# Homological mirror symmetry

Recall that we wanted to associate to a Calabi-Yau 3-fold $X$ categories $\mathcal{C}_A$ and $\mathcal{C}_B$ where

(A) $\mathrm{Ob}\,\mathcal{C}_A(X,\omega) \supset$ Lagrangians on $(X,\omega) \ni L_i$

(B) $\mathrm{Ob}\,\mathcal{C}_B(X,J) \supset$ Bundles on $(X,J)$

We can now sketch the construction of $\mathcal{C}_A$:

For $L_0, L_1$ we set $\mathrm{Mor}(L_0,L_1) = HF_*(L_0,L_1)$.

Now to make a category we require composition morphisms $HF_*(L_0,L_1) \otimes HF_*(L_1,L_2) \to HF_*(L_0,L_2)$

There are many subtleties in this construction: for instance the restrictions on $M$ and $L$ which we needed to define $HF_*$.

Also, establishing an associativity property for the composition is difficulty: usually, it is only associative up to homotopy, leading to study of "$A^\infty$ structures".

We now give background to the construction of the category $\mathcal{C}_B$.

# Coherent sheaves

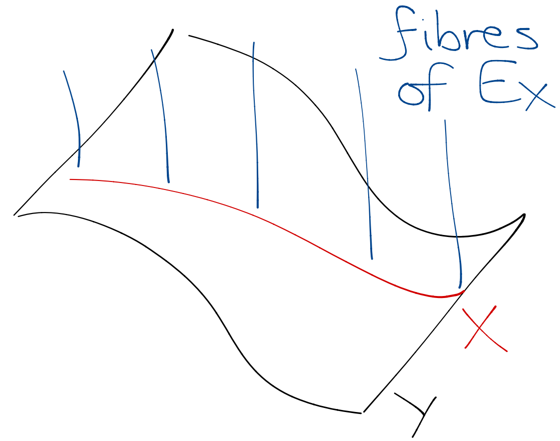
Take a complex manifold $(X, J)$. Then these are a generalization of (complex, holomorphic) vector bundles on $X$. Recall first that for

a morphism $\quad f: X \to Y$

and vector bundles $\quad E_X \quad E_Y$

we have a natural notion of the pullback $f^*E_Y$, with fibre at $x \in X$ defined to be the fibre of $E_Y$ at $f(x) \in Y$. Coherent sheaves have a notion of pullback, and also (under restrictions on $f$) a notion of pushforward.

**Ex** For $f$ an embedding of a submanifold, we have pushforward $f_* E_X$ as a coherent sheaf on $Y$, which should be thought of as a "vector bundle on $X \subset Y$". We explain below.




## From vector bundles

Recall the following.

**Def** For a vector bundle $E$ on $M$, get a sheaf $\mathcal{E} = \mathcal{S}(E)$ where $\mathcal{E}(U)$ is (holomorphic) sections of $E|_U$ for each open $U \subset M$, and morphisms are restrictions.

**Ex** For $L$ the trivial line bundle (rank 1 vector bundle) we write $\mathcal{O} = \mathcal{S}(L)$ and call $\mathcal{O}$ the structure sheaf

The sheaves $\mathcal{S}(E)$ have a special structure:

**Def** Say $\mathcal{E}$ is a <u>sheaf of modules</u> on $M$, or an <u>$\mathcal{O}$-module</u>, if $\mathcal{E}(U)$ has structure of an $\mathcal{O}(U)$-module for each open $U \subset M$, with a compatibility of the restriction morphisms.

**Ex** $\mathcal{S}(E)$ is an $\mathcal{O}$-module: $\mathcal{O}(U)$ is the functions on $U$, and we may multiply sections of $E|_U$, and thence elements of $\mathcal{S}(E)(U)$, by these functions.

$\mathcal{O}$-modules form a category, which we write as $\mathrm{Mod}(\mathcal{O}_M)$. Via $\mathcal{S}$, the category $\mathrm{Bun}(M)$ of bundles forms a full subcategory.

non-Ex The locally constant sheaf $\underline{\mathbb{C}}$ is not an $\mathcal{O}$-module (because multiplying a locally constant function by a general function does not give a locally constant function).

## Coherent sheaves

Note that there is an operation of direct sum $\oplus$ on bundles (fibrewise direct sum) which extends to $\mathrm{Mod}(\mathcal{O}_M)$ (direct sum of sections on each $U$)

We thence write $\mathcal{E}^{\oplus I}$ for $\underbrace{\mathcal{E} \oplus \_ \oplus \mathcal{E}}_{I \text{ times}}$ where $I$ is a (possibly infinite) set.

<u>Note</u>: In particular, for $n \in \mathbb{Z}_{\geq 0}$ we have

$\mathcal{E}^{\oplus\{1-n\}} = \underbrace{\mathcal{E} \oplus \_ \oplus \mathcal{E}}_{n \text{ times}}$, which we abbreviate

to $\mathcal{E}^{\oplus n}$, and let $\mathcal{E}^{\oplus 0} = 0$ for $I = \emptyset$.

We then make the following definition.

<u>Def</u> Say $\mathcal{E} \in \mathrm{Mod}(\mathcal{O}_M)$ is quasi-coherent if it locally has a finite presentation, that is for every $m \in M$ have open $\mathcal{U} \ni m$ and $\mathcal{E}|_{\mathcal{U}} = \mathrm{coker}(\mathcal{O}_{\mathcal{U}}^{\oplus I} \to \mathcal{O}_{\mathcal{U}}^{\oplus J})$.

**Rem** We do not recall the definition of cokernel of a morphism of sheaves here. But note that we have a morphism

$$\mathcal{O}_{\mathcal{U}}^{\oplus J} \to \mathcal{E}|_{\mathcal{U}}, \text{ thence } \mathcal{O}_{\mathcal{U}}(\mathcal{U})^{\oplus J} \to \mathcal{E}(\mathcal{U}),$$

so the images $e_j$ for $j \in J$ of

$$0 \oplus \_ \oplus \underset{j^{th}\text{ place}}{1} \oplus \_ \oplus 0 \in \mathcal{O}_{\mathcal{U}}(\mathcal{U})^{\oplus J}$$

generate $\mathcal{E}(\mathcal{U})$, and the set $I$ indexes relations between the $e_j$.

**Ex** $S(E)$ is quasi-coherent, indeed it is locally trivial, that is we may always choose $\mathcal{U}$ such that $I = \emptyset$ in the above.

**Notation**: Write $\mathrm{QCoh}(M)$ for the full subcategory of $\mathrm{Mod}(\mathcal{O}_M)$ consisting of quasi-coherent sheaves.

**Note**: $\mathrm{QCoh}(M)$ includes infinite-rank generalizations of vector bundles.

Coherence is a finiteness condition: in particular, it requires that in the above $|J| < \infty$. We omit the details, but give some examples.

**Notation**: Write $\mathrm{Coh}(M)$ for the full subcategory of coherent sheaves on $M$.

We have the following operation.

**Def** Given a sheaf $\mathcal{E}$ on $X$ and a morphism $f: X \to Y$, the pushforward sheaf $f_*\mathcal{E}$ on $Y$ is defined by its sections

$$(f_*\mathcal{E})(U) = \mathcal{E}(f^{-1}U)$$

**Rem** With suitable assumptions, this can be made into a functor $f_*: \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$. In particular, in the algebraic context, we include a comorphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ in the data of the morphism $f: X \to Y$. This induces an $\mathcal{O}_X$-module structure on $f_*\mathcal{E}$ (via its natural $f_*\mathcal{O}_Y$-module structure).

Ex For $f$ an embedding of a submanifold, and $E$ a vector bundle on it, the sheaf

$$\mathcal{F} = f_* \mathcal{S}(E) \text{ on } Y$$

is coherent (and should be thought of as a vector bundle on $X \subset Y$, as discussed)

Rem For two such $\mathcal{F}_i$, bundles $E_i$ on $X_i \subset Y$, we have $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) =$

$$\begin{cases} \mathrm{Hom}(E_1|_{X_2}, E_2) & \text{if } X_2 \subseteq X_1 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathrm{Hom}(B_1, B_2)$ between bundles on some space is defined as the global sections of the bundle $B_1^\vee \otimes B_2$. The zero here follows from the observation that there are no non-trivial morphisms from torsion to torsion-free modules.

## The derived category

Note that the proposed morphism sets in $\mathcal{M}_A$, namely $HF_*(L_0, L_1)$ are naturally graded (by $\mathbb{Z}_{\geq 0}$, or indeed $\mathbb{Z}$). But the sets $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ for $\mathrm{Coh}(X)$ are not naturally graded.

To get graded objects from sheaves, we can take (co)homology of complexes of them. The natural setting for this is the derived category, developed by Grothendieck and Verdier in the 60s.

Surprisingly, this gives a way to construct categories $\mathcal{M}_B$ for mirror symmetry.

I give a characterisation before discussing construction and properties.