Universal property

Recall that for a complex of sheaves

$$\cdots \to \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \mathcal{F}^{i+2} \to \cdots$$

we require $d^{i+1} \circ d^i = 0$. Then we define cohomology sheaves $\mathbb{H}^i(\mathcal{F}^*) = \ker(d^i)/\mathrm{im}(d^{i-1})$

Write $C = \mathrm{Mod}(\mathcal{O}_X)$. Then complexes $\mathrm{Com}(C)$ form a category, and $\mathbb{H}^i$ is a functor

$$\mathrm{Com}(C) \to \mathrm{Vect}, \text{ vector spaces} / \mathbb{C}.$$

To make this work, we define morphisms f^* to be commutative diagrams as follows.

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}^i \arrow[r, "d^i"] \arrow[d, "f^i"] & \mathcal{F}^{i+1} \arrow[r, "d^{i+1}"] \arrow[d, "f^{i+1}"] & \mathcal{F}^{i+2} \arrow[r] \arrow[d, "f^{i+2}"] & \cdots \\
\cdots \arrow[r] & \mathcal{G}^i \arrow[r, "e^i"'] & \mathcal{G}^{i+1} \arrow[r, "e^{i+1}"'] & \mathcal{G}^{i+2} \arrow[r] & \cdots
\end{tikzcd}
$$

Then we have $H^i(f^*)$ for $i \in \mathbb{Z}$. When these are isomorphisms, we say f^* is a <u>quasi-isomorphism</u>

In particular, then we have $\mathbb{H}^i(\mathcal{F}^*) \cong \mathbb{H}^i(\mathcal{G}^*)$.

<u>Idea</u> To construct the derived category, quasi-isomorphisms are made into isomorphisms.

This is implemented by functors as follows.

Def Say a functor $P : \mathrm{Com}(C) \to D$ has <u>property ⊛</u> if f^* quasi-iso $\Rightarrow Pf^*$ iso

Def The <u>derived category</u> $D(C)$ is a category with a functor

$$Q : \mathrm{Com}(C) \to D(C)$$

such that

1. Q has property ⊛
2. Any functor P with property ⊛ factors uniquely through Q that is there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Com}(C) \arrow[rr, "P"] \arrow[dr, "Q"'] & & D \\ & D(C) \arrow[ur, dashed, "R"'] & \end{tikzcd}$$

First properties of $D(C)$

By the characterization above, we always have cohomology functors $H^i: D(C) \to C$.

<u>Note</u>: these arise by taking $P = H^i: Com(C) \to C$.

The functor Q allows us to consider objects in $Com(C)$ as objects in $D(C)$ (with same cohomology H^i)

<u>Note</u> We may furthermore consider an object $A \in C$ as an object of $Com(C)$ as follows

$$\cdots \to 0 \to \underset{i\text{th place}}{A} \to 0 \to \cdots$$

and thence of $D(C)$. For this, we write $A[-i]$

Recall the following.

<u>Def</u> a <u>resolution</u> $\mathcal{F}^*$ of a sheaf $\mathcal{G}$ is a sequence $\mathcal{F}^0 \to \mathcal{F}^1 \to \dots$ such that
$$0 \to \mathcal{G} \xrightarrow{r} \mathcal{F}^0 \to \mathcal{F}^1 \to \dots \text{ is exact}$$

This corresponds to a morphism $\mathcal{G} \to \mathcal{F}^*$ in $\mathrm{Com}(\mathcal{C})$ induced by r. By exactness, this is a quasi-iso. Viewing it as a morphism in $D(\mathcal{C})$ via P, it becomes an iso by property (*).

<u>Ex</u> Taking resolutions $\mathcal{F}_1^*, \mathcal{F}_2^*$ of $\mathcal{G}$ we have $\mathcal{G} \xrightarrow{\sim} \mathcal{F}_i$ in $D(\mathcal{C})$, thence $\mathcal{F}_1 \cong \mathcal{F}_2$.

<u>Homotopy category</u> $Com_\sim(C)$

This is useful for constructing $D(C)$. We consider morphisms in $Com(C)$ "up to homotopy equivalence". This is an algebraic version of a notion from topology.

<u>Def</u> For $A^*, B^* \in Com(C)$ and $f^*, g^* \in Mor(A^*, B^*)$ say <u>homotopy equivalent</u> $f \sim g$ if have morphisms $h^i : A^i \to B^{i-1}$ such that

$$f^i - g^i = h^{i+1} d_A^i + d_B^{i-1} h^i$$

$$\begin{tikzcd}
{} \arrow[r] & A^{i-1} \arrow[r] \arrow[dl] \arrow[d, shift right, "f^{i-1}"'] \arrow[d, shift left, "g^{i-1}"] & A^i \arrow[r] \arrow[dl, "h^i"'] \arrow[d, shift right, "f^i"'] \arrow[d, shift left, "g^i"] & A^{i+1} \arrow[r] \arrow[dl, "h^{i+1}"'] \arrow[d, shift right, "f^{i+1}"'] \arrow[d, shift left, "g^{i+1}"] & {} \arrow[dl] \\
{} \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & {}
\end{tikzcd}$$

Def homotopy category $\mathrm{Com}_{\sim}(\mathcal{C})$, same objects as $\mathrm{Com}(\mathcal{C})$, same morphisms modulo equiv relation $\sim$.

Construction of $D(\mathcal{C})$

Def derived category $D(\mathcal{C})$.

objects: same as $\mathrm{Com}(\mathcal{C})$

morphisms: diagrams as follows in $\mathrm{Com}_{\sim}(\mathcal{C})$ up to an equivalence relation given below

$$\begin{tikzcd} & C^* \arrow[dl, "q"', "f^*"] \arrow[dr, "g^*"'] & \\ A^* & & B^* \end{tikzcd}$$

where "q" indicates a quasi-isomorphism.

Rem These diagrams are sometimes called "roofs"

Rem Think of the morphism $A^* \to B^*$ in $D(C)$ as like a formal fraction $g^*/f^* = g^* f^{*-1}$

In particular, we want to take the following diagrams to correspond to quasi-iso f^* and its inverse

$$\begin{tikzcd} & C^* \arrow[dl, "q"', "1"] \arrow[dr, "f^*"] & \\ C^* & & A^* \end{tikzcd} \qquad \begin{tikzcd} & C^* \arrow[dl, "q"', "f^*"] \arrow[dr, "1"] & \\ A^* & & C^* \end{tikzcd}$$

For this remark to make sense, we need to give a notion of <u>composition</u> of morphisms.

First, we take two "roofs" C_i^* to be equivalent if they can be "joined" by a further roof, that is there exists a commutative diagram

$$\begin{tikzcd}
& C^* \arrow[dl] \arrow[dr] & \\
C_1^* \arrow[d] \arrow[drr] & & C_2^* \arrow[dll] \arrow[d] \\
A^* & & B^*
\end{tikzcd}$$

where the composition $C^* \to C_1^* \to A^*$ is a quasi-isomorphism. Now we have :

Def Given roofs C_1^*, C_2^* below, they may be composed to give a roof C_3^* by drawing a diagram in $\mathrm{Com}_{\sim}(\mathcal{C})$

$$\begin{tikzcd}
& & C_3^* \arrow[dl] \arrow[dr] & & \\
& C_1^* \arrow[dl] \arrow[dr] & & C_2^* \arrow[dl] \arrow[dr] & \\
A^* & & B^* & & C^*
\end{tikzcd}$$

Rem For the construction of this diagram, see for instance Gelfand-Manin, "Methods of Homological Algebra". In particular, this construction gives a commutative diagram in $\mathrm{Com}_{\sim}(\mathcal{C})$ but not in $\mathrm{Com}(\mathcal{C})$.

Morphisms in $D(C)$

When $C = Coh(X)$, morphism sets between objects $A[k]$ for $A \in C$ may be calculated using Ext-groups, that is:

$$\mathrm{Hom}_{D(C)}(A, B[k]) = \mathrm{Ext}^k_X(A, B) \text{ for } A, B \in C$$

Indeed we may construct endofunctors $[j]$, $j \in \mathbb{Z}$ of $Com(C)$ which descend to $D(C)$ such that $A[i][j] = A[i+j]$. Thence we have

$$\mathrm{Hom}_{D(C)}(A[j], B[k]) = \mathrm{Ext}^{k-j}_X(A, B) \text{ for } A, B \in C$$

Rem To calculate Hom between other objects, we may use "spectral sequences".

Complexes in $D(\mathcal{C})$

From the definition, $\mathbb{H}^j(A[-i]) = \begin{cases} A & \text{if } j=i \\ 0 & \text{otherwise} \end{cases}$

For $E \in D(\mathcal{C})$ with $\mathbb{H}^j(E) = $ same, $E \cong A[-i]$.

However, not all objects are of this form.

Ex For $E = \bigoplus_{i \in \mathbb{Z}} A^i[-i]$ we have $\mathbb{H}^j(E) = A^j$

Indeed, objects of $D(\mathcal{C})$ are not determined by their cohomology.

Note This is true in $\mathrm{Com}(\mathcal{C})$ too, for instance

$$\to 0 \to \mathbb{k} \xrightarrow{1} \mathbb{k} \to 0 \to \text{ has trivial } H^*.$$

Ex Take $E \in D(\mathcal{C})$ with $H^j(E) = 0$ for $j \neq 0, 1$

Then we have objects and morphisms in $\mathrm{Com}(\mathcal{C})$ as follows.

$$\begin{tikzcd}
F_0 \arrow[d, "f"] & \to E^{-1} \arrow[r] \arrow[d] & \ker d \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
E \arrow[d, "g"] & \to E^{-1} \arrow[r] \arrow[d] & E^0 \arrow[r, "d"] \arrow[d] & E^1 \arrow[r] \arrow[d] & E^2 \arrow[r] \arrow[d] & {} \\
F_1 & \to 0 \arrow[r] & 0 \arrow[r] & \operatorname{coker} d \arrow[r] & E^2 \arrow[r] & {}
\end{tikzcd}$$

Note that f, g are <u>not</u> quasi-isos, but they induce quasi-isos as follows.

$$F_0 \xrightarrow{\text{qi}} H^0(E) = \ker d / \ldots \qquad H^1(E)[-1] \xrightarrow{\text{qi}} F_1$$

In $D(C)$ therefore we have morphisms

$$H^0(E) \to E \to H^1(E)[-1]$$

It turns out, using a structure on $D(C)$ called "distinguished triangles" that E is determined by $H^0(E)$, $H^1(E)$, and an element (up to scale) of

$$\mathrm{Hom}_{D(X)}(H^1(E), H^0(E)[2]) = \mathrm{Ext}^2_X(H^1(E), H^0(E))$$

So, in this example and more generally, an object of $D(C)$ is determined by its homology, plus some further data.