Entanglement Bootstrap lectures in Beijing 2025

December 9, 2025

Introduction

This lecture series provides an introduction to the Entanglement Bootstrap program for quantum many-body systems. These lectures will be given by Bowen Shi, from UIUC.

- 6 lectures on Dec. 15, 19, 20, 22, 23, 24
- each lecture is 90 minutes plus questions (room reserved for 2 hours)
- location: Shuangqing complex
  12.15 Monday 10:00-12:00 C548;
  12.19 Friday 13:00-15:00 C546;
  12.20 Saturday 10:00-12:00 C654;
  12.22-24 Monday- Wednesday 10:00-12:00 C548.
  Online : There will be online broadcast – we will announce tencent meeting numbers one day before each lecture.
- Language of the lecture: English

There will be tutorial and discussion sessions led by Yizhou Ma from the Chinese University of Hong Kong. Topics include (i) simple illustrative proofs in entanglement bootstrap, (ii) categorical descriptions of topologically ordered systems in 2+1D and higher dimensions, and (iii) exactly solvable wave functions and analytical computations of information convex sets.

Lecture 1: Entanglement Bootstrap for Gapped Systems (I)

Overview. This lecture introduces the motivation and framework of the entanglement bootstrap program. We review the physical context of gapped quantum phases, the notion of emergence in interacting many-body systems, and the philosophy of extracting universal data directly from a single ground state wave function. We then formulate the basic axioms and the reference (vacuum) state, and highlight several quantities that can already be computed at this stage.

Suggested reading. "Fusion rules from entanglement," arXiv 1906.09376 [1].

Things to cover:

- **Physical motivation.** Quantum phases of matter and emergent phenomena. Universal properties of gapped phases, with emphasis on $(2+1)$D topological order, anyons, and quantized chiral response.

- **Emergence and the classification problem.** Why emergence in interacting systems is difficult. The conjectured classification of $(2+1)$D topological orders: unitary modular tensor category plus the chiral central charge. The idea of coarse-graining and the philosophy of "starting from the middle length scale."

- **Quantum-information tools.** Density matrices, von Neumann entropy, multipartite systems, and strong subadditivity. Quantum Markov states and their key structural properties.

- **Information convex set and anyons (based on intuition):** From a single ground state, we can extract universal information finner than the total quantum dimension (from the topological entanglement entropy γ [2, 3]). To illustrate this, we introduce the information convex set [4, 5] and provide its definition. We focus on the annulus and its detection of anyon superselection sectors until; more in tutorial 1 and the next lecture.

- **Entanglement-bootstrap axioms and setup.**

  1. Axioms **A0** and **A1**: a concrete formulation of the entanglement area law.
  2. The reference (vacuum) state σ

- **Implications of the axioms.** Quantities that can be extracted directly from a single wave function, such as the topological entanglement entropy γ [2, 3] and the modular commutator J [6]. Examples in $(2+1)$D, together with theorems or conjectures regarding their quantization.

Tutorial 1: Make sense of the information convex set

In this lecture, we would try to make sense of the definition of information convex sets. First, we would review the TQO conditions introduced in 1001.0344 and explain the connection. We would then compute information convex set of the toric code in detail and outline how to compute the information convex set of qunatum doubles. Finally, we will prove that there is only a unique element in the information convex set of a disk and a sphere.

Lecture 2: Entanglement Bootstrap for Gapped Systems (II)

Overview. This lecture develops the logical consequences of the axioms of entanglement bootstrap introduced in Lecture 1, with emphasis on the information convex set (ICS), the merging theorem, and the isomorphism theorem. We examine how entanglement constraints allows us to define anyon superselection sectors, anti-sectors and fusion rules. We explain a bootstrap question that leads to the quantization of the quantum dimensions of anyons from fusion rules. Suggested reading.

- "Fusion rules from entanglement," arXiv 1906.09376 [1].

- Information convex set in solvable models [4]

  - Other physical contexts: gapped domain walls [7], 3+1D topological order [8] and topological mixed states [9].

- **Conceptual highlights.** "Local vacuum is not vacuum": the role of ICS in capturing nonlocal order. Consequences of A1, including the topological meaning of deformation.

- **Information convex set (ICS).** Equivalent definitions of ICS; the structure of toy examples such as locally indistinguishable states and Bell pairs.

- **Merging and isomorphism theorems.** Statement of the merging theorem; why the isomorphism theorem follows as a corollary. First look at immersed regions and the motivating question: *what are the most general regions?*

- **Extreme points and factorization.** Factorization properties of extreme points and their significance for the global structure of the ICS.

- **Structure theorems of ICS**

  - The simplex theorem of annulus and anyon types
  - Fusion spaces of on the two-hole disk.

- **Quantum dimensions from bootstrap equations.** Definition of quantum dimensions in terms of entropy differences $2f(a) = S(\rho^a) - S(\rho^1)$ and the bootstrap equation

$$e^{f(a)}e^{f(b)} = \sum_c N_{ab}^c \, e^{f(c)},$$

  demonstrating quantization of quantum dimensions.

- **Consequences in broader physical contexts.** Information-convex structures (ICS) on higher-dimensional manifolds, sectorizable regions, and gapped domain walls. The goal is not only to illustrate the richness of the resulting physical structures, but also to emphasize that each conclusion is tied to its assumptions: once an assumption is relaxed, the corresponding local conclusions can change as well.

Tutorial 2: Why entanglement bootstrap: A categorical perspective

In this lecture, we would try to answer why entanglement bootstrap (EB) works from a categorical perspective. After a quick recap of unitary modular tensor category (UMTC), we would explain the idea of remote detectability. We would further illustrate the connection between remote detectability and entanglement bootstrap by delving into (3+1)D scenarios. Braiding non-degeneracy in 3+1D is more subtle. We discuss if 3+1D EB can detect the Cheshire charge in 3+1D toric code.

Lecture 3: Entanglement Bootstrap for Gapped Systems (III)

Overview. This lecture explores how the entanglement bootstrap of (2+1)D gapped systems extends beyond fusion rules to incorporate braiding data and the chiral central charge. The guiding theme is that "everything comes from a ball". simply connected regions. Suggested reading:

- About braiding: non-degeneracy [10, 11]; immersed figure-8 and topological spin [12].
- Chiral central charge: modular commutator [6, 13]; a related no-go theorem [14]; instantaneous modular-flow generalization of the modular commutator [15].

Topics

- **Conceptual highlights.** Why we believe "everything is from a ball": how immersion ($X \looparrowright$ ball) provides a bridge between local entanglement constraints and global topological features.
- **Immersion.** Topological meaning of immersions and regular homotopy. Immersion of punctured surfaces into a ball and why this plays a central role in reconstructing topological information.
- **States on immersed regions.** Information convex sets on immersed regions. Why braiding non-degeneracy follows from the entanglement structure. The immersed figure-8 annulus and the strong-isomorphism conjecture.
- **Smooth deformations.** Introduction of the "space of immersed regions"—a topological space capturing allowed deformations of regions consistent with the bootstrap axioms.
- **Chiral central charge.** Extraction of the chiral central charge from a *single* wave function. Relation to modular flows and modular commutators.
- **Remarks.** Failure of braiding non-degeneracy in mixed topological states that satisfy volume-law versions of the entanglement bootstrap axioms.

Tutorial 3: Domain walls, entanglement bootstrap and category theory

In this tutorial, we would approach domain walls from both the entanglement bootstrap perspective and categorical perspective. We would compute simple examples in the toric code.

Lecture 4: Entanglement Bootstrap for Gapped Systems (IV)

We discuss deeper aspects of why the entanglement bootstrap provides a concrete approach to classifying gapped phases of matter. We contrast this viewpoint with the modern classification program based on finite-depth local unitaries (FDLU). We also explain the emergence of deformable string operators that create anyons, and show why the required circuit depth depends on whether the excitations are Abelian or non-Abelian.

Tutorial 4: Symmetric entanglement bootstrap

In this tutorial, we will introduce the ongoing development for symmetric entanglement bootstrap and its relationship to different descriptions of SPT/SET phases.

Lecture 5: Entanglement Bootstrap for Gapless Fixed Points

Overview. This lecture extends the entanglement bootstrap philosophy to gapless phases, focusing 1+1D conformal field theory (CFT) and the chiral edge of 2+1D topological order.

Things to cover:

- **Locality of modular Hamiltonians.** Modular Hamiltonians are local not only for gapped systems but also for $(1+1)$D CFTs.
- **Vector, scalar, and operator equations.** Distinction between vector equations, scalar equations, and operator-level constraints. The vector fixed-point equation for 1+1D CFTs [16]. Operator strong subadditivity inequality in [17].
- **Renormalization-group (RG) flow and the c-function.** The idea of c-function and constraints on monotonicity under RG. Also explained is the idea that entanglement quantity could be a c-function.
- **Stationarity condition for gapless edges.** The *stationarity condition* for gapless (and chiral) edges introduced in [18]. Its role as an axiom in the gapless version of entanglement bootstrap. Equivalence between the stationarity condition and the vector fixed-point equation. (This implies that the stationarity condition holds on 1+1D CFT fixed points.) Verification that the stationarity condition is satisfied on irregular and disordered chiral edges.
- **Consequence of gappless EB.** Emergence of conformal cross-ratio and the unformity of central charge are followed from the stationarity condition.
- **Open problems.**

Tutorial 5: Categorical description for gapless systems and a simple example

In this tutorial, we will explain the categorical description for gapless systems [19–21], and illustrate this in the example of the critical Ising chain. We will discuss difficulties of defining information convex set in this scenario.

Lecture 6: Open Questions in the Entanglement Bootstrap

Overview. We present a selection of open problems in the entanglement bootstrap across a variety of physical settings. For each question we discuss why it is physically well motivated and/or mathematically compelling.

References

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