2020-11-06 Kähler geometry

1

$\tilde{r} = re^{\varphi}$

$$\tilde{\omega} = \frac{1}{2} i \partial\bar{\partial} \tilde{r}^2 = \frac{1}{2} i \partial\bar{\partial} (r^2 e^{-\varphi})$$

$$\tilde{\omega}^T = d\tilde{\eta} = d(d^c \log re^{\varphi})$$

$$= dd^c \log r + dd^c \varphi = \omega^T + i\partial\bar{\partial}\varphi$$

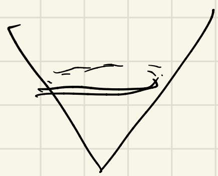
$[\tilde{\omega}^T] = [\omega^T]$ if φ basic

$\Downarrow$

(Excercise: ξ does not change)

If you change the Reeb vector field the $\underline{\varphi \text{ is not basic}}$.

$$\bar{g} = dr^2 + r^2 g \quad \rightsquigarrow \quad \tilde{\bar{g}} = d\tilde{r}^2 + \tilde{r}^2 g$$



$r \to \tilde{r}$

We now follow the paper of de Borbon-Legendre.

C(S) Toric Kähler cone.

$X = C(S)$ Kähler cone of complex dimension $n = m+1$, with compact cross section.

$$g_X = dr^2 + r^2 g_S$$

J complex structure

ω Kähler form : $\omega = \frac{i}{2} \partial\bar{\partial} r^2$

radial vector field $r\frac{\partial}{\partial r}$.

This induces $\mathbb{R}^+$-action (free)

$$L_{r\frac{\partial}{\partial r}} \omega = 2\omega ,$$

$X/\mathbb{R}^{+} =: S$ Sasakian manifold

compact cross section

$\cong \{r = 1\}$

$\xi := J r \frac{\partial}{\partial r}$ Reeb vector field.

Def X Toric $\Leftrightarrow$ $(m+1)$-dim real Torus T^{m+1} acts on X effectively, Hamiltonian, holomorphically (isometrically).

Def Sasakian mfd S Toric $\Leftrightarrow$ $C(S) = X$ is toric.

Ex. $S = S^{2m+1} \subset \mathbb{C}^{m+1}$

$\uparrow$ toric $\qquad \curvearrowleft$ toric T^{m+1}

By definition, $\exists$ moment map $\mu : X \to \mathfrak{t}^*$

$\mathfrak{t} = \mathrm{Lie}(T^{m+1})$, $\mathfrak{t}^*$ = the dual space of $\mathfrak{t}$.

If $v \in \mathfrak{t}$ then v is identified with a vector field on X

$$i(v)\,\omega = -d\mu_v.$$

$$\langle \mu(p), v \rangle \overset{\mathrm{def}}{=} \mu_v(p).$$

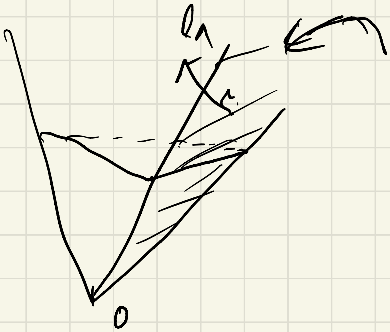
Fact (Lerman) $\leftarrow$ Theorem.

$C = \mu(X)$ is a strictly convex rational polyhedral cone which is good in the following sense.

(Cone version of Atiyah, Guillemin - Sternberg)

$$C = \{ x \in \mathfrak{t}^* \setminus \{0\} \mid \langle x, \lambda_a \rangle \geq 0,\ a = 1, \dots, d\}$$

$\lambda_a \in \mathfrak{t}$



$F_a = \{ x \in t^* \mid \langle x, l_a \rangle = 0 \}$

"facet"

$d = \#$ "facets"

t^*

$\Lambda = \ker(\exp : t \to T) \cong 2\pi \mathbb{Z}^{m+1}$

$2\pi l_1, \dots, 2\pi l_d$ primitive in the lattice Λ

s.t. $t/\Lambda \cong T$ $\to$ "rational"

What is "good"?

l_a generates an S^1-action which fixes

$\mu^{-1}(\langle x, l_a \rangle = 0)$: codim$_{\mathbb{R}}$ 1 submfd in X.

Identity

$$\begin{array}{ccccc} t & \cong & \mathbb{R}^{m+1} & \cong & t^* \\ \cup & & & & \cup \\ \Lambda & \cong & \mathbb{Z}^{m+1} & \cong & \Lambda^* \\ \wr\Vert & & & & \\ \cancel{2\pi}\mathbb{Z}^{m+1} & & & & \end{array}$$

dual lattice of Λ.

$\leftarrow$ (Better to forget 2π)

There is a natural Euclidean structure

$\rightsquigarrow$ inner product.

$\langle x, l_a \rangle = \langle x, v_a \rangle \qquad v_a \in \mathbb{R}^{n+1}$

$x \in$

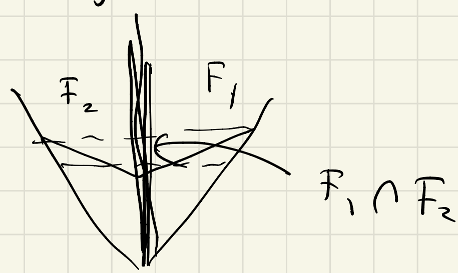
identify $\quad t \times t^* \longrightarrow \mathbb{R}$

$\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \nearrow$ standard inner product

rationality

$\longrightarrow v_a \in \mathbb{Z}^{n+1}, \quad x \in \mathbb{R}^{n+1}$

Def A face $F = \bigcap_{j=1}^{k} \{ l_{a_j} = 0 \}$



$\mu^{-1}(F) \subset X \quad$ $\mathrm{codim}_{\mathbb{C}}\, k$ submanifolds.

Fixed-point-set of $(\mathbb{C}^*)^k$ action generated by $l_{a_1}, \dots, l_{a_k}$.

Def C is good in the sense of Lerman

$\iff$ For any face $F = \bigcap_{j=1}^{k} \underbrace{\{ l_{a_j} = 0 \}}_{F_{a_j}}$

$$(\mathbb{R} v_{a_1} + \cdots + \mathbb{R} v_{a_k}) \cap \mathbb{Z}^{m+1}$$

$$= \mathbb{Z} v_{a_1} + \cdots + \mathbb{Z} v_{a_k}.$$

$(2, 1, 0, \ldots, 0)$
$(0, 1, 0, \ldots, 0)$
not good

Fact (Cone version of Delzant construction).

- X smooth $\Longrightarrow$ $C(X)$ good.
- C good $\Longrightarrow$ $\exists$ smooth X s.t. $\mu(X) = C$.

Let C_0 be the interior of C.


C_0

$X_0 = \mu^{-1}(C_0) \cong C_0 \times T \leftarrow$ free action

$T^{m+1} = (S^1)^{m+1}$

$\curvearrowleft \mathbb{T} = T^{\mathbb{C}}$ algebraic torus $(\mathbb{C}^\times)^{m+1}$

$(\theta_0, \theta_1, \dots, \theta_m) \in \mathfrak{t} \cong \mathbb{R}^{m+1}$

$i\left(\frac{\partial}{\partial\theta_i}\right)\omega = -d\mu_i$ $\quad T^{m+1} \ni (e^{i\theta_0}, \dots, e^{i\theta_m})$

μ_i Hamiltonian function for $\frac{\partial}{\partial\theta_i} \in \mathfrak{t}$

$(0, \dots, 0, 1, 0, \dots, 0) \overset{\text{identify}}{=} \frac{\partial}{\partial\theta_i}$

We write $\mu_i = y_i$

$$\omega = \sum_{i=0}^{m} dy_i \wedge d\theta_i \leftarrow i\left(\frac{\partial}{\partial\theta_i}\right)\omega = -dy_i$$

$(y_0, y_1, \dots, y_m, \theta_0, \theta_1, \dots, \theta_m)$ is called the action-angle coordinates.

$\boxed{\text{$\sum dy_i \wedge d\theta_i$ is the standard symplectic form on $\mathbb{R}^{2m+2} \ni (y_0, \theta_0, \dots, y_m, \theta_m)$}}$

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

not standard $\quad$ not standard

For any element $v = b_0 \frac{\partial}{\partial \theta_0} + \cdots + b_m \frac{\partial}{\partial \theta_m}$

its Hamiltonian function is expressed by

$$i\left(\sum b_i \frac{\partial}{\partial \theta_i}\right)\left(\sum dy_i \wedge d\theta_i\right) = -d\left(\sum_{i=0}^{m} b_i y_i\right)$$

$$\mu_v = \sum_{i=0}^{m} b_i y_i + \underbrace{c_v}_{\text{const}} \longrightarrow 0 \ \text{normalized}$$

That is, any Hamiltonian function μ_v for $v \in \mathfrak{t}$ is a linear function on $C = \mu(X)$.

In Sasakian situation $\xi \in \mathfrak{t}$. Reeb v.f.

$\mathfrak{t} \supset C^*$

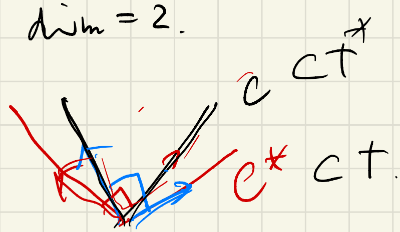
$C \subset \mathfrak{t}^*$
$\parallel$
$\mu(X)$

$X = C(S)$

$$C^* = \{ q \in \mathfrak{t} \mid \langle q, y \rangle \geq 0 \ \text{for } \forall y \in C \}$$

dual cone of C.

Ex. dim = 2.

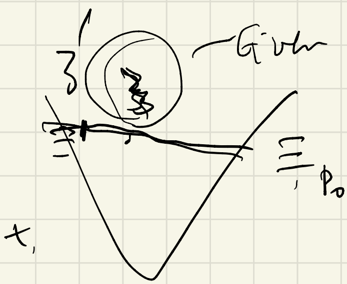


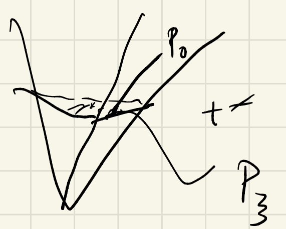
$\xi \in \mathfrak{t}$ Reeb vector field.

normalization

Martelli - Sparks - Yau

$$P_\xi = \{ y \in \mathfrak{t}^* \cong \mathbb{R}^{m+1} \mid \langle \xi, y \rangle = \frac{1}{2}, \ l_a(y) \geq 0, \ a = 1, \dots, d \}$$





l_a "label" de Borbon - Legendre.

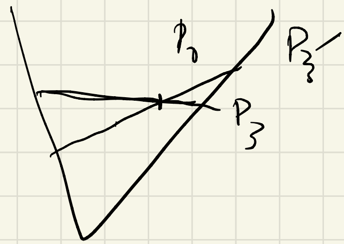
P_ξ with l_a = "labelled polytope".

We fix $p_0 \in P_\xi$.

$$\Xi_{p_0} \overset{\text{def}}{=} \{ \xi' \in C^* \subset \mathfrak{t} \mid \langle \xi', p_0 \rangle = \frac{1}{2} \}$$

$\uparrow$
ξ

We regard Ξ_{p_0} as the deformation space of Reeb vector fields.



$$\begin{aligned} vol : \Xi_{P_0} &\longrightarrow \mathbb{R}^+ \\ \xi' &\longmapsto vol(P_{\xi'}). \end{aligned}$$

<u>Prop 1.</u> Let $\xi_t = \xi + tv$ be a path in Ξ_{P_0}. Then

$$\frac{d}{dt} vol(\xi_t)\Big|_{t=0} = -\frac{2(m+1)}{|\xi|} \int_{P_\xi} \langle v, p \rangle dp.$$

where dp is a natural measure on P_ξ
(later explained).

<u>Prop 2.</u> Hess(vol) is positive definite and vol is proper. So $\exists 1$ critical pt.

Prop 1. Prop 2 $\longleftarrow$ Volume minimization of Martelli–Sparks–Yau.

$\longrightarrow$ $Fut_\xi = 0$.