N=2 SQM

It is based on so-called extended superspace

$t, \theta_+, \theta_-, \bar\theta_+, \bar\theta_-$

$Q_\pm = \frac{\partial}{\partial\theta_\pm} + \bar\theta_\pm \frac{\partial}{\partial t}$, $\bar Q_\pm = \frac{\partial}{\partial\bar\theta_\pm} + \theta_\pm \frac{\partial}{\partial t}$

$D_\pm = \frac{\partial}{\partial\theta_\pm} - \bar\theta_\pm \frac{\partial}{\partial t}$, $\bar D_\pm = \frac{\partial}{\partial\bar\theta_\pm} - \theta_\pm \frac{\partial}{\partial t}$

$\{Q_+, \bar Q_+\} = \{Q_-, \bar Q_-\} = \frac{\partial}{\partial t}$ ← to be 4 SUSY of the theory

$\{D_+, \bar D_+\} = \{D_-, \bar D_-\} = \frac{\partial}{\partial t}$

othere anticommutators of Q's and D's are equal to zero.

New concept - chiral superfields.
just the name

$X^i(t, \theta_\pm, \bar\theta_\pm)$ is called chiral if it is annihilated by $\bar D_\pm$: $\boxed{\bar D_\pm X^i = 0}$ Example

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$$\left(\frac{\partial}{\partial\bar\theta_+} - \theta_+ \frac{\partial}{\partial t}\right) f(t, \theta_+, \bar\theta_+) = 0 \quad (\mathcal{E})$$

θ_+ is a chiral

t - is not chiral, $\bar\theta_+$ - is not

chiral $t_c = t + \bar\theta_+\theta_+$ is chiral

we may solve equation $(\mathcal{E})$ considering $F(\theta_+, t_c) \to$ is chiral.

The chiral field is a function of θ_+, θ_- and $t_c = t + \bar\theta_+\theta_+ + \bar\theta_-\theta_-$

Soleetion to chirality equations

$\hat{\Phi}^i = X^i(t_c) + \psi_+^i(t_c)\theta_+ + \psi_-^i(t_c)\theta_- + F^i\theta_+\theta_-$

How Q and $\bar{Q}$ are acting on chiral fields. Action of $\bar{Q}_\pm$ is easy to compute - really, $\bar{Q}_\pm$ is different from $\bar{D}_\pm$ by a sign

$$Q_+ = \frac{\partial}{\partial\theta_+} + \bar{\theta}_+\frac{\partial}{\partial t}$$

$$Q_\pm F(t_c, \theta_+, \theta_-) = \frac{\partial F}{\partial\theta_\pm} + \cancel{\bar{\theta}_+\frac{\partial F}{\partial t_c}} + \cancel{\theta_+\frac{\partial F}{\partial t_c}} = \frac{\partial F}{\partial\theta_\pm}$$

$$\bar{Q}_\pm = -\bar{D}_\pm + 2\theta_\pm\frac{\partial}{\partial t}$$

$\bar{Q}_\pm F = 2\theta_\pm\frac{\partial F}{\partial t}$, so there is the following representation

$$Q_\pm \rightarrow \frac{\partial}{\partial\theta_\pm} \qquad \bar{Q}_\pm = 2\theta_\pm\frac{\partial}{\partial t_c}$$

they actually ~~commute~~ anticommute as they should anticommute.

Action of $Q_\pm$ on components:

$Q_+\hat{\Phi}^i = \psi_+^i + \theta_- F^i$ $\qquad \delta_{Q_+}X^i = \psi_+^i \quad \delta_{Q_+}\psi_-^i = F^i$

$Q_-\hat{\Phi}^i = \psi_-^i - \theta_+ F^i$ $\qquad \delta_{Q_-}X^i = \psi_-^i \quad \delta_{Q_-}\psi_+ = -F^i$

$\bar{Q}_+\hat{\Phi}^i = \theta_+\frac{\partial X^i}{\partial t} + \theta_+\theta_-\frac{\partial\psi_-^i}{\partial t}$ $\qquad \delta_{\bar{Q}_+}\psi_+^i = \frac{\partial X^i}{\partial t}$

$\qquad \delta_{\bar{Q}_+}F^i = \frac{\partial\psi_-^i}{\partial t}$

similarly $\delta_{\bar{Q}_-}\psi_- = \frac{\partial X^i}{\partial t}$

$$\delta_{\bar{Q}_-} F^i = \frac{\partial \psi_+^i}{\partial t}$$

Let us check the pair δQ_+, $\delta \bar{Q}_+$:

$$\left\{ \frac{\delta}{\delta Q_+}, \frac{\delta}{\delta \bar{Q}_+} \right\} X^i = \frac{\partial X^i}{\partial t}$$

$N=2$ SQM we also need antichiral field $\bar{\Phi}^{\bar{i}}$: $D_\pm \bar{\Phi}^{\bar{i}} = 0$

$$\bar{\Phi} = f(t_a, \bar{\theta}_+, \bar{\theta}_-) \qquad t_a = t + \theta_+ \bar{\theta}_+ + \theta_- \bar{\theta}_-$$

$$\bar{\Phi}^{\bar{i}} = \bar{X}^{\bar{i}}(t_a) + \bar{\psi}_+^{\bar{i}}(t_a)\bar{\theta}_+ + \bar{\psi}_-^{\bar{i}}(t_a)\bar{\theta}_- + \bar{F}(t_a)\bar{\theta}_+\bar{\theta}_-$$

The role of $Q_\pm$ and $\bar{Q}_\pm$ would be interchanged: exchange $\theta \leftrightarrow \bar{\theta}$, $Q \leftrightarrow \bar{Q}$.

Action - invariant of the supertransformation.

in $N=1$ case we had $\int dt \int d\theta d\bar{\theta}\, W(\Phi)$

in $N=2$ case we have similarly the following terms:

$$\int \underline{\underline{dt_c}} \int d\theta_+ d\theta_- \, W(\Phi_c) =$$

$$= \int dt \left(\frac{\partial W}{\partial \phi^i} F^i + \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi_+^i \psi_-^j \right)$$

if $W = \Phi$ it is just $\int dt\, F$ and is called F-term

similarly $\int dt_a \int d\bar\theta_+ d\bar\theta_- \bar W(\bar\Phi_a) =$

$$= \int dt \left(\frac{\partial \bar W}{\partial \bar\Phi^i} \bar F^i + \frac{\partial^2 \bar W}{\partial \bar\Phi^i \partial \bar\Phi^j} \bar\psi_+^i \bar\psi_-^j \right)$$

Before, in $N=1$ SQM we got kinetic term from $g_{ij} D\Phi^i \bar D\Phi^j$ $\leftrightarrow$ but not now!

$$\int dt\, d\theta_+ d\theta_- d\bar\theta_+ d\bar\theta_- \, K(\Phi_c^i, \bar\Phi_a^j) \overset{?}{=}$$

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simplest case $K = \Phi_c \bar\Phi_a$, I need 4-θ's.

easiest structure is - get 2θ from $F\theta_+\theta_-$ in Φ_c and $2\bar\theta$ from $\bar\theta_+\bar\theta_- \bar F$ in $\bar\Phi_a$: $\int F\bar F dt$ (it is an analogue of F^2 term in $N=1$ SQM).

Now we need derivatives

Take $\theta_+ \psi_+(t_c)$ and $\bar\theta_+ \bar\psi_+(t_a)$

from Φ_c from $\bar\Phi_a$ $\to$

we already have $\theta_+\bar\theta_+$, how to get $\theta_-\bar\theta_-$? from the difference in $\theta_-\bar\theta_-$ denendence of t_c and t_a!

I may do tedious calculations exp

$\psi_+(t_c)$ as $\psi_+(t) + \bar\theta_-\theta_- \frac{\partial\psi_+}{\partial t} + \ldots$

$\bar\psi_+(t_a)$ as $\bar\psi_+(t) + \theta_-\bar\theta_- \frac{\partial\bar\psi_+}{\partial t} + \ldots$

$\underline{\psi_+ \frac{\partial \bar\psi_+}{\partial t} + \bar\psi_+ \frac{\partial \psi_+}{\partial t}}$ - not a total derivative due to fact that $\psi_+ \; \bar\psi_+$ are anticommuting

Kinetic term for fermions, sim for $\psi_-, \bar\psi_-$

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what about bosons

$X(\underline{t}_c)\,\bar X(\underline{\underline{t}}_a)$ - similarly, I expand $\theta\bar\theta$ dependence of t_c and t_a. But now I get 2 time derivatives. to get 4 θ from

$$\int dt\,(\partial_t X\, \partial_t \bar X)$$

(after we eliminate total derivatives in t).

So we get the action in components similar to $N=1$ SQM, with the dif. that we have COMPLEX STRUCTURE ON THE TARGET.

Hol. coordinates are chiral fields, antihol. coordinates are antichiral fields.

Remark, in $N=1$ theory we may try to int. only one constr. $\bar D f_c = 0$, then field would go $f_c = \psi + \theta \tilde F$; how to write kinetic term? $\int d\theta \int W(f_c)$?

probably, it is a way to 1-order formulation of

SQM. - I never tried. -
- good question

$$\int d^2\theta_+ d^2\theta_- \, K(\Phi_a^i, \bar{\Phi}_c^{\bar{j}}) = \frac{\partial^2 K}{\partial \Phi^i \partial \bar{\Phi}^{\bar{j}}} \partial_- \chi^i \partial_+ \bar{\chi}^{\bar{j}} + \dots$$

D-term

$$\boxed{\frac{\partial^2 K}{\partial \Phi^i \partial \bar{\Phi}^{\bar{j}}} \Longleftrightarrow g_{i\bar{j}}}$$

- metric on the target space.

But is exactly the condition for metric to be Kahler.

Really, $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}} \rightarrow$

$$g_{i\bar{k}} = \partial_i (v_{\bar{k}}) \qquad (1)$$

similarly, if we also impose

$$\partial_{\bar{i}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{i}} \rightarrow$$

$$g_{i\bar{k}} = \partial_{\bar{k}} (\tilde{v}_i) \qquad (2)$$

$$(1) \; \& \; (2) \Longrightarrow g_{i\bar{j}} = \frac{\partial K}{\partial X^i \partial \bar{X}^{\bar{j}}}$$

exactly what we have from the D-term