψ_s).

After untwisting Frobenius, we have a finite étale map $\psi_s\colon \mathfrak{U}_s \to \mathfrak{D}_{f;s}^{(-N)}$, which we still denote by ψ_s by a slight abuse of the notation. It lifts to a finite étale map

$$\widehat{\psi}\colon \mathfrak{U} \to \mathfrak{D}_f^{(-N)} \coloneqq \mathfrak{D}_f^{\circ} \times_{\operatorname{Spf} W(k), \mathrm{Frob}^{-N}} \operatorname{Spf} \mathcal{O}_K$$

and thus gives rise to a finite étale map $\psi\colon U \to X_{1,f}^{(-N)}$ on the adic generic fibers. Here $X_{1,f}^{(-N)}$ denotes the adic generic fiber of $\mathfrak{D}_f^{(-N)}$; in particular, we have

$$X_{1,f}^{(-N)} \cong X_{1,f'}$$

with $f' \coloneqq \prod_i (T - [c_i'])$, where $c_i' = \mathrm{Frob}^{-N}(c_i)$. Thus, we obtain a finite étale map $\widehat{\psi}\colon \mathfrak{U} \to \mathfrak{D}_{f'} = \mathfrak{D}_f^{(-N)}$ that gives rise to a finite étale map

$$\psi\colon U \to X_{1,f'} \tag{9.3}$$

on the adic generic fibers. This map sends ξ_{II} to the Gaussian point δ_{II} of the disk.

Step 3 (Construction of the desired map).

Next, we show that, up to enlarging K, shrinking U, V, and re-centering the unit disk $\mathbb{D}_K$ (and thus modifying the c_i''s) if necessary, there exists a finite étale map

$$g\colon V \to X_{\sigma', f'}$$

for some $\sigma' \in (0,1) \cap p^{\mathbb{Q}}$, which satisfies the requirement of the proposition. In particular, it sends the type V point ξ to the type V point δ on $\mathbb{D}_K$ corresponding to the open unit disk centered at 0. We will construct this map g from ψ in (9.3).

First, by Theorem 9.2, up to shrinking U and $X_{1,f'}$, there exist a smooth projective curve X' over K (viewed as an adic space) and a finite map $\pi\colon X' \to \mathbb{P}^1_K$, together with an isomorphism

$$i\colon U \xrightarrow{\sim} (\pi)^{-1}(X_{1,f'})$$

satisfying $\pi \circ i = \psi$, where we have identified $X_{1,f'}$ with a subspace of $\mathbb{P}^{1,\mathrm{an}}_K$ via $X_{1,f'} \subset \mathbb{D}_K \subset \mathbb{P}^{1,\mathrm{an}}_K$. In the rest of this proof, for any smooth adic space Z over $\mathrm{Spa}(K, \mathcal{O}_K)$, let $Z_{\mathscr{B}}$ denote the associated Berkovich space over K; in particular, the underlying topological space of $Z_{\mathscr{B}}$ is the maximal Hausdorff quotient of that of Z. By [Ber07, Proposition 2.2.1], up to enlarging K if necessary, there exist a Berkovich open neighborhood $V'_0 \subset X'_{\mathscr{B}}$ of $i(\xi_{\mathrm{II}})$ and a smooth projective curve $\mathcal{Y}'$ over $\mathcal{O}_K$ (with adic generic fiber Y') such that V'_0 can be identified with a Berkovich open neighborhood W_0 of the Shilov point of $Y'_{\mathscr{B}}$.[38] Let

$$\lambda_0\colon W_0 \xrightarrow{\sim} V'_0$$

denote the corresponding isomorphism. Let V' (resp. W) denote the pre-image of V'_0 (resp. W_0) under the natural projection $X' \to X'_{\mathscr{B}}$ (resp. $Y' \to Y'_{\mathscr{B}}$). Then λ_0 induces an isomorphism $\lambda\colon W \xrightarrow{\sim} V'$ of adic spaces. By construction, V' contains every type V specialization of $i(\xi_{\mathrm{II}})$. After shrinking $\mathfrak{U}$ (and hence shrinking U) if necessary, we may assume that $i(U) \subset V'$. By further shrinking, we may assume that the isomorphism $\lambda^{-1} \circ i : U \xrightarrow{\sim} \lambda^{-1} \circ i(U)$ comes from an isomorphism $\iota : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U}'$ on smooth integral models where $\mathfrak{U}' \subset \mathcal{Y}'$ is an open affine formal subscheme, such that the reduction of ι extends to an isomorphism[39]

$$\iota\colon \mathcal{Y}_s \xrightarrow{\sim} \mathcal{Y}'_s.$$

Let $x_0 \in \mathcal{Y}_s$ be a closed point away from $\{z, z_1, \dots, z_m\}$ and let $x'_0 = \iota(x_0) \in \mathcal{Y}'_s$. Let $D_0 \subset Y$ be the preimage of x_0 under the specialization map $Y \to \mathcal{Y}_s$, and similarly define $D'_0 \subset Y'$ using x'_0. Since the formal deformation of smooth affine schemes is unobstructed and in fact trivial, the isomorphism $\iota\colon \mathcal{Y}_s \backslash \{x_0\} \xrightarrow{\sim} \mathcal{Y}'_s \backslash \{x'_0\}$ lifts to an isomorphism

$$j_0\colon Y \backslash D_0 \xrightarrow{\sim} Y' \backslash D'_0$$

of affinoid adic spaces on the generic fiber.

After shrinking V, the composition $j \coloneqq \lambda \circ j_0$ embeds V into Y'. By construction, on U, we have $j = i \circ a$ for a map a induced by an automorphism of the formal scheme $\mathfrak{U}$. In particular, $\pi \circ j|_U$ agrees with $\psi \circ a$, which still has a finite étale integral model. Finally, after enlarging K, shrinking V (i.e., enlarging σ_0), and choosing a different coordinate of $\mathbb{D}_K$ if necessary, we obtain the desired finite étale map

$$g \coloneqq \pi \circ j\colon V \to X_{\sigma',f'}$$

that sends ξ to the type V point δ corresponding to the open unit disk in $\mathbb{D}_K$ centered at 0. This finishes the proof of the proposition. □

9.4. **Proof of p-adic monodromy for type V points.**

Finally, we prove Theorem 9.1.

Proof of Theorem 9.1. Let ξ be the type V point as in the statement of the theorem and let ξ_{II} be the associated type II point. Recall that both ξ and ξ_{II} are K-rational. We want to show that, up to a finite extension of K, there exist an open neighborhood W of ξ and a finite étale cover $W' \to W$ such that the pullback of $\mathbb{L}|_W$ to W' is pointwise semistable. By Proposition 5.2, we may assume that $\mathbb{L}$ is already semistable in an open neighborhood of ξ_{II}.

Consider the map

$$g\colon V \to X_{\sigma',f'}$$

given by Proposition 9.8. In particular, we have $g^{-1}(\delta_{\mathrm{II}}) = \{\xi_{\mathrm{II}}\}$ and $g^{-1}(\delta) = \{\xi\}$ where δ_{II} is the Gaussian point of $\mathbb{D}_K$ and δ is the type V point in $X_{\sigma',f'}$ that corresponds to the open unit disk in $\mathbb{D}_K$ centered at 0. Consider the p-adic local system $g_*\mathbb{L}$ on $X_{\sigma',f'}$. By assumption (3) in Proposition 9.8, g restricts to

[38] Here, the Shilov point of $Y'_{\mathscr{B}}$ is the preimage of the generic point of the special fiber $\mathcal{Y}'_s$ under the specialization map $Y'_{\mathscr{B}} \to \mathcal{Y}'_s$. It is a point of type II.

[39] Notice that the function fields of both $\mathcal{Y}_s$ and $\mathcal{Y}'_s$ are determined by the residue field of the completed stalk $\mathscr{H}(\xi_{\mathrm{II}})$.

$U \to X_{1,f'}$ which admits a finite étale integral model $\mathfrak{U} \to \mathfrak{D}_{f'}$. Since $\mathbb{L}$ is assumed to be semistable in a neighborhood of ξ_{II}, we deduce that $g_*\mathbb{L}$ is semistable in a neighborhood of δ_{II} by applying [GY24, Theorem 1.6].

By Theorem 8.1, up to enlarging K, up to further shrinking $X_{\sigma',f'}$ to a smaller neighborhood of δ, there exists a finite étale cover $\psi\colon X'_{\sigma',f'} \to X_{\sigma',f'}$ such that the local system $(g_*\mathbb{L})|_{X_{\sigma',f'}}$ becomes pointwise semistable after pulling back along the map ψ. Let $W \coloneqq g^{-1}(X_{\sigma',f'})$ and $W' \coloneqq W \times_{X_{\sigma',f'}} X'_{\sigma',f'}$. They fit into a commutative diagram

$$\begin{array}{ccc} W' & \xrightarrow{\psi'} & W \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X'_{\sigma',f'} & \xrightarrow{\psi} & X_{\sigma',f'}. \end{array}$$

Then $(\psi')^*(g^*g_*\mathbb{L}) = (g')^*\psi^*(g_*\mathbb{L})$ is pointwise semistable on W'. Note that for any classical point $x \in W$, the restriction $\mathbb{L}|_x$ is a direct summand of $(g^*g_*\mathbb{L})|_x$ as $\mathrm{Gal}_{\kappa(x)}$-representations. It follows that $(\psi')^*\mathbb{L}$ is also pointwise semistable. This finishes the proof of Theorem 9.1 (that is, the general case of Theorem 12 for type V points). □

Remark 9.9. Combining the results established in §5.1, §5.2, §6, §7, §8, and §9, the proof of Theorem 12 is now complete.

10. Proof of the main theorem

We are ready to prove the main theorem.

Proof of Theorem 1. Applying Theorem 12 and Theorem 9.2 to each (adic) point $\xi \in X_{\mathbb{C}_K}$, we obtain a triple $(L(\xi), U_\xi, f_\xi\colon Y_\xi \to X_{L(\xi)})$ where

- $L(\xi)$ is a finite extension of K;
- $U_\xi \subset X_{L(\xi)}$ is an open neighborhood of the image of ξ in $X_{L(\xi)}$;
- $f_\xi\colon Y_\xi \to X_{L(\xi)}$ is a finite cover between smooth projective curves over $L(\xi)$;

such that the pullback of $\mathbb{L}$ to $Y_\xi \times_{X_{L(\xi)}} U_\xi$ is pointwise semistable. Notice that $\{U_{\xi,\mathbb{C}_K}\}$ covers the adic space $X_{\mathbb{C}_K}$. By compactness, $X_{\mathbb{C}_K}$ is covered by a finite sub-collection $\{U_{\xi,\mathbb{C}_K}\}_{\xi\in\mathscr{C}}$. Pick a finite extension L/K containing $L(\xi)$ for all $\xi \in \mathscr{C}$, so that $\{U_{\xi,L}\}_{\xi\in\mathscr{C}}$ cover X_L.

Now, consider the normalization Y of the fiber product of all $Y_{\xi,L}$'s over X_L. By construction, the pullback of $\mathbb{L}$ to Y is pointwise semistable. By enlarging K, we may further assume that Y has semistable reduction. It then follows from the pointwise criterion (Theorem 13) that the pullback of $\mathbb{L}$ to Y is semistable. □

Finally, we conclude the article with the proof of the $\mathbb{Q}_p$-version of the main result.

Proof of Theorem 7. Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on X as in the statement of the theorem. Let $\xi \in X_{\mathbb{C}_K}$ be an (adic) point on $X_{\mathbb{C}_p}$. By [DJ95, Corollary 4.4], there exists an open neighborhood U of the image of ξ in X, such that the restriction $\mathbb{L}|_U$ of $\mathbb{L}$ to U admits a $\mathbb{Z}_p$-lattice. That is, there exists a $\mathbb{Z}_p$-local system Λ on U such that $\mathbb{L}|_U \cong \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. By Theorem 12 and its proof, we know that, after shrinking U and replacing K by a finite extension, there exists a finite étale cover $f\colon V \to U$ such that $f^*(\mathbb{L}|_U)$ is pointwise semistable. The rest of the argument is identical to that of Theorem 1. □

Appendix A. Notations for period rings

In this appendix, we gather a list of notations of period rings that are commonly used in p-adic Hodge theory. As the table below suggests, for the period rings in the classical setting (cf. §3), our notation is in the same spirit as in the work of Kedlaya–Liu ([KL15, KL19]).

Our choice	[CC98]	[Ber02]	[Col08]	[KL15, KL19]
$\mathbf{A}_K$	—	—	—	—
$\mathbf{A}_K^r$	$\mathbf{A}^\dagger_{r'-,K}, (\mathbf{A}^\dagger_{r'-})^{H_K}$	$\mathbf{A}_K^{\dagger,r'}, \mathbf{A}^{H_K}_{[r';+\infty]}$	$\mathbf{A}_K^{(0,r]}$	—
$\mathbf{A}_K^\dagger$	—	—	—	—
$\mathbf{B}_K$	—	—	—	—
$\mathbf{B}_K^r$	$\mathbf{B}^\dagger_{r'-,K}, (\mathbf{B}^\dagger_{r'-})^{H_K}$	$\mathbf{B}_K^{\dagger,r'}, \mathbf{B}^{H_K}_{[r';+\infty]}$	$\mathbf{B}_K^{(0,r]}$	—
$\mathbf{B}_K^\dagger$	—	—	—	—
$\mathbf{C}_K^{[s,r]}$	n/a	$\mathbf{B}^{H_K}_{[r';s']}$	n/a	—
$\mathbf{C}_K^r$	n/a	$\mathbf{B}^{\dagger,r'}_{\mathrm{rig},K}, \mathbf{B}^{H_K}_{[r';+\infty[}$	$\mathbf{B}_K^{]0,r]}$	—
$\mathbf{C}_K^\dagger$	n/a	$\mathbf{B}^\dagger_{\mathrm{rig},K}$	$\mathbf{B}^\dagger_{\mathrm{rig},K}$	$\mathbf{C}_K$

- In the table above, a dash "—" indicates that the notation there agrees with the notation used in this paper.
- s, r are real numbers with $0 < s \leqslant r$ and $s, r \in p^{\mathbb{Q}}$; while $r' = \frac{1}{r} \cdot \frac{p-1}{p}$ and $s' = \frac{1}{s} \cdot \frac{p-1}{p}$.
- As the table suggests, we make one slight modification to the notation in [KL15, KL19] — we write $\widetilde{\mathbf{C}}_K^\dagger$ and $\mathbf{C}_K^\dagger$ for the rings denoted by $\widetilde{\mathbf{C}}_K$ and $\mathbf{C}_K$ in [KL15, KL19] to emphasize that these rings are obtained by taking the union over $r > 0$, as in Definitions 3.2 and 3.6, corresponding to "approaching the boundary of the annulus".
- There are similar tables for perfect period rings and for period rings without the subscript K, which we omit here.
- For relative period rings, we use $\widetilde{\mathbf{A}}_X^r, \widetilde{\mathbf{A}}_X^\dagger, \widetilde{\mathbf{B}}_X^r, \widetilde{\mathbf{B}}_X^\dagger, \widetilde{\mathbf{C}}_X^{[s,r]}, \widetilde{\mathbf{C}}_X^r, \widetilde{\mathbf{C}}_X^\dagger$ to denote perfect period rings, as opposed to the notation $\widetilde{\mathbf{A}}_\psi^r, \widetilde{\mathbf{A}}_\psi^\dagger, \widetilde{\mathbf{B}}_\psi^r, \widetilde{\mathbf{B}}_\psi^\dagger, \widetilde{\mathbf{C}}_\psi^{[s,r]}, \widetilde{\mathbf{C}}_\psi^r, \widetilde{\mathbf{C}}_\psi$ used in [KL19]. In fact, in both [KL15] and [KL19], Kedlaya and Liu use $\widetilde{\mathbf{A}}_X^r, \widetilde{\mathbf{A}}_X^\dagger, \widetilde{\mathbf{B}}_X^r, \widetilde{\mathbf{B}}_X^\dagger, \widetilde{\mathbf{C}}_X^{[s,r]}, \widetilde{\mathbf{C}}_X^r, \widetilde{\mathbf{C}}_X$ to refer to sheaves of period rings.

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