Long range order for random field Ising and Potts models

Zijie Zhuang (Upenn)

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joint work with Jian Ding

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Random field Ising model on $\mathbb{Z}^d$ $(d \geq 2)$:

- Λ_N : $[-N, N]^d$ box with side length $2N$
- h_u : i.i.d. $\mathcal{N}(0, \varepsilon^2)$ on each vertex u

  ε: magnitude of random field
- configuration $\sigma \in \{-1, 1\}^{\Lambda_N}$

- Hamiltonian with plus (resp. minus) boundary condition:

$$H^{\pm}(\sigma,\Lambda_N) = -\Big(\sum_{\substack{u\sim v \\ u,v\in\Lambda_N}} \sigma_u\sigma_v \pm \sum_{\substack{u\sim v \\ u\in\Lambda_N, v\notin\Lambda_N}} \sigma_u + \sum_{u\in\Lambda_N} h_u\sigma_u\Big)$$

- Ising measure for temperature $T\geq 0$

$$\mu^{\pm}_{h,\Lambda_N}(\sigma) \propto \exp\Big(-\frac{1}{T}\cdot H^{\pm}(\sigma,\Lambda_N)\Big)$$

Observation: $\mu^{\pm}_{h,\Lambda_N}$ favours the configuration with lower Hamiltonian

i.e. with more agreeing neighbouring pairs

Question: does long range order exist for different $T, \varepsilon \geq 0$?

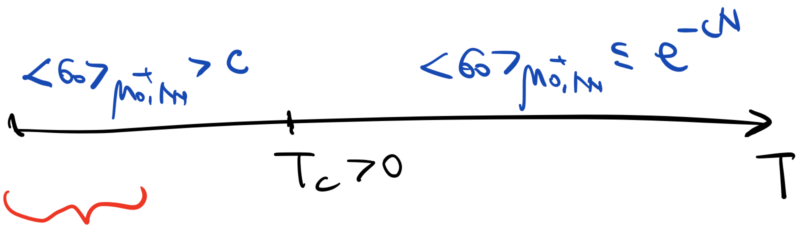
- Boundary influence: $\mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}}$
- We will study how $\mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}}$ behaves with N ?

  for different T and ε

When $\varepsilon = 0$, random field Ising model = Ising model

$$\mathbb{E} \langle \sigma_0 \rangle_{\mu^+_{h,N}} = \langle \sigma_0 \rangle_{\mu^+_{0,N}}$$

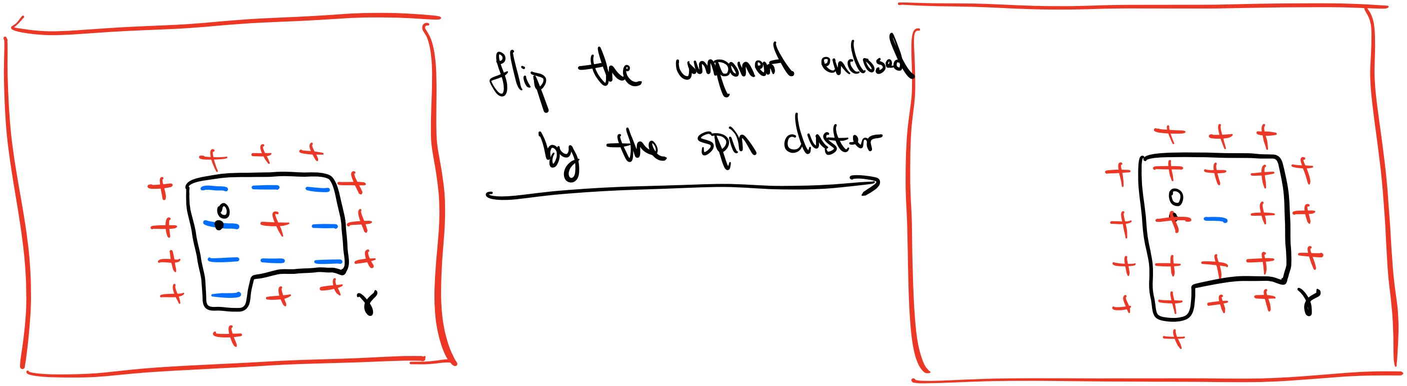
Classic result: $(d \geq 2)$



Peierls argument

What's Peierls argument?

red = + blue = −



- Hamiltonian decreases by $2|\gamma|$

  $\Rightarrow P[\text{ left }] \le \exp(-2|\gamma|/T)$

- $\# \{\gamma : |\gamma| = n\} = \exp(O(n))$

- summing over all γ $\Rightarrow \langle \sigma_0 = -1 \rangle_{\mu^+_{0,N}} \le \exp(-c/T)$

  i.e. $\langle \sigma_0 \rangle_{\mu^+_{0,N}} \ge 1 - \exp(-c/T)$

Random field Ising model overview:

①. ε is large $\Rightarrow$ $\mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}} \leq e^{-cN}$

$$H^+(\sigma, \Lambda_N) = -\Big(\sum_{u \sim v} \sigma_u \sigma_v + \sum_{\substack{u \sim v \\ u \in \Lambda_N, v \notin \Lambda_N}} \sigma_u + \sum_{u \in \Lambda_N} h_u \sigma_u \Big).$$

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dominate the first two terms!

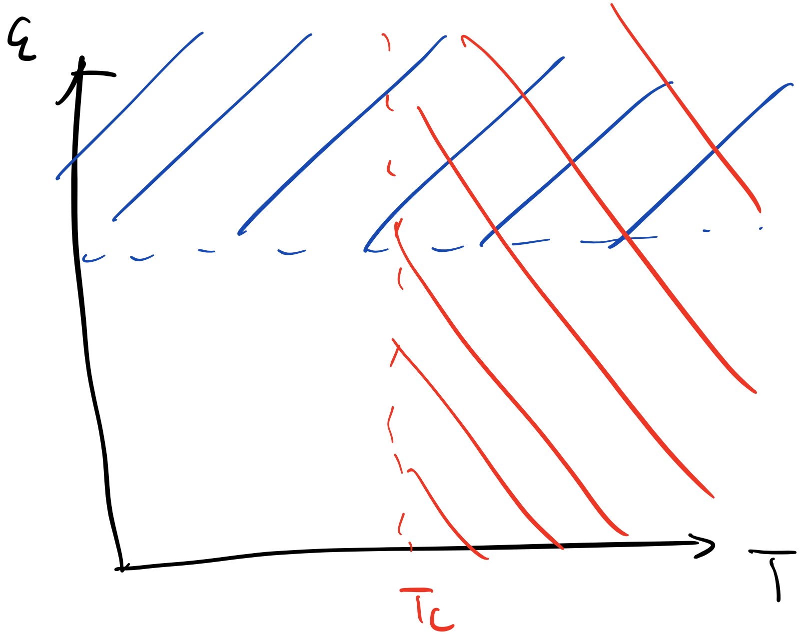
②. $T > T_c$

Thm (Ding, Song, Sun 2021) $\forall$ fixed T, h :

$$\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}} - \langle \sigma_0 \rangle_{\mu^-_{h,\Lambda_N}} \leq \langle \sigma_0 \rangle_{\mu^+_{0,\Lambda_N}} - \langle \sigma_0 \rangle_{\mu^-_{0,\Lambda_N}}$$

+ result of Ising model

$$\Rightarrow \mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}} = \frac{1}{2}\mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}} - \langle \sigma_0 \rangle_{\mu^-_{h,\Lambda_N}} \leq e^{-cN}$$



$$\mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,N}} \le e^{-cN}$$

$d \ge 2$

When ε, T are small, what happens?

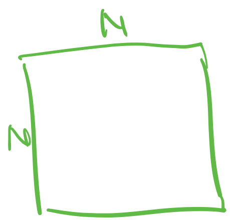
Imry-Ma prediction ('75): $d \geq 3$ long range order exists!

$$\mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}} \geq c$$

$d = 2$ long range order not exist!

$$\mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}} \to 0 \quad \text{as } N \to \infty$$

Reason:



boundary effect $O(N^{d-1})$

random field contribution $O(N^{d/2})$

$d \geq 3$, $N^{d-1} \gg N^{d/2}$

$d = 2$, $N^{d-1} \asymp N^{d/2}$

Mathematical results :

$d \geq 3$: Thm (Imbrie 85', Bricmont-Kupiainen 88')

When ε and T are small, $\boxed{\text{we will reprove it}}$

long range order exists ! $\left(\mathbb{E} \langle \sigma_0 \rangle^{+}_{\mu_{h,\Lambda_N}} \geq c \right)$

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When ε and T are small, we will reprove it

long range order exists ! $(\mathbb{E}\langle\sigma_0\rangle^+_{\mu_{h,\Lambda_N}} \geq c)$

$d = 2$: Thm (Aizenman-Wehr 90')

For any $\varepsilon, T > 0$,

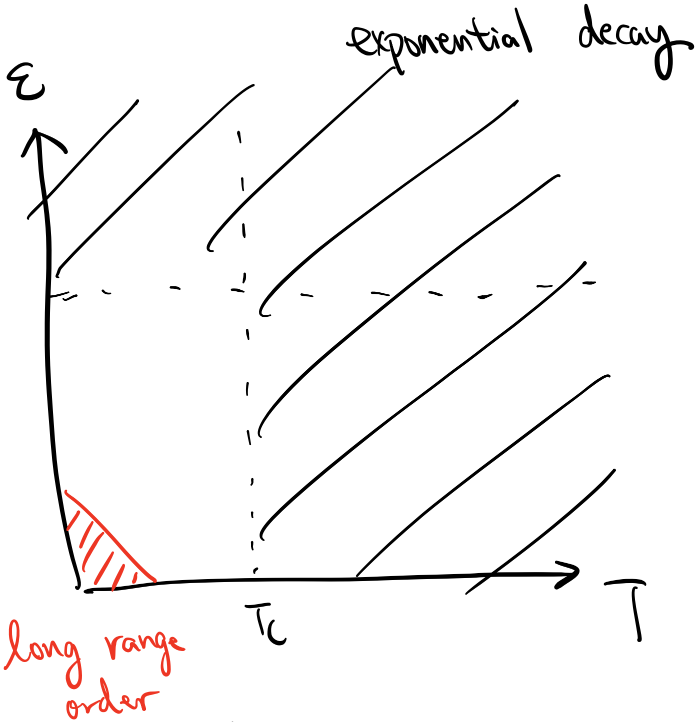
long range order does not exist ! $(\mathbb{E}\langle\sigma_0\rangle^+_{\mu_{h,\Lambda_N}} \to 0$ as $N \to \infty)$

(Chatterjee 18', Aizenman-Peled 19', Ding-Xia 21', Aizenman-Harel-Peled 20')

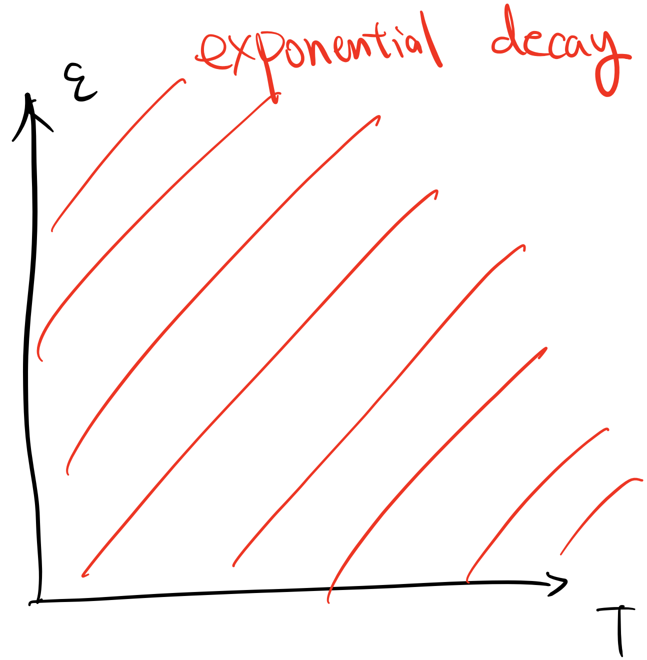
Thm: $\mathbb{E}\langle\sigma_0\rangle^+_{\mu_{h,\Lambda_N}} \leq e^{-cN}$

d \ T, ε	both are small	one of them is large
≥ 3	long range order	exponential decay
2	exponential decay	exponential decay

behaviour of $\mathbb{E}\langle \sigma_0 \rangle_{\mu^{+}_{h,\Lambda_N}}$


exponential decay
ε
T
T_c
long range order

$d \geq 3$


exponential decay
ε
T

$d = 2$

Correlation length for 2D RFIM

- When $\varepsilon = 0$, we can see long range order
- decrease ε with N $\Longleftrightarrow$ let N not be too large

  we shall still see long range order

- Correlation length

$$\psi(\varepsilon) := \sup_N \{ N : \mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h, \Lambda_N}} \geq \frac{1}{2} \}$$

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$$\psi(\varepsilon) := \sup_N \{ N : \mathbb{E}\langle \sigma_0 \rangle_{\mu^+_{h,\Lambda_N}} \geq \tfrac{1}{2} \}$$

Thm (Ding-Wirth 20') $\psi(\varepsilon) \leq \exp(\varepsilon^{-4/3 + o(1)})$ $T \geq 0$.

$\psi(\varepsilon) \geq \exp(\varepsilon^{-4/3 + o(1)})$ $T = 0$

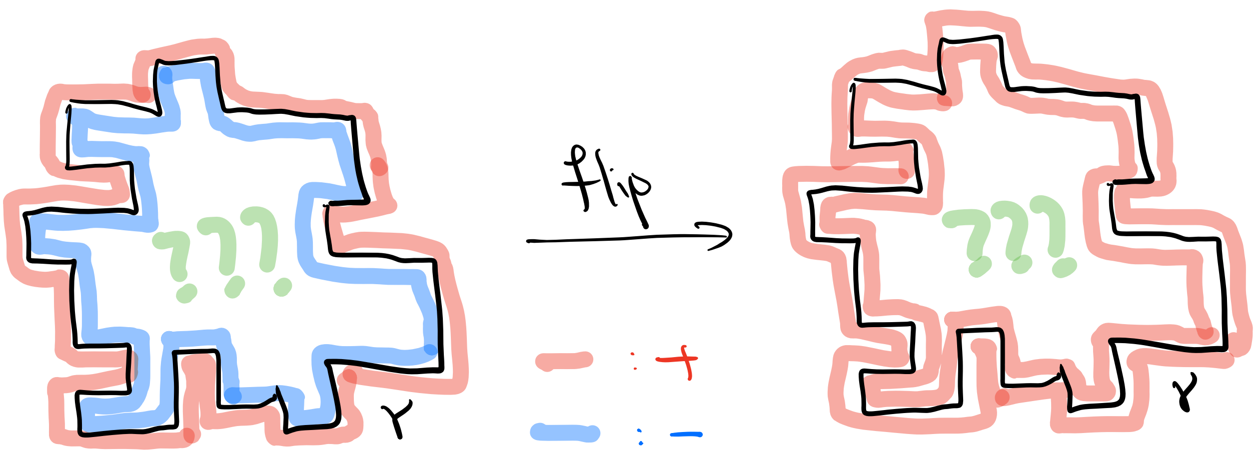
Thm (Ding-Z. 21') $\psi(\varepsilon) \geq \exp(\varepsilon^{-4/3 + o(1)})$,

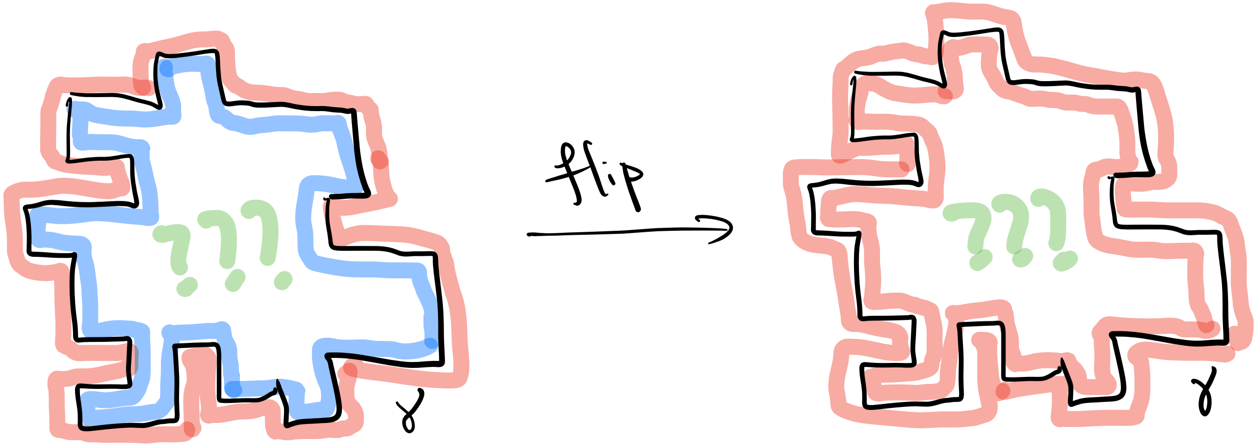
for all small T

Main difficulty when applying Peierls argument to RFIM

- change of Hamiltonian involves external field

$$H^{+}(\sigma,\Lambda_N) = -\Big(\sum_{\substack{u\sim v\\ u,v\in\Lambda_N}} \sigma_u\sigma_v + \sum_{\substack{u\sim v\\ u\in\Lambda_N, v\notin\Lambda_N}} \sigma_u + \sum_{u\in\Lambda_N} h_u\sigma_u\Big).$$





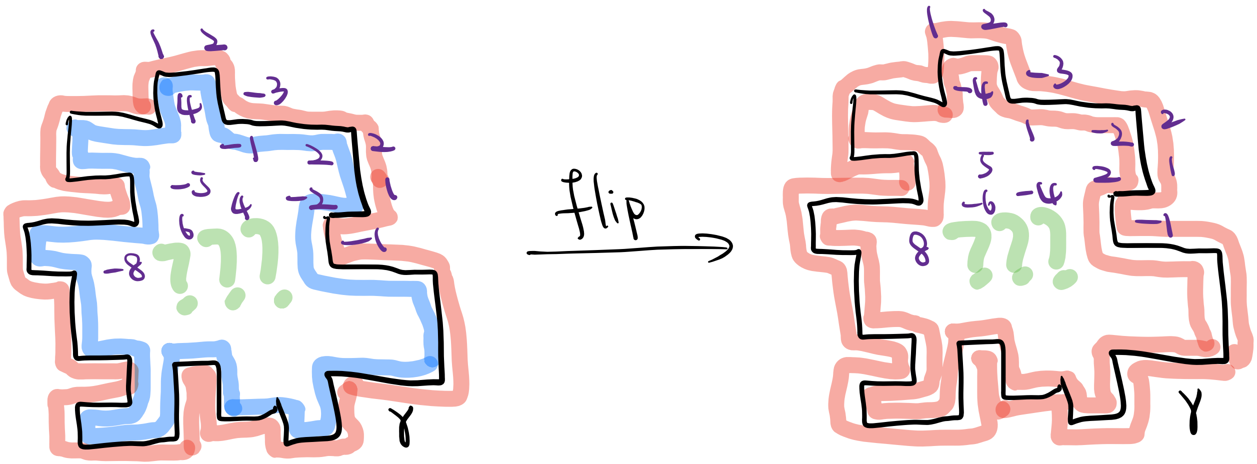
- Change of Hamiltonian can be

$$-2|\gamma| + \sum_{u \in \mathrm{int}\,\gamma} |h_u| \gg 0$$

- renormalization group method : (Imbrie 85', Bricmont-Kupiainen 88')

  keep track of the sign clusters inside γ

☆ Key idea : What about flipping both the spins and external fields ?



$$H^{+}(\sigma,\Lambda_N) = -\Big(\sum_{\substack{u\sim v\\ u,v\in\Lambda_N}}\sigma_u\sigma_v + \sum_{\substack{u\sim v\\ u\in\Lambda_N, v\notin\Lambda_N}}\sigma_u + \underbrace{\sum_{u\in\Lambda_N} h_u\sigma_u}_{\text{unchanged}}\Big)$$

Hamiltonian decreases by $2|\gamma|$!!!

One subtle thing:

$$\mu^+(h,\sigma) = \prod_u \frac{1}{\sqrt{2\pi}} e^{-\frac{h_u^2}{2\varepsilon^2}} \cdot \frac{e^{-\frac{1}{T}H^+(\sigma,\Lambda_N)}}{Z^+(h,\Lambda_N)}$$

h^A: h after flipping the fields in A

- $Z^+(h,\Lambda_N) \neq Z^+(h^A,\Lambda_N)$

but we can control $\dfrac{Z^+(h^A,\Lambda_N)}{Z^+(h,\Lambda_N)}$ over all $0 \in A$

Prop: ① $d \geq 3$

$$\mathbb{P}\left[\forall\, 0 \in A,\ \frac{Z^+(h^A, \Lambda_N)}{Z^+(h, \Lambda_N)} \leq e^{|\partial A|/T}\right] \geq 1 - e^{-q\varepsilon^2}$$

② $d = 2$ $\qquad \varepsilon = o\left((\log N)^{-3/4 + o(1)}\right)$

$$\mathbb{P}\left[\forall\, 0 \in A \subset \Lambda_N,\ \frac{Z^+(h^A, \Lambda_N)}{Z^+(h, \Lambda_N)} \leq e^{|\partial A|/T}\right] \geq 1 - e^{-c/\varepsilon^2}$$

Prop: ①. $d \geq 3$

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+ extension of Peierls argument $\Rightarrow$

Thm: ①. $d \geq 3$ $\quad \mathbb{E}\langle \sigma_o \rangle_{\mu^+_{h,\Lambda_N}} \geq 1 - e^{-c/\varepsilon^2} - e^{-c/T}$

②. $d = 2$ $\quad \varepsilon = o\left((\log N)^{-3/4 + o(1)}\right)$

$$\mathbb{E}\langle \sigma_o \rangle_{\mu^+_{h,\Lambda_N}} \geq 1 - e^{-c/\varepsilon^2} - e^{-c/T}$$

Why this order?

(a) $T\log Z^{+}(h^{A},\Lambda_N) - T\log Z^{+}(h,\Lambda_N)$ "=" $\sum_{u\in A} h_u$

(b) We know that for $d\geq 3$, $\mathbb{P}\left[\forall 0\in A, \frac{\sum h_u}{|\partial A|}\leq 1\right]\geq 1-e^{-c/\varepsilon^2}$

(Fisher-Frölich-Spencer 84')

for $d=2$, $\varepsilon = o((\log N)^{-3/4+o(1)})$,

(Ding-Wirth 20')

$$\mathbb{P}\left[\forall 0\in A\subset\Lambda_N, \frac{\sum h_u}{|\partial A|}\leq 1\right]\geq 1-e^{-c/\varepsilon^2}$$

Random field Potts model :

- q : # of spin choices
- $(h_u^1, h_u^2, \ldots, h_u^q)$ i.i.d. $\mathcal{N}(0, \varepsilon^2)$ at each vertex u
- $\sigma \in \{1, 2, \ldots, q\}^{\Lambda_N}$

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- $\sigma \in \{1, 2, \ldots, q\}^{\Lambda_N}$
- Hamiltonian with boundary condition k

$$H^k(\sigma, \Lambda_N) = -\Big(\sum_{\substack{u \sim v \\ u, v \in \Lambda_N}} \mathbb{1}_{\sigma_u = \sigma_v} + \sum_{\substack{u \sim v \\ u \in \Lambda_N,\, v \notin \Lambda_N}} \mathbb{1}_{\sigma_u = k} + \sum_{u \in \Lambda_N} h_u^{\sigma_u} \Big)$$

- Potts measure for temperature $T \geq 0$

$$\mu^k_{h, \Lambda_N}(\sigma) \propto \exp\left(-\frac{1}{T} H^k(\sigma, \Lambda_N)\right)$$

Random field Potts model overview

①. when ε is large or T is large,

$$\mathbb{E} \sup_{\eta} \langle \sigma_0 = 1 \rangle_{\mu^{\eta}_{h,\Lambda_N}} - \inf_{\eta} \langle \sigma_0 = 1 \rangle_{\mu^{\eta}_{h,\Lambda_N}} \leq e^{-cN}$$

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③. $d = 2$.

Thm (Aizenman-Wehr 90') For any $\varepsilon, T > 0$.

$$\mathbb{E} \sup_{\eta} \langle \sigma_0 = 1 \rangle_{\mu^{\eta}_{h,\Lambda_N}} - \inf_{\eta} \langle \sigma_0 = 1 \rangle_{\mu^{\eta}_{h,\Lambda_N}} \to 0 \quad \text{as} \quad N \to \infty$$

Thm (Ding - Z. 21):

① $d \geq 3$, when T, ε are small,

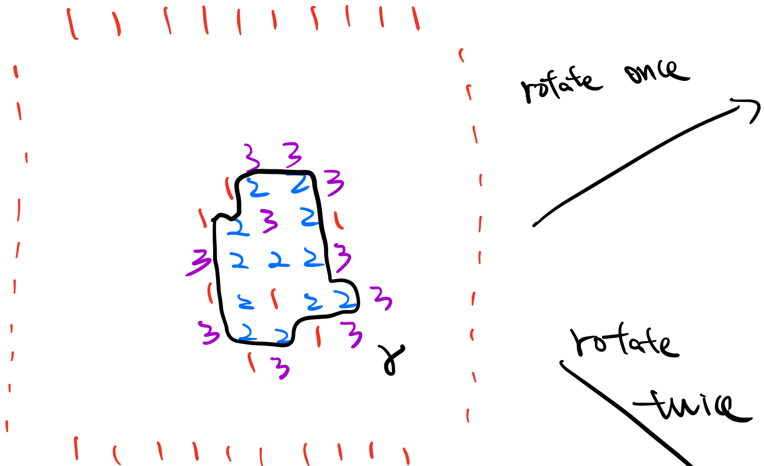
$$\mathbb{E}\langle \sigma_0 = 1 \rangle_{\mu^{i}_{n,\Lambda_N}} \geq 1 - e^{-c/\varepsilon^2} - e^{-c/T}$$

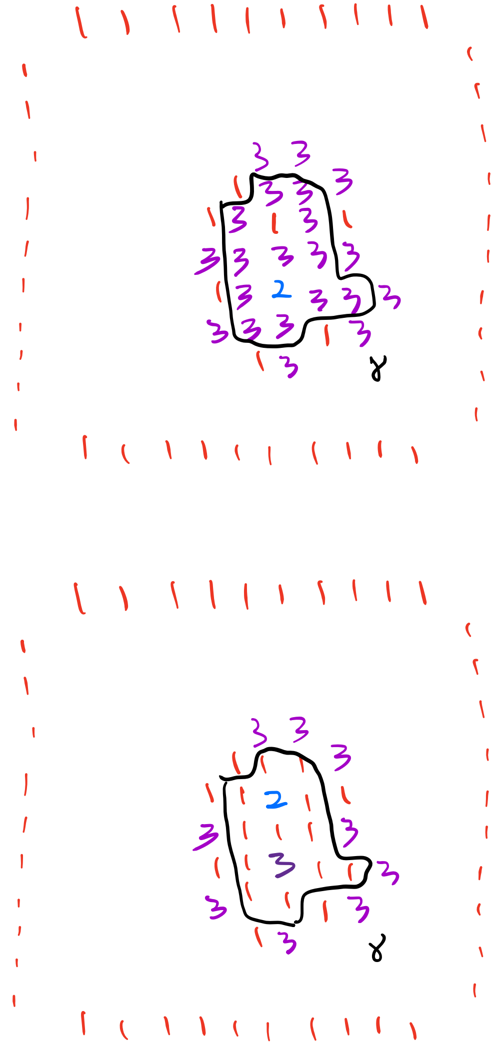
② $d = 2$, $\psi(\varepsilon) \geq \exp(-\varepsilon^{4/3 + o(1)})$ for small T,

where $\psi(\varepsilon) := \sup_N \{ N : \mathbb{E}\langle \sigma_0 = 1 \rangle_{\mu^{i}_{n,\Lambda_N}} \geq \frac{1}{2} \}$

Proof strategy:

When $\varepsilon = 0$





- at least one of their Hamiltonian decreases by $\frac{|\gamma|}{q-1}$

For random field Potts model,

we rotate both spins and external fields

$$H^{K}(\sigma, \Lambda_N) = -\Big(\underbrace{\sum_{u\sim v} \mathbb{1}_{\sigma_u=\sigma_v} + \sum_{\substack{u\sim v \\ u\in\Lambda_N,\, v\notin\Lambda_N}} \mathbb{1}_{\sigma_u=K}}_{\text{decrease by } |H|/(q-1)} + \sum_{u\in\Lambda_N} h_u^{\sigma_u} \Big)$$

unchanged.

- Controls on partition functions

$\Rightarrow$ We are done!

Open questions:

1. Prove exponential decay for 2d - RFPM.
   (and upper bound on correlation length)

2. in 2d-RFIM, prove that

$$\mathbb{E}\langle \sigma_0 \rangle_{\mu^{+}_{h,\Lambda_N}} \approx \exp\left(-\frac{N}{\xi(\varepsilon)}\right)$$

3. Other spin models?