<u>lecture 3</u>:

Compactness of Kahler-Ricci solitons

based on works of
G. - Phong - Song - Sturm
Phong - Song - Sturm
& Z. Zhang

- X compact Kahler manifold, $\dim_{\mathbb{C}} X = n$
- X Fano, i.e. $c_1(X) > 0$. $-K_X$ ample
- a metric $\omega = \omega_{KS} \in c_1(X)$ is a (shrinking) Kahler-Ricci soliton, if

$$\lambda = \begin{cases} +1 & \text{shrinking} \\ 0 & \text{steady} \\ -1 & \text{expand.} \end{cases}$$

$$\begin{cases} Ric(\omega) = +\lambda\omega + L_V\omega \\ V \text{ a holomorphic vector field} \end{cases}$$

- let $u = u_\omega$ be the normalized Ricci potential of $\omega = \omega_{KS}$,

ie. $\int_X e^{-u}\omega^n = \int \omega^n = V_g$

$$Ric(\omega) = \omega - i\partial\bar\partial u_\omega$$

$g \longleftrightarrow \omega$

- $V = -\nabla_g u$

soliton equation

$$\begin{cases} R_{i\bar j} = g_{i\bar j} - u_{i\bar j}, \\ u_{ij} = u_{\bar i\bar j} = 0, \text{ ie } V \text{ is holomorphic.} \end{cases}$$

- the Futaki invariant w.r.t. V is

$$\mathrm{Fut}_X(V) = \frac{1}{V_g}\int_X |V|_g^2\,\omega^n$$

$$= \frac{1}{V_g}\int_X |\nabla u|^2\,\omega^n$$

- one <u>goal</u>: $\mathrm{Fut}_X(V)$ is uniformly bounded, $\forall\,(X,\omega,V)\in KR(n)$.

- $KR(n) = \left\{ (X,\omega,V) \;\middle|\; \begin{array}{l} \dim X = n \quad c_1(X) > 0 \\ \omega \text{ is a KR soliton w/} \\ \text{soliton vector field } V \end{array} \right\}$

Theorem: $\exists\, C = C(n) > 0$, s.t. $\forall\, (X, \omega, V) \in KR(n)$

$$\begin{cases} Fut_X(V) \le C \\ \|u\|_{C^0} + \|\nabla u\|_{C^0} + \|R_\omega\|_{C^0} \le C. \\ diam(X, g) \le C \end{cases}$$

RK: this is trivial if $V = 0$

$$\frac{1}{C(n)} \le \frac{Vol_g(B_g(p, r))}{r^{2n}} \le C(n)$$

RK: Ivey, Z. Zhang $R_\omega > 0$

<u>Corollary</u>: let $(X_i, \omega_i, V_i) \in KR(n)$, up to subsequence.

$$(X_i, g_i) \xrightarrow{d_{GH}} (X_\infty, d_\infty)$$ a compact metric length space

① X_∞^{sing} a closed subset of codim ≥ 4 Hausdorff $\qquad X_\infty = X_\infty^{reg} \cup X_\infty^{sing}$

② $$(X_i, g_i, V_i) \xrightarrow{C^\infty_{loc}(X_\infty^{reg})} (X_\infty^{reg}, g_\infty, V_\infty)$$

(g_∞, V_∞) a Kahler-Ricci solitor on X_∞^{reg}

③ $$\overline{(X_\infty^{reg}, g_\infty)} \overset{\text{isometric}}{=} (X_\infty, d_\infty)$$

X_∞ normal projective variety w/ klt sing

$|V_\infty|_{g_\infty} \leq C$ & V_∞ extends to a hol v.f. on X_∞

• <u>Preliminaries</u>:

<u>Fact</u>: ① smooth Fano manifolds of dim n have only finitely many deformation types, hence

$$c_1(X)^n = \int_X \omega^n = (-K_X)^n \leq C(n).$$

② $\frac{1}{2\pi} c_1(X)$ is integral, so

$$c_1(X)^n \geq c(n) > 0$$

$\Rightarrow$ the total volume

$$0 < c(n) \leq V_g \leq C(n) < \infty$$

Definition: (α-invariant). $0 < \omega \in c_1(X)$

$$\alpha(X) = \alpha_\omega(X) = \sup\left\{ \alpha > 0 \;\middle|\; \begin{array}{l} \exists\, C_\alpha > 0 \text{ s.t.} \\ \int_X e^{-\alpha(\varphi - \sup_X \varphi)} \omega^n \leq C_\alpha \end{array} \right.$$

$$\left. \forall\, \varphi \in PSH(X, \omega) \cap C^\infty(X) \right\}.$$

Rk: • $\alpha_\omega(X)$ is independent of the choice of $\omega \in c_1(X)$

& holomorphic invariant

• $\alpha(X) > 0$ by Hörmander, Tian.

Def: (log canonical).

X proj normal, $\Delta \geq 0$ effective $\mathbb{Q}$-Cartier divisor w/ coefficients ≤ 1

the pair (X, Δ) is log canonical (l.c.) if

$\exists$ a log resolution of (X, Δ); $\pi : Y \to X$

$\pi^{-1}(\Delta) \cup \mathrm{exc}(\pi)$ is a divisor of normal crossing

$$K_Y = \pi^*(K_X + \Delta) + \sum a_i F_i$$

$$\mathbb{Q} \ni a_i \geq -1$$

(if $a_i > -1$, klt singularity)

Def: X, proj normal variety. D $\mathbb{Q}$-Cartier div.

log canonical threshold of D is

$$\mathrm{lct}(X, D) = \sup\{t \mid (X, tD) \text{ is log canonical}\}.$$

Theorem: ① (Demailly) X Fano

$$\alpha(X) = \mathrm{lct}(X, -K_X)$$

$$=: \inf_{m \in \mathbb{Z}_{>0}} \sup_{D \in |-mK_X|} \mathrm{lct}(X, \tfrac{1}{m} D)$$

$\frac{D}{m} \in |-K_X|$

② (Birkar). $\exists\ \delta(n) > 0$. s.t.

$$\mathrm{lct}(X, -K_X) \geq \delta(n)$$

$\Rightarrow \alpha(X) \geq \delta(n) > 0$

Def: greatest Ricci lower bound. X Fano manifold.

$$R(X) = \sup\left\{ t > 0 \;\middle|\; \begin{array}{l} \exists\, \omega \in c_1(X),\ \text{s.t.} \\ Ric(\omega) > t\,\omega \end{array} \right\}$$

$\in (0,1]$

Lemma: $R(X) \geq \min\left(1, \dfrac{n+1}{n}\alpha(X)\right)$ if $\alpha(X) > \dfrac{n}{n+1}$ then $R(X) = 1$, ie $\exists$ KE

outline of the proof: consider the "continuity method"

Fix a metric $\omega_0 \in c_1(X)$

$$(MA_t) \quad \begin{cases} (\omega_0 + i\partial\bar\partial \varphi_t)^n = e^{-t\varphi_t + F_0}\,\omega_0^n \\ Ric(\omega_0) - \omega_0 = i\partial\bar\partial F_0 \end{cases}$$

① $(MA)_t$ can be solved up to $\min\left(\frac{n+1}{n}\alpha(X), 1\right)$

$$Ric(\omega_t) = t\omega_t + (1-t)\omega_0 \underset{> \; t\omega_t}{,} \qquad \omega_t = \omega_0 + i\partial\bar{\partial}\varphi_t$$

(see Tian's 1987 paper) $\Rightarrow R(X) \geq \min\left(1, \frac{n+1}{n}\alpha(X)\right)$

② (Szekelyhidi) $[0, R(X))$ maximal existence interval of $(MA)_t$

$$\Rightarrow \quad R(X) \geq \min\left(\frac{n+1}{n}\alpha(X), 1\right)$$

$\square$

$\alpha(X) \geq \delta(n) \quad \Rightarrow \quad R(X) \geq \varepsilon(n)$

Corollary: $\exists\ \varepsilon(n)>0$, s.t. $\forall$ Fano manifold X,

$\exists\ \hat{\omega}\in c_1(X)$, s.t.

$$Ric(\hat{\omega})\geq \varepsilon(n)\,\hat{\omega}.$$

Rk: let $G < Aut(X)$ be any compact subgp.

averaging $\hat{\omega}$ over G, we may take $\hat{\omega}$ to be G-invariant.

In particular, $\hat{\omega}$ is $Im V$-invariant if $(X,\omega,V)\in KR(n)$.

Perelman's W-, μ- functionals:

$$W_X(g,f) = \frac{1}{V_g}\int_X \left(R_g + |\nabla f|^2 - f - n\right) e^{-f} dV_g$$

$$\left[\ f \in C^\infty(X) \quad \frac{1}{V_g}\int_X e^{-f} dV_g = 1 \ \right]$$

μ- functional

$$\mu_X(g) = \overset{\text{min}}{\cancel{\inf}}\left\{ W_X(g,f) \ \middle|\ \begin{array}{l} f \in C^\infty \\ \int_X e^{-f} dV_g = V_g \end{array} \right\}$$

Well-known fact:

$\mu_X(g_t)$ is $\nearrow$ along (normalized) KR flow.

Lemma: $\exists\ A(n) > 0$, s.t. $\mu_X(\hat{g}) \geq -A(n)$.

Proof: Recall $Ric(\hat{g}) \geq \varepsilon(n)\,\hat{g}$.

- Myers' theorem: $D = diam(X, \hat{g}) \leq \dfrac{2n-1}{\sqrt{\varepsilon(n)}} = C(n)$

- $c(n) \leq Vol(X, \hat{g}) \leq C(n)$

- Bishop-Gromov volume comparison

$$\frac{Vol_{\hat{g}}(B_{\hat{g}}(p, r))}{r^{2n}} \geq \frac{Vol_{\hat{g}}(B(p, D))}{D^{2n}} \geq K(n) > 0$$

$Vol_{\hat{g}}(B(p,D)) = Vol_{\hat{g}}(X)$

noncollapsing

- Groke's theorem: Sobolev constant C_S of $(X, \hat{g})$ is bounded by $C(n)$

- $\forall f \in C^\infty$, $\int_X e^{-f} dV_g = V_g$.

Write $\phi = e^{-f/2}$, $\frac{1}{V_g}\int \phi^2 dV_g = 1$

by Jensen's inequality

$$\frac{1}{V_g}\int_X \phi^2 \log \phi^{\frac{2}{n-1}} dV_g \leq \log \frac{1}{V_g}\int_X \phi^{\underbrace{2+\frac{2}{n-1}}_{=\frac{2n}{n-1}}} dV_g$$

$$\leq \log \left[C_S \left(\int_X |\nabla \phi|^2 + \phi^2 \right) \right]$$

$$\leq \frac{4}{n-1} \frac{1}{V_g} \int_X |\nabla \phi|^2 dV_g + C(n)$$

$$\Rightarrow \frac{1}{V_g}\int_X \phi^2 \log \phi^2 \, dV_g \le \frac{4}{V_g}\int |\nabla \phi|^2 dV_g + C(n)$$

$$\Rightarrow \mathcal{W}(\hat{g}, f) = \frac{1}{V_g}\int_X \Big(\underbrace{R_{\hat{g}}}_{\geq 0}\phi^2 + 4|\nabla\phi|^2 - \phi^2\log\phi^2\Big) dV_{\hat{g}} - n$$

$$\geq -C(n)$$

$\square$

<u>Recall</u>: $\mathrm{Ric}(\hat{\omega}) \geq \varepsilon(n)\,\hat{\omega}$

$$\Rightarrow R_{\hat{g}} \geq n\,\varepsilon(n) > 0.$$

Corollary: $\forall\, (X, \omega, V) \in KR(n)$.

$$\mu_X(g) \geq -A(n)$$

Proof: ① we consider the KR flow

$$\begin{cases} \dfrac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \omega_t \\ \omega_t\big|_{t=0} = \hat{\omega} \in G(X) \end{cases}$$

$\hat{\omega}$ is $Im V$ invariant

- since $\omega = \omega_{KS}$ is a KR soliton, by smooth convergence of $\eta_t^* \omega_t$. $\eta_t = \exp(tV)$.

$$\eta_t^* \omega_t \xrightarrow{C^\infty} \omega'_{KS}$$

② $\mu_X(g_t) = \mu_X(\eta_t^* g_t) \nearrow$ in t

& $\lim_{t\to\infty} \mu_X(\eta_t^* g_t) = \underbrace{\mu_X(g'_{KS}) = \mu_X(g_{KS})}_{\text{by uniqueness of KR solitons}}$ ie $g'_{KS} = \sigma^* g_{KS}$

Tian - Zhu

$\Rightarrow \quad \mu_X(g_{KS}) \geq \mu_X(\hat{g}) \geq -A(n)$

$\square$

<u>Lemma</u>: $\forall\ (X, \omega, V) \in \mathcal{KR}(n)$, $\exists\ C(n) > 0$

$$\begin{cases} \|u\|_{C^0} + \|\nabla u\|_{C^0(\omega)} + \|R_\omega\|_{C^0} \leq C \\ \operatorname{diam}(X, \omega) \leq C \end{cases}$$

$\Rightarrow$ Theorem 1.

<u>Recall</u>: $V = -\nabla u$

$R_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}}$

$u_{ij} = u_{\bar{i}\bar{j}} = 0$.

$\int e^{-u}\omega^n = \int \omega^n$.

- Recall the well-known identities:

(g, V) Kahler-Ricci soliton. then

$$R_{i\bar{j}} = g_{i\bar{j}} - u_{i\bar{j}} \quad \& \quad u_{ij} = 0$$

① $R = n - \Delta u$ ✓

② $R + |\nabla u|_g^2 - u = \text{const}.$

Pf of ②: $R_{i\bar{j},j} = g_{i\bar{j},j} - u_{i\bar{j}j} = -\left(u_{ij\bar{j}} - u_k R_{i\bar{k}j\bar{j}}\right)$

$\quad \parallel$ Bianchi

$\quad R_{,i}$

$$= u_k R_{i\bar{k}}$$

$$= u_k \left(g_{i\bar{k}} - u_{i\bar{k}}\right)$$

$$= u_i - \nabla_i |\nabla u|^2$$

$$\Rightarrow \nabla_i \left(R + |\nabla u|^2 - u\right) = 0$$

Integrating ② against $\frac{1}{V_g} e^{-u} \omega^n$ $\qquad \underline{\frac{1}{V_g} \int e^{-u} \omega^n = 1}$

$$\text{const} = \frac{1}{V_g} \int \left(R + |\nabla u|^2 - u \right) e^{-u} \omega^n$$

$$= \frac{1}{V_g} \int \left(R + \Delta u - u \right) e^{-u} \omega^n$$

$$= n - \frac{1}{V_g} \int u e^{-u} \omega^n$$

$$\Rightarrow \boxed{R + |\nabla u|^2 - u = n - \underbrace{\frac{1}{V_g} \int_X u e^{-u} \omega^n}_{\text{entropy of } e^{-u}}}$$

Fact: If $(\overset{=g}{\omega}, V)$ is a KR soliton, u is a minimizer of $W(g,f)$

$$\Rightarrow \mu_X(g) = W_X(g,u)$$

$dV_g = \omega^n$

$$= \frac{1}{V_g}\int_X \left(R + |\nabla u|^2 + u - n\right)e^{-u}\, dV_g$$

$$= \frac{1}{V_g}\int_X 2u e^{-u}\, dV_g - \frac{1}{V_g}\int u e^{-u}\, dV_g$$

$$= \frac{1}{V_g}\int_X u e^{-u}\, dV_g$$

$$\mu_X(g) \geq -A(n)$$

$$\Rightarrow \quad \frac{1}{V_g}\int_X u e^{-u}\,\omega^n \geq -A(n).$$

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the identity

$$R + |\nabla u|^2 = u - \frac{1}{V_g}\int u e^{-u}\,dV_g + n$$

$$\Rightarrow \quad \min_X u \geq -A(n) - n$$

- use Perelman's uniform estimates (Sesum - Tian)

<u>Recall</u>: $\omega = \omega_{KS}$ is a KR soliton, it can be viewed as a solution to a KR flow.

let $\sigma_t = \exp(-tV)$, $\omega_t = \sigma_t^* \omega_{KS}$

$$\begin{cases} \dfrac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \omega_t \\ \omega_t \big|_{t=0} = \omega_{KS}. \end{cases}$$

$u_{\omega_t} = u_\omega \circ \sigma_t = \sigma_t^* u_\omega$, normalized Ricci potential.

$\Vert\!\!\!\!\sqsubseteq u_t$

Apply Maximum principle to the functions as in Perelman

$$\frac{|\nabla u_t|^2_{\omega_t}}{u_t - \min_X u + 1} \le C(n), \qquad \frac{-\Delta_{\omega_t} u_t}{u_t - \min_X u_t + 1} \le C(n)$$

these two functions are uniformly bounded.

then from Perelman's techniques & $\mu_X(g_t) \ge -A(n)$

$$\Rightarrow \quad \max_X u_t \le C(n) \quad \& \quad \operatorname{diam}(X, g_t) \le C(n).$$

$$\Rightarrow \quad |\nabla u_t|^2 \le C(n)$$

$$\& \quad \underbrace{-\Delta_{\omega_t} u_t}_{" R_{\omega_t} - n} \le C(n).$$

$\square$

- Gromov-Hausdorff (pre) compactness : (Z. Zhang).

idea : conformal transformations.

let $(X, \omega, V) \in KR(n)$. consider

$$\tilde{g} = e^{-u/n-1} g$$

u the Ricci potential as before

$\tilde{g}$ a Riemannian metric (not Kahler)

since u & $|\nabla u|$ are bounded

$\tilde{g}$ is equivalent to g in $C^1(X, g)$

$$c(n) g \le \tilde{g} \le C(n) g$$

well-known formula: the Ricci curvatures of $\tilde{g}$ & g are related by ($i, j = 1, 2 \dots 2n$ real coordinates indices)

$$\tilde{R}_{ij} = \underbrace{R_{ij} + \nabla_i \nabla_j u}_{= g_{ij} \text{ by soliton equation.}} + \frac{1}{2(n-1)} \underline{u_i u_j} - \frac{1}{2(n-1)} \left(\underline{|\nabla u|_g^2} - \underline{\Delta_g u} \right) g_{ij}$$

bounded terms

$\Rightarrow \exists\, C(n) > 0$, s.t

$$|\widetilde{Ric}|_{\tilde{g}} \leq C(n)$$

$$\mathrm{diam}(X, \tilde{g}) \leq C(n), \qquad \frac{1}{C(n)} \leq \mathrm{vol}(X, \tilde{g}) \leq C(n)$$

so applying Cheeger-Colding-Naber's theorem to $(X_i, \tilde{g}_i)$

& the C^1 equivalence of $\tilde{g}_i$ & g_i, we conclude

the GH seq. convergence of (X_i, g_i) & structure of the limit space

<u>In sum</u>: $(X_i, g_i) \xrightarrow{d_{GH}} (X_\infty, d_\infty)$

- $\operatorname{codim} X_\infty^{sing} \geq 4$, X_∞^{reg} open & dense
- $g_i \xrightarrow{C^\infty_{loc}(X_\infty^{reg})} g_\infty$, g_∞ a KR soliton on X_∞^{reg}
- $\overline{(X_\infty^{reg}, g_\infty)} \overset{\text{isometric}}{=} (X_\infty, d_\infty)$

- claim: X_∞ is a normal projective variety.

  (Donaldson - Sun)

main tool: partial C^0 estimates.

- $k \in \mathbb{Z}_{>0}$, $(X, \omega) \in KR(n)$, $\omega \in c_1(X)$

- Fix an hermitian metric h on $-K_X$ s.t.

$$\Theta_h = -i\partial\bar{\partial} \log h = \omega \in c_1(-K_X)$$

(ω is a KR soliton
$h = e^{-u}\omega^n$)

- denote
$$\begin{cases} -K_X^\# = (-K_X)^{\otimes k} \\ h^\# = h^{\otimes k} \\ \omega^\# = k\omega \\ L^{2,\#} = L^2(X, \omega^\#) \end{cases}$$

$$\Theta_{h^\#} = k\omega = \omega^\#$$

define an inner product on $H^0(X,(-K_X)^{\#}) \ni s, \tilde{s}$

$$(s,\tilde{s}) = \int_X \langle s,\tilde{s}\rangle_{h^{\#}} (\omega^{\#})^n = k^n \int_X \langle s,\tilde{s}\rangle_{h^{\otimes k}} \omega^n$$

the Bergman kernel (diagonal) is defined

$$\rho_{X,k}(z) = \sum_{j=0}^{N_k} |S_j(z)|^2_{h^{\#}} = \sup\left\{ |s|^2_{h^{\#}(z)} \;\middle|\; \|s\|^2_{L^{2,\#}} = 1 \right\}$$

where $\{S_j\}_{j=0}^{N_k}$ is an orthonormal basis of $H^0(X,(-K_X)^{\#}))$

$\rho_{X,k}$ well-defined, indep of the choice of o.b.

Theorem (partial C^0):

$\exists\, k = k(n) \in \mathbb{Z}_{>0}$, $\varepsilon(n) > 0$, s.t

$\forall\, (X, \omega, V) \in \mathcal{KR}(n)$, the Bergman kernel of $H^0(X, -kK_X)$

$$\inf_{z \in X} \rho_{X,k}(z) \geq \varepsilon(n) > 0$$

- main idea: the structure of tangent cone of (X_∞, d_∞) & Hörmander L^2-technique to solve $\bar\partial$-equations.

One key step is to show mean value type inequality for $s \in H^0(X, -kK_X)$

Prop: $\exists\, C = C(n) > 0$, s.t. $\forall\, (X, g) \in KR(n)$, $s \in H^0(X, -kK_X)$

① $\|s\|_{L^{\infty}, \#} \leq C \|s\|_{L^2, \#}$

② $\|\nabla s\|_{L^{\infty}, \#} \leq C \|s\|_{L^2, \#}$

③ $\lambda_1(\Delta^{\#}_{\bar\partial, u}) \geq \frac{1}{C(n)} > 0$

where $\Delta^{\#}_{\bar\partial, u} = \bar\partial \bar\partial^*_u + \bar\partial^*_u \bar\partial : (-K_X)^{\otimes k} \otimes \Omega^{0,1}(X)$

$\bar\partial^*_u$ adjoint of $\bar\partial$ w.r.t. $\frac{1}{V_g}\int_X |\sigma|^2_{\omega^{\#}, h^{\#}} e^{-u} (\omega^{\#})^n$

Proof: $\forall (X, g) \in \mathcal{KR}(n)$

Observe: $\tilde{g} = e^{-\frac{u}{n-1}} g$

$C_S(\tilde{g})$ uniformly bounded. $\Rightarrow C_S(g) \leq C(n)$.

Sobolev constant of $\tilde{g}$

Sobolev constant scaling invariance $\Rightarrow C_S(g^{\#}) \leq C(n)$

For a fixed k, omit $\#$ in $h^{\#}, g^{\#}$ ...

① $s \in H^0(X, -kK_X)$. $\Delta = \Delta_{\omega^\#}$

$$\Delta |s|^2_{h^\#} = |\nabla s|^2_{h^\#} - |s|^2_{h^\#}$$

Kato inequality $\Rightarrow$ $\Delta |s|_{h^\#} \geq - |s|_{h^\#}$

Moser iteration $\Rightarrow$

$$\|s\|_{L^{\infty,\#}} \leq C(n) \|s\|_{L^{2,\#}}$$

$\square$

② from the Bochner formula

$$\Delta|\nabla s|^2 = |\nabla\nabla s|^2 - 2|\nabla s|^2 + n|s|^2 + Ric(\nabla s, \overline{\nabla s})$$

$$= |\nabla\nabla s|^2 - |\nabla s|^2 + n|s|^2 - i\partial\bar\partial u(\nabla s, \overline{\nabla s})$$

Multiplying both sides by $(|\nabla s|^2)^p \quad p \geq 1$

& running the usual process of Moser iteration.

unbounded term

$$\int -i\partial\bar\partial u(\nabla s, \overline{\nabla s})_{h^\#} |\nabla s|^{2p} \omega^n$$

$$= \int u_{\bar j}\left[(\nabla_j s \overline{\nabla_i s}) h^\# |\nabla s|^{2p}\right]_i \omega^n \sim \int \nabla\nabla s * (\nabla s)^{2p+1} + (\nabla s)^{2p+2} \quad \square$$

③ Recall

$$(\sigma, \sigma)_u = \int_X |\sigma|^2_{h^\#, \omega^\#} e^{-u} (\omega^\#)^n$$

$$\sigma \in (-kK_X) \otimes \Omega^{0,1}(X)$$

let D_u be the covariant derivative on $(-kK_X) \otimes \Omega^{0,1}$ w.r.t. $g^\#$, & $h^\# e^{-u}$

$$\begin{cases} \Delta_{\bar\partial, u} = \bar\partial \bar\partial^*_u + \bar\partial^*_u \bar\partial \\ \Delta_{D_u} = -(D_u)^{\bar j} (D_u)_{\bar j} \end{cases}$$

Bochner-Kodaira formula

$$\left(\Delta_{\bar\partial,u}\sigma\right)_{\bar j} = \left(\Delta_{D_u}\sigma\right)_{\bar j} + \sigma_{\bar q}\left(g^\#\right)^{i\bar q}\left(g^\#_{i\bar j} + u_{i\bar j} + Ric(g)_{i\bar j}\right)$$

$$= \left(\Delta_{D_u}\sigma\right)_{\bar j} + \left(1+\frac{1}{k}\right)\sigma_{\bar j}$$

$$\Rightarrow \quad \left(\Delta_{\bar\partial,u}\sigma, \sigma\right)_u \geq \left(1+\frac{1}{k}\right)\int |\sigma|^2 e^{-u}\left(\omega^\#\right)^n$$

$$\geq c(n)\,\|\sigma\|^2_{L^{2,\#}}$$

$\square$

Bergman projection, $\forall\, \sigma \in -kK_X$.

$s = \sigma - \tau$ is hol. $\tau = \bar\partial_u^*\left(\Delta_{\bar\partial,u}\right)^{-1}\bar\partial\sigma$

w/ uniform estimates.

$\square$

Thank You!