Combinatorics Lecture 11, 2022/6/28

§ 1.

Markov's inequality. Let $X \geq 0$ be random variable and $t > 0$. Then $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$.

Pf:
$$\mathbb{E}[X] = \sum_{\omega} X(\omega) \cdot P(\{\omega\})$$
$$\geq \sum_{\omega : X(\omega) \geq t} t \cdot P(\{\omega\})$$
$$= t \cdot P(X \geq t). \qquad \boxtimes$$

Corollary. Let $X_n \geq 0$ be an integer-value random variable. If $\mathbb{E}[X_n] \to 0$ as $n \to +\infty$, then $\mathbb{P}(X_n = 0) \to 1$ as $n \to +\infty$

Lemma 1. Let $p = p(n)$ be a function of n and let G be $G(n, p)$. Then

$$\mathbb{P}\left(\alpha(G) \le \left\lceil \frac{2\ln n}{p} \right\rceil \right) \to 1 \quad \text{as } n \to +\infty.$$

pf. Let $t = \left\lceil \frac{2\ln n}{p} \right\rceil$. Let X_n be the number of independent set of size $t+1$ in $G = G(n,p)$. Then we have

$$\mathbb{E}[X_n] = \sum_{S \in \binom{[n]}{t+1}} \mathbb{E}\left[\mathbb{1}_{\{S \text{ is independent}\}} \right]$$

$$= \binom{n}{t+1} \cdot (1-p)^{\binom{t+1}{2}} \le \frac{n^{t+1}}{(t+1)!} e^{-p\binom{t+1}{2}}$$

$$= \frac{1}{(t+1)!} \left(n e^{-p\frac{t}{2}} \right)^{t+1}$$

$$\overset{t = \left\lceil \frac{2\ln n}{p} \right\rceil}{\le} \frac{1}{(t+1)!} \to 0 \quad (n \to +\infty)$$

$$\Downarrow$$

$$\left(t \ge 2\ln n \to +\infty \right)$$

∎

Def. The chromatic number of a graph G, denoted by $\chi(G)$, is the least integer k such that $V(G)$ can be partitioned into k independent set.

Fact 1. $\forall G$, $\chi(G)\cdot\alpha(G) \geq |V(G)|$.

$$\boxed{\text{Thm 1 (Erdős) For any fixed } k \in \mathbb{N}^+, \text{ there exists a graph } G \text{ with } \chi(G) \geq k \text{ and } g(G) \geq k.}$$

Here, $g(G)$ denotes the girth of G, i.e. the length of the shortest cycle in G.

Pf. Consider a random graph $G = G(n, p)$, where p will be determined later and $n \gg k$.

Let $t = \left\lceil \frac{2 \ln n}{p} \right\rceil$. By Lemma 1,

$$\boxed{\mathbb{P}(\alpha(G) \leq t) \to 1 \quad \text{as } n \to +\infty.}$$

Let Y_n be the total number of cycles of length less than k in G.

Then $\mathbb{E}[Y_n] = \sum_{i=3}^{k-1} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i$

$$\leq \sum_{i=3}^{k-1} (np)^i \leq \frac{(np)^k - 1}{np - 1}$$

By Markov's inequality

$$\mathbb{P}\left(Y_n > \frac{n}{2}\right) \leq \frac{\mathbb{E}[Y_n]}{n/2} \leq \frac{2\left((np)^k - 1\right)}{n(np-1)}.$$

Let $p = n^{-\frac{k-1}{k}}$. So $np = n^{\frac{1}{k}}$

$$\Rightarrow \boxed{\mathbb{P}\left(Y_n > \frac{n}{2}\right) \leq \frac{2(n-1)}{n(n^{1/k}-1)} \to 0 \quad \text{as } n \to +\infty}$$

Then when n is sufficiently large, there exist a graph G with $\alpha(G) \leq \lceil \frac{2\ln n}{p} \rceil \leq 3 n^{\frac{k-1}{k}} \ln n$

and $Y_n \leq \frac{n}{2}$.

By deleting one vertex from each cycle of length at most $k-1$, we can find an induced ~~subgraph~~ subgraph G^* of G, which has at least $\frac{n}{2}$ vertices and has no cycle of length $\leq k-1$.

Then $g(G^*) \geq k$ and

$$\alpha(G^*) \leq \alpha(G) \leq 3n^{\frac{k-1}{k}} \ln n.$$

By Fact 1 $\Longrightarrow$

$$\chi(G^*) \geq \frac{|V(G^*)|}{\alpha(G^*)} \geq \frac{n/2}{3n^{\frac{k-1}{k}} \ln n} = \frac{n^{\frac{1}{k}}}{6 \ln n}$$

$$\geq k \quad (\text{as } n \gg k)$$

$\boxtimes$

Def. An k-uniform hypergraph $G = (V, E)$ (k-graph, for short) if $E \subseteq \binom{V}{k}$.

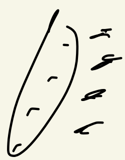
Def We say a k-graph G is r-colorable if its vertices can be colored by r colors so that no edge is monochromatic.

Def. Let $m(k)$ be the minimum number of edges in a k-graph which is not 2-colorable.

Thm 2 (Erdős) $m(k) \geq 2^{k-1}$, i.e. any k-graph with fewer than 2^{k-1} edges is 2-colorable.

pf. Consider a random 2-coloring on vertices.

The probability that an edge is monochromatic

k  is 2^{1-k}.

By union bound, $\mathbb{P}(\exists \text{ a monochromatic edge})$

$$\leq \sum_{e} 2^{1-k} < 1. \quad \blacksquare$$

---

§2. Lovász Local Lemma

Consider "bad events" $A_1, \ldots, A_n$.

We want to avoid all, i.e. $\mathbb{P}(\bar{A}_1 \cap \cdots \cap \bar{A}_n) > 0$.

① If all $\mathbb{P}(A_i)$ are small, say $\sum_i \mathbb{P}(A_i) < 1$, then we can avoid all.

② If they are all independent, then $\mathbb{P}(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n) = (1-\mathbb{P}(A_1)) \cdots (1-\mathbb{P}(A_n))$

$$> 0 \quad (\text{if } \mathbb{P}(A_i) < 1)$$

The LLL deals with the case when each bad event is independent with most other events, but possible dependent with a small number of events.

Def. An event A_0 is <u>independent</u> from $\{A_1, \dots, A_m\}$ if $\mathbb{P}(A_0 \cap B_1 \cap \cdots \cap B_m) = \mathbb{P}(A_0)\,\mathbb{P}(B_1 \cap \cdots \cap B_m)$, where each $B_i \in \{A_i, \bar{A}_i\}$.

<u>Lovász Local Lemma (symmetric form)</u> Let $A_1, A_2, \dots, A_n$ be events with $\mathbb{P}(A_i) \le p \;\; \forall i$.

Suppose that each A_i is independent from a set of all other A_j except for at most d of them. If $\boxed{e p (d+1) \leq 1}$, then none of the events A_i occur with a positive probability;

Lovász Local Lemma (The general form)

Let $A_1, \dots, A_n$ be events. For $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent from $\{A_j : j \notin N(i) \cup \{i\}\}$. If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\boxed{\mathbb{P}(A_i) \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \text{for } \forall i \in [n],}$$

then none of the events A_i occur with probability at least $\prod_{i=1}^{m} (1 - x_i)$.

Pf (The general form $\Rightarrow$ Symmetric form)

Suppose we have $\boxed{e\, p\, (d+1) \le 1}$.

Let $x_i = \frac{1}{d+1} < 1$ for all $i \in [n]$.

Then
$$x_i \prod_{j \in N(i)} (1 - x_j) = \frac{1}{d+1} \prod_{j \in N(i)} \left(1 - \frac{1}{d+1}\right)$$

$$\geq \frac{1}{d+1} \left(1 - \frac{1}{d+1}\right)^d > \frac{1}{e(d+1)} \geq p \geq p(A_i)$$

So the conditions of the general form are satisfies $\Rightarrow$ $\mathbb{P}(\overline{A_1} \cap \cdots \cap \overline{A_n}) > 0$ ☒

Pf of the general form. We will prove that

for any $i \notin S \subseteq [n]$,

$$\boxed{\mathbb{P}\left(A_i \,\middle|\, \bigcap_{j \in S} \overline{A_j}\right) \le x_i . \quad (*)}$$

$\triangle$

Once (*) is proved, we have

$$\mathbb{P}(\bar{A}_1 \cap \bar{A}_2 \cdots \cap \bar{A}_n) = \mathbb{P}(\bar{A}_1) \cdot \mathbb{P}(\bar{A}_2 \mid \bar{A}_1) \cdot \mathbb{P}(\bar{A}_3 \mid \bar{A}_1 \cap \bar{A}_2) \cdots \mathbb{P}(\bar{A}_n \mid \bar{A}_1 \cap \cdots \cap \bar{A}_{n-1})$$

$$\geq (1-x_1)(1-x_2)\cdots(1-x_n)$$

Now we prove (*) by induction on $|S|$.

Base case: when $|S|=0$. It is trivial to get $\mathbb{P}(A_i) \leq x_i$

Consider $i \notin S$. Let $S_1 = S \cap N(i)$
$S_2 = S \setminus N(i)$.

We have

$$\mathbb{P}\left(A_i \,\middle|\, \bigcap_{j \in S} \bar{A}_j\right) = \frac{\mathbb{P}\left(A_i \cap \bigcap_{j \in S_1} \bar{A}_j \,\middle|\, \bigcap_{j \in S_2} \bar{A}_j\right)}{\mathbb{P}\left(\bigcap_{j \in S_1} \bar{A}_j \,\middle|\, \bigcap_{j \in S_2} \bar{A}_j\right)}.$$

The numerator of the above formular ①

$$\le \mathbb{P}\left(A_i \,\middle|\, \bigcap_{j \in S_2} \overline{A_j}\right) = \mathbb{P}(A_i)$$

$$\le x_i \prod_{j \in N(i)} (1 - x_j) \qquad ②$$

,

Let $S_1 = \{j_1, \dots, j_r\} \subseteq N(i)$.

Then the denominator of ①   Let $B = \bigcap_{j \in S_2} \overline{A_j}$.

$$= \mathbb{P}(\overline{A_{j_1}} \mid B) \cdot \mathbb{P}(\overline{A_{j_2}} \mid \overline{A_{j_1}} \cap B) \cdots$$

$$\mathbb{P}(\overline{A_{j_r}} \mid \overline{A_{j_1}} \cap \cdots \cap \overline{A_{j_{r-1}}} \cap B)$$

by induction

$$\ge (1 - x_{j_1})(1 - x_{j_2}) \cdots (1 - x_{j_r})$$

$$\ge \prod_{j \in N(i)} (1 - x_j) \qquad ③$$

By ①, ②, ③

$$\mathbb{P}\left(A_i \,\middle|\, \bigcap_{j\in S} \overline{A_j}\right) \leq \frac{x_i \prod_{j\in N(i)} (1-x_j)}{\prod_{j\in N(i)} (1-x_j)} = x_i$$

$\boxed{\#2}$

this prove (*)

Thm 3 (Spencer, 1977) If $\boxed{e\left(\binom{k}{2}\binom{n}{k-2}+1\right)2^{1-\binom{k}{2}}} < 1$,

then $R(k,k) > n$.

Pf: Random 2-edge-coloring on K_n.

For each $R \in \binom{[n]}{k}$, Let E_R be the event that R induces a monochromatic K_k.

$$\Rightarrow \mathbb{P}(E_R) = 2^{1-\binom{k}{2}} \triangleq p.$$

For each R, it is independent of all $\bar{E}_S$

if $|S\cap R| \leq 1$.

$\Rightarrow$ There are at most $\binom{k}{2}\binom{n}{k-2}$ $S \in \binom{[n]}{k}$ satisfying $|S \cap R| \geq 2$.

$\Rightarrow$ $d = \binom{k}{2}\binom{n}{k-2}$.

By LLL (symmetric form), $e\,p(d+1) < 1$

$\Rightarrow$ $\mathbb{P}\left(\bigcap_{\forall R} \overline{E_R}\right) > 0$. $\Rightarrow$ $R(k,k) > n$ $\boxed{\Xi}$

Corollary. $R(k,k) > \left(\frac{\sqrt{2}}{e} + o(1)\right) k\, 2^{k/2}$.

This still is the best lower bound for $R(k,k)$!

Thm 9 A k-graph is 2-colorable if every edge intersects at most $d = e^{-1}2^{k-1} - 1$ other edges.

Pf: For each edge f, let A_f be the event that f is monochromatic (for random 2-coloring on vertices).

$\Rightarrow$ $\mathbb{P}(A_f) = 2^{1-k}$.

Each A_f is independent from all $A_{f'}$

where $f' \cap f = \phi$.

By condition, at most $d = e^{-1} 2^{k-1} - 1$ edges

intersecting with f, and $e\,p(d+1) = e\,2^{1-k}(e^{-1} 2^{k-1})$
≤ 1.

By LLL, none of the events A_f occur

with positive probability.

$\Rightarrow$ $\exists$ 2-coloring on vertices such that

No edge is monochromatic

$\Rightarrow$ this k-graph is 2-colorable. $\boxtimes$