The Chern-Ricci Flow

Valentino Tosatti

McGill University

Chern: a Great Geometer of the 20th century
A conference for the 110th anniversary
of the birth of Professor Shiing-Shen Chern
Tsinghua University

October 12, 2021

joint work with Ben Weinkove and others; arXiv:2107.12928

Setup

X^n compact complex manifold, $\dim_{\mathbb{C}} X = n$, $g = \sum_{j,k} g_{j\bar{k}}(z) dz_j \otimes d\bar{z}_k$ a Hermitian metric

$$\omega = i \sum_{j,k} g_{j\bar{k}} dz_j \wedge d\bar{z}_k, \quad g \text{ is Kähler if } d\omega = 0, \quad \Leftrightarrow \quad \frac{\partial g_{j\bar{k}}}{\partial z_i} = \frac{\partial g_{i\bar{k}}}{\partial z_j}$$

The first Chern form of a Hermitian metric g is given by $\mathrm{Ric}(\omega) = i \sum_{j,k} R_{j\bar{k}} dz_j \wedge d\bar{z}_k$, where locally $R_{j\bar{k}} = -\frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} \log \det(g_{p\bar{q}})$ is the Chern-Ricci curvature. It is a closed real $(1,1)$-form.

If $\tilde{g}$ is another Hermitian metric then

$$\mathrm{Ric}(\tilde{\omega}) = \mathrm{Ric}(\omega) + i\partial\overline{\partial} \log \frac{\omega^n}{\tilde{\omega}^n}$$

Hence the first Chern class $c_1(X) = [\mathrm{Ric}(\omega)/2\pi] \in H^2(X, \mathbb{R})$ is independent of ω. Similarly for its refined version $c_1^{\mathrm{BC}}(X)$ in the Bott-Chern cohomology $H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ of closed real $(1,1)$-forms modulo $i\partial\overline{\partial}$-exact ones

The Chern-Ricci flow

(X^n, ω_0) compact Kähler manifold. A smooth family of Kähler metric $\omega(t)$ on X, $t \in [0, T)$ solves the Kähler-Ricci flow if

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$$

This is the same as Hamilton's Ricci Flow when the initial metric is Kähler. However, if ω_0 is only Hermitian, its evolution by the Ricci flow is not Hermitian anymore, in general. Thus Ricci flow is not a useful tool to study non-Kähler complex manifolds.

T.-Weinkove in 2011 proposed a way to fix this: consider the same evolution equation above, where $\mathrm{Ric}(\omega(t))$ is the first Chern form of the Hermitian metric $\omega(t)$. We called this flow the *Chern-Ricci Flow*. It turns out that the flow had been studied in a special case and in a different guise by Gill, Ben's PhD student, in 2010.

Short time existence

Theorem (T.-Weinkove 2011)

(X^n, ω_0) compact Hermitian manifold. The Chern-Ricci flow has a unique solution on a (forward) maximal time interval $t \in [0, T)$, $0 < T \leqslant \infty$.

Given $\omega(t)$ Hermitian metrics on X for $t \in [0, T)$ some interval, $\omega(0) = \omega_0$, then they solve CRF $\Leftrightarrow$ there is $\varphi(t) \in C^\infty(X \times [0, T), \mathbb{R})$ such that $\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t)$ and

$$\begin{cases} \dfrac{\partial \varphi(t)}{\partial t} = \log \dfrac{(\omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t))^n}{\omega_0^n} \\ \omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t) > 0 \\ \varphi(0) = 0 \end{cases}$$

holds on $X \times [0, T)$. This is a strictly parabolic scalar complex Monge-Ampère equation so short-time solution follows from standard theory.

Given $\omega(t)$ Hermitian metrics on X for $t \in [0, T)$ some interval, $\omega(0) = \omega_0$, then they solve CRF $\Leftrightarrow$ there is $\varphi(t) \in C^\infty(X \times [0, T), \mathbb{R})$ such that $\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t)$ and

$$\begin{cases} \dfrac{\partial\varphi(t)}{\partial t} = \log \dfrac{(\omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t))^n}{\omega_0^n} \\ \omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t) > 0 \\ \varphi(0) = 0 \end{cases}$$

holds on $X \times [0, T)$.

$\Leftarrow$ compute $\frac{\partial\omega(t)}{\partial t} = -\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\left(\frac{\partial\varphi(t)}{\partial t}\right) = -\mathrm{Ric}(\omega(t))$

$\Rightarrow$ define $\varphi(t) = \int_0^t \log \frac{\omega(s)^n}{\omega_0^n} ds$, satisfies $\frac{\partial\varphi(t)}{\partial t} = \log \frac{\omega(t)^n}{\omega_0^n}, \varphi(0) = 0$

$$\frac{\partial}{\partial t}\left(\omega(t) - (\omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t))\right) = 0, \quad \left(\omega(t) - (\omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t))\right)\Big|_{t=0} = 0$$

so $\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t)$ on $X \times [0, T)$.

Maximal existence time

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)), \quad \omega(0) = \omega_0, \quad \omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t)$$

As long as a solution exists, $\omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\varphi(t) > 0$. Conversely

Theorem (T.-Weinkove 2011)

(X^n, ω_0) compact Hermitian manifold. The maximal time interval where the Chern-Ricci flow starting at ω_0 exists is $[0, T)$ where

$$T = \sup\left\{t > 0 \,|\, \exists\psi \in C^\infty(X) \text{ with } \omega_0 - t\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\psi > 0\right\}.$$

When ω_0 Kähler, this condition is the same as $[\omega_0] - 2\pi t c_1(X)$ being a Kähler class.

Corollary

$T = \infty \Leftrightarrow -c_1^{\mathrm{BC}}(X)$ *nef: for every* $\varepsilon > 0$ *there is* $\psi_\varepsilon \in C^\infty(X)$ *s.t.* $-\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}\psi_\varepsilon \geqslant -\varepsilon\omega_0$

Equivalently, K_X is nef (in the analytic sense)

Explicit solutions

Suppose that (X, ω_0) admits a geometric structure in the sense of Thurston, i.e. that it is locally homogenous. Its universal cover is then $\Gamma\backslash G$, G Lie group with left-invariant complex structure, Γ discrete subgroup, ω_0 induced by a left-invariant Hermitian metric on G.

Lauret-Valencia 2013: in this case, $\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0)$ solves CRF. Explicitly computed by T.-Weinkove 2012 when $n = 2$.

Ex. Hopf surface $X = \mathbb{C}^2\backslash\{0\}/\mathbb{Z}$ with action generated by $(z_1, z_2) \mapsto (2z_1, 2z_2)$. Then $G = S^3 \times \mathbb{R}$ and $\omega_0 = \frac{\delta_{ij}}{|z_1|^2+|z_2|^2}\sqrt{-1}dz_i \wedge dz_j$ evolves by

$$\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) = \frac{1}{|z_1|^2 + |z_2|^2}\left((1-2t)\delta_{ij} + 2t\frac{\overline{z}_i z_j}{|z_1|^2 + |z_2|^2}\right)\sqrt{-1}dz_i \wedge d\overline{z}_j,$$

$0 \leqslant t < \frac{1}{2}$. As $t \to \frac{1}{2}$ these converge smoothly to a degenerate Hermitian form, whose kernel restricted to S^3 is the standard contact distribution. From this, we show that $(X, \omega(t)) \to S^1$ in Gromov-Hausdorff.

Calabi-Yau and canonically polarized

Two important classes of manifolds where $T = \infty$

Theorem (Gill 2010)

X compact complex manifold with $c_1^{\mathrm{BC}}(X) = 0$ (e.g. K_X holomorphically torsion). Then the Chern-Ricci flow starting at any Hermitian metric ω_0 exists for all time and converges smoothly to ω_∞ Hermitian metric with $\mathrm{Ric}(\omega_\infty) = 0$.

These Chern-Ricci flat metrics were first constructed by T.-Weinkove 2009, generalizing the celebrated Yau's Theorem 1976.

Theorem (T.-Weinkove 2011)

X compact Kähler manifold with K_X ample. Then the Chern-Ricci flow $\omega(t)$ starting at any Hermitian metric ω_0 exists for all time and $\omega(t)/t$ converges smoothly exponentially fast to ω_∞ the unique Kähler-Einstein metric with $\mathrm{Ric}(\omega_\infty) = -\omega_\infty$.

Generalizes Cao's result from 1985. Existence of the KE metric was proved by Aubin and Yau in 1976.

Compact complex surfaces

Gauduchon 1977: (X^n, ω) compact Hermitian, then we can conformally rescale $\tilde{\omega} = e^u \omega$ so that $\partial\overline{\partial}(\tilde{\omega}^{n-1}) = 0$ (Gauduchon metric).

When $n = 2$ this means $\partial\overline{\partial}\tilde{\omega} = 0$, which is preserved by the CRF. In this case

$$\mathrm{Vol}(X, \omega(t)) = \frac{1}{2}\int_X \omega(t)^2 = \frac{1}{2}\int_X (\omega_0 - t\mathrm{Ric}(\omega_0))^2$$
$$= \mathrm{Vol}(X, \omega_0) - t\int_X \omega_0 \wedge c_1^{\mathrm{BC}}(X) + 2\pi^2 t^2 (K_X^2)$$

explicitly computable. Similarly, if $C \subset X$ is a complex curve then

$$\mathrm{Vol}(C, \omega(t)) = \int_C \omega(t) = \int_C (\omega_0 - t\mathrm{Ric}(\omega_0)) = \mathrm{Vol}(C, \omega_0) + 2\pi t(K_X \cdot C)$$

Theorem (T.-Weinkove 2011)

(X^2, ω_0) Gauduchon surface. Then $\omega(t)$ exists until either $\mathrm{Vol}(X, \omega(t)) \to 0$ or $\mathrm{Vol}(C, \omega(t)) \to 0$ for some curve $C \subset X$ with $(C^2) < 0$.

Minimal complex surfaces

X a compact complex surface is minimal if it does not contain (-1)-curves $C \subset X, C \cong \mathbb{P}^1, (C^2) = -1$. Every compact complex surface is an iterated blowup of a minimal one (Castelnuovo-Enriques 1901, Grauert 1962).

The Kodaira-Enriques classification of minimal complex surfaces (1968) divides them into classes which are mostly well-understood. The basic invariant is the Kodaira dimension

$$\kappa(X) = \limsup_{m\to\infty} \frac{\log \dim H^0(X, mK_X)}{\log m} \in \{-\infty, 0, 1, 2\}$$

It is invariant under blowups.

Kodaira-Enriques classification

$\kappa(X) = 2$	Minimal surface of general type (projective algebraic)	$T = \infty, \mathrm{Vol}(X, \omega(t)) \sim t^2$
$\kappa(X) = 1$	Minimal properly elliptic surfaces (may be Kähler or non-Kähler)	$T = \infty, \mathrm{Vol}(X, \omega(t)) \sim t$
$\kappa(X) = 0$	Kähler Calabi-Yau (tori, K3, Enriques and bielliptic)	$T = \infty, \mathrm{Vol}(X, \omega(t)) =$const
	Kodaira surfaces (non-Kähler)	
$\kappa(X) = -\infty$	$\mathbb{P}^2$	$T < \infty, \mathrm{Vol}(X, \omega(t)) \to 0$
	ruled (projective algebraic)	
	Class VII ($b_1(X) = 1$)	
	•Hopf surfaces ($b_2(X) = 0$)	$T < \infty, \mathrm{Vol}(X, \omega(t)) \to 0$
	•Inoue surfaces ($b_2(X) = 0$)	$T = \infty, \mathrm{Vol}(X, \omega(t)) \sim t$
	• Kato surfaces, ??? ($b_2(X) > 0$)	$T < \infty, \mathrm{Vol}(X, \omega(t)) \to 0$

What is the behavior of $\omega(t)$ as $t \to T$ in each of these classes?

In the cases when $\mathrm{Vol}(X, \omega(t)) \to \infty$ we consider instead $\omega(t)/t$

$\kappa(X) = 2$

X^2 minimal surface of general type (or more generally X^n compact complex manifold with $-c_1^{\mathrm{BC}}(X)$ nef and $\kappa(X) = n$), ω_0 any Hermitian metric on X and $\omega(t), t \geqslant 0$, CRF

Let $E = \bigcup_{V \subset X, (V \cdot K_X^{\dim V}) = 0} V$, the union of all irreducible closed analytic subvarieties V with $\mathrm{Vol}(V, \omega(t)/t) \to 0$ as $t \to \infty$. Collins-T. 2013: E is a proper closed analytic subvariety of X. When $n = 2$ this is the union of all (-2)-curves in X.

Theorem (Gill 2013)

$\frac{\omega(t)}{t} \to \omega_{\mathrm{KE}}$ in $C^\infty_{\mathrm{loc}}(X \backslash E)$, where ω_{KE} Kähler metric on $X \backslash E$ with $\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}$.

Extends Cao 1985 and Tsuji 1988 from Kähler case.

Conjecture (Known in Kähler case Guo-Song-Weinkove, Tian-Zhang, Wang)

$$\mathrm{diam}\left(X, \frac{\omega(t)}{t}\right) \leqslant C, \quad \left(X, \frac{\omega(t)}{t}\right) \to (Z, d)$$

in Gromov-Hausdorff, where (Z, d) is the metric completion of $(X \backslash E, \omega_{\mathrm{KE}})$.

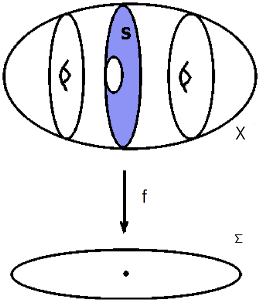
$\kappa(X) = 1$

Kodaira: $f : X^2 \to \Sigma$ minimal properly elliptic surface, in general with singular fibers $S \subset X$.

Theorem (T.-Weinkove-Yang 2013, Angella-T. 2021)

Suppose X is a non-Kähler minimal properly elliptic surface, and ω_0 Gauduchon. Then $\omega(t)/t \to f^*\omega_{\mathrm{KE}}$ uniformly on X (in particular GH), where ω_{KE} orbifold KE metric on Σ, and for any smooth fiber F, $\omega(t)|_F \to \omega_F$ smoothly where ω_F flat metric. If $\omega_0 = \omega_{\mathrm{TV}} + i\partial\overline{\partial}\psi$ then $\omega(t)/t$ converge smoothly and with uniformly bounded (Chern) curvature tensor.

These last items are conjectured to hold for any ω_0 Gauduchon X non-Kähler$\Rightarrow$there is an étale cover of X which is an elliptic bundle. When X is a Kähler minimal properly elliptic surface, much is known when ω_0 is Kähler (Song-Tian,...), but nothing when it is not. In this case we conjecture that analogous results hold on compact sets away from S, where ω_{KE} now satisfies a twisted Kähler-Einstein equation on $\Sigma \backslash f(S)$. The collapsed Gromov-Hausdorff limit should be the metric completion of $(\Sigma \backslash f(S), \omega_{\mathrm{KE}})$.



$\kappa(X) \leqslant 0$ (except Inoue)

•Kodaira: X^2 minimal surface with $\kappa(X) = 0 \Rightarrow mK_X \cong \mathcal{O}_X$ for some $m \geqslant 1$. In particular $c_1^{\mathrm{BC}}(X) = 0$ so Gill's result mentioned earlier shows that $\omega(t) \to \omega_\infty$ smoothly where $\mathrm{Ric}(\omega_\infty) = 0$. Since $\omega_\infty = \omega_0 + i\partial\overline{\partial}\varphi_\infty$, it is Kähler iff ω_0 is.

• On $\mathbb{P}^2$ the flow starting at any ω_0 Hermitian has finite-time volume collapsing with $\omega(t) \leqslant C\omega_0$ for $0 \leqslant t < T$ (T.-Weinkove-Yang 2015). When ω_0 is Kähler, the Gromov-Hausdorff limit is a point (Perelman 2003). We don't know what the limit looks like when ω_0 is non-Kähler. $\mathrm{Vol}(X, \omega(t)) \sim (T - t)$, unlike Kähler case where $\sim (T - t)^2$.

• If X is a minimal ruled surface, we only know that in general we will have finite-time volume collapsing. Even when ω_0 is Kähler the behavior of the flow is not well-understood.

• If X is any Hopf surface ($\tilde{X} \cong \mathbb{C}^2 \backslash \{0\}$) then we have finite-time volume collapsing. For a class of Hopf surfaces and initial metrics, T.-Weinkove 2011 and Edwards 2019 show that $\omega(t) \leqslant C\omega_0$ for $0 \leqslant t < T$. The Gromov-Hausdorff limit is conjectured to be S^1

• If X is a class VII surface with $b_2(X) > 0$ then we have finite-time volume collapsing. These surfaces are not completely classified, and we don't know what the Gromov-Hausdorff limit is.

Inoue surfaces

Inoue surfaces were constructed by Inoue and Bombieri in 1974, as quotients $(\mathbb{C} \times \mathbb{H})/\Gamma$, and come in 3 families, S_M and $S^{\pm}$. Bogomolov, Li-Yau-Zheng, Teleman later proved that every minimal class VII surface with $b_2(X) = 0$ is either Hopf (if it contains a curve) or Inoue (if not).

X: any Inoue surface, then $-c_1^{\mathrm{BC}}(X)$ nef, so $T = \infty$. A multiple of the Poincaré metric on $\mathbb{H}$ descends to a closed nonnegative form ω_∞ on X which represents $-c_1^{\mathrm{BC}}(X)$.

Theorem (Fang-T.-Weinkove-Zheng 2015, Angella-T. 2021)

Let ω_0 be a Gauduchon metric on an Inoue surface X. Then $\omega(t)/t \to \omega_\infty$ uniformly as $t \to \infty$, and $(X, \omega(t)/t) \to S^1$ in Gromov-Hausdorff. If $\omega_0 = \omega_{\mathrm{TV}} + i\partial\overline{\partial}\psi$ then $\omega(t)/t$ converge smoothly and with uniformly bounded (Chern) curvature tensor.

Again these last two items are conjectured to hold for general ω_0 Gauduchon.

Curvature bounds at infinity

Let $(X^n, \omega(t))$ be a solution of the CRF with $T = \infty$. Following Hamilton, we say that the solution is of type III if $\omega(t)/t$ has uniformly bounded Chern curvature tensor as $t \to \infty$, and of type IIb if not.

Theorem (T.-Weinkove 2021)

Let $(X^2, \omega(t))$ be a solution of the CRF with $T = \infty$ and ω_0 Gauduchon. Then

- *X minimal general type: type III $\Leftrightarrow$ K_X ample*
- ** X minimal properly elliptic: type III $\Leftrightarrow$ only singular fibers are type $mI_0, m \geqslant 2$*
- *X Kähler Calabi-Yau: type III $\Leftrightarrow$ X finitely covered by a torus and ω_0 Kähler*
- *X Kodaira surface: always type IIb*
- ** X Inoue surface: always type III*

The items with asterisk are conditional on the earlier conjectures on the boundedness of the Chern curvature tensor for $\omega(t)/t$ on Inoue surfaces and elliptic bundles.

Non-minimal surfaces

If X is a compact complex surface which is not minimal, then X contains some (-1)-curve.

If $\kappa(X) \geqslant 0$ then every CRF solution with ω_0 Gauduchon must have $T < \infty$ and $\mathrm{Vol}(X, \omega(t)) \geqslant c > 0$. If $\kappa(X) = -\infty$ then we could also have $T < \infty$ and $\mathrm{Vol}(X, \omega(t)) \to 0$, which only happens when X is birational to ruled or to a class VII which is not Inoue. Finite time collapsing is poorly understood. On the other hand:

Theorem (T.-Weinkove 2012)

Suppose $T < \infty$ and $\mathrm{Vol}(X, \omega(t)) \geqslant c > 0$. Then the set E of all curves in X whose volume shrinks to zero is a finite disjoint union of (-1)-curves, and $\omega(t) \to \omega_T$ smoothly on $X \backslash E$ where ω_T Gauduchon metric.

Let $\pi : X \to Y$ be the blowdown of E. If one assumes that $d\omega_0 = \pi^* \beta$ for some 3-form β, then $(X, \omega(t)) \to (Y, d_T)$ in Gromov-Hausdorff (T.-Weinkove 2012) and the flow can be restarted on Y and is continuous in Gromov-Hausdorff (Nie, Tô 2017). It is still not known whether (Y, d_T) is the metric completion of $(X \backslash E, \omega_T)$, or whether these results hold without the assumption on $d\omega_0$. Kähler case: Song-Weinkove 2010

Finite-time singularities

Let X^n be a compact complex manifold, ω_0 Hermitian such that the evolution $\omega(t)$ exists on $[0, T)$, $T < \infty$ and $\mathrm{Vol}(X, \omega(t)) \geqslant c > 0$.

$$E = \bigcup_{V \subset X, \mathrm{Vol}(V, \omega(t)) \to 0} V$$

the union of all irreducible closed analytic subvarieties whose volume shrinks as $t \to T$.

$$\Sigma = X \backslash \left\{ \begin{array}{l} x \in X \mid \exists U \ni x \text{ open}, \exists C > 0, \\ \text{ s.t. } |\mathrm{Rm}(\omega(t))|_{g(t)} \leqslant C \text{ on } U \times [0, T) \end{array} \right\}$$

Gill-Smith 2013: $\omega(t)$ converge locally smoothly away from Σ. This easily implies that $E \subset \Sigma$.

Conjecture (T.-Weinkove 2021; Feldman-Ilmanen-Knopf 2003 in Kähler case)

We have $E = \Sigma$, which is a closed proper analytic subvariety of X.

Collins-T. 2013: true when ω_0 Kähler

Gill-Smith 2013: true when $n = 2$ and ω_0 Gauduchon.

Two extensions

•Chern-Ricci flow on non-compact Hermitian manifolds considered by Lee-Tam 2017 who used it to give another proof of a theorem of Liu: a complete Kähler manifold with maximal volume growth and nonnegative bisectional curvature is biholomorphic to $\mathbb{C}^n$. In their proof, they wish to run the Kähler-Ricci flow, but the curvature may be unbounded. By a conformal rescaling, which destroys the Kähler property, they obtain Hermitian metrics with bounded curvature, and then run the CRF. The rescaling is trivial on a compact set, so there CRF=KRF.

•Non-integrable almost-complex structures. Here (X^{2n}, J) is a compact almost-complex manifold and ω_0 a J-Hermitian metric. Chu-T.-Weinkove 2016 proposed the extension of the Chern-Ricci flow

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}^{(1,1)}(\omega(t)) \\ \omega(0) = \omega_0 \end{cases}$$

where Ric is still the first Chern form (closed real 2-form representing $c_1(X, J)$). We still have the same characterization of the maximal existence time (Zheng 2017) and convergence when there is some ω with $\mathrm{Ric}^{(1,1)}(\omega)$ zero (Chu 2016) or negative (Zheng 2017).

Thank You !