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Reference

- Burghelea, "on the homotopy type of $\mathrm{Diff}(M^n)$ and connected problems", 1973
- Burghelea, Lashof, "The homotopy type of the space of diffeomorphisms.", 1974
- Ferry, Weinberger, "Curvature, tangentiality, and controlled topology", 1991

Now show condition (*) holds for some i, n.

(Recall condition (*):

$$\begin{tikzcd} \pi^{\mathbb{Q}}_{i+1}\mathrm{Diff}(D^{n-1},\partial) \arrow[r, "\alpha^{\mathbb{Q}}_{n*}"] & \pi^{\mathbb{Q}}_{i}\mathrm{Diff}(D^{n},\partial) \arrow[r, "\alpha^{\mathbb{Q}}_{n+1*}"] \arrow[d, phantom, "\|\wr" description] & \pi^{\mathbb{Q}}_{i-1}\mathrm{Diff}(D^{n+1},\partial) \\ & \pi_i\mathrm{Diff}(D^n,\partial)\otimes\mathbb{Q} & \end{tikzcd}$$

is nontrivial.)

Consider

$$\begin{tikzcd} \text{Th3.6} \leftarrow & \pi^{\mathbb{Q}}_{i}\mathrm{Diff}(D^{n},\partial) \arrow[r, "\alpha^{\mathbb{Q}}_{n+1*}"] \arrow[d, "\phi^{\mathbb{Q}}_*", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \pi^{\mathbb{Q}}_{i-1}\mathrm{Diff}(D^{n+1},\partial) \arrow[d, "\phi^{\mathbb{Q}}_*", "\cong"'] & \text{cf.[B73]} \\ & \pi^{\mathbb{Q}}_{i+n+1}\mathrm{Top}(n)/O(n) \arrow[r] & \pi^{\mathbb{Q}}_{i+n+1}\mathrm{Top}(n+1)/O(n+1) & (**) \\ \text{for } i \geq n. & \pi^{\mathbb{Q}}_{i+n+2}B\mathrm{Top}(n) \arrow[u, "\partial^{\mathbb{Q}}_*", "② \cong"'] \arrow[r, "j^{\mathbb{Q}}_{n*}", "①"'] \arrow[ur, phantom, "\circlearrowright"] & \pi^{\mathbb{Q}}_{i+n+2}B\mathrm{Top}(n+1) \arrow[u, "\partial^{\mathbb{Q}}_*", "\cong"'] & \text{for } i \geq n+1 \end{tikzcd}$$

where

① $j_n : B\mathrm{Top}(n) \longrightarrow B\mathrm{Top}(n+1)$ is induced by $\mathrm{Top}(n) \longrightarrow \mathrm{Top}(n+1)$

② fiber bundle

$$Top(n)/O(n) \hookrightarrow BO(n) \longrightarrow BTop(n)$$

⇝ exact sequence

$$\begin{tikzcd}
\pi^{\mathbb{Q}}_{i+n+2} BO(n) \arrow[r] \arrow[d, equal, "i\geq n"] & \pi^{\mathbb{Q}}_{i+n+2} BTop(n) \arrow[r, "\partial^{\mathbb{Q}}_{*}", "\cong"'] & \pi^{\mathbb{Q}}_{i+n+1} Top(n)/O(n) \arrow[r] & \pi^{\mathbb{Q}}_{i+n+1} BO(n) \arrow[d, equal, "i\geq n"] \\
0 & & & 0
\end{tikzcd}$$

(Reason for $\pi^{\mathbb{Q}}_i BO(n)=0$ for $i\geq 2n-1$

fiber bundle

$$S^{n-1}=O(n)/O(n-1) \hookrightarrow BO(n-1) \longrightarrow BO(n)$$

⇝ exact sequence

$$\begin{tikzcd}
\pi^{\mathbb{Q}}_i S^{n-1} \arrow[r] \arrow[d, equal, "i\geq 2n-1"] & \pi^{\mathbb{Q}}_i BO(n-1) \arrow[r] & \pi^{\mathbb{Q}}_i BO(n) \arrow[r] & \pi^{\mathbb{Q}}_{i-1} S^{n-1} \arrow[d, equal, "i\geq 2n-1"] \\
0 & & & 0
\end{tikzcd}$$

Th (Serre). $\pi_j S^n$ = torsion except $j=n$ or n even and $j=2n-1$.

By diagram (**), to show α_{n+1*} is nontrivial for some i,n, it suffices to show

$$j_{n*}: \pi^{\mathbb{Q}}_{i+n+2} BTop(n) \longrightarrow \pi^{\mathbb{Q}}_{i+n+2} BTop(n+1)$$

is nontrivial for some i,n.

For this, recall

Th 3.7 $\exists$ a constant $c>0$. s.t. for all $m\geq c$, $\exists\, k\in\mathbb{Z}_{\geq 0}$ s.t. the rational Pontrjagin class

$$p_{k+m}\in H^{4k+4m}(BTop(2m);\mathbb{Q})$$

evaluates nontrivially on $\pi^{\mathbb{Q}}_{4k+4m} BTop(2m)$

$$\Downarrow$$

$\exists\, x \in \pi^{\mathbb{Q}}_{4m+4k} BTop(2m)$ s.t.

$$\langle p_{k+m}, h(x) \rangle \neq 0$$

where $h : \pi^{\mathbb{Q}}_{*} BTop(2m) \longrightarrow H_*(BTop(2m)) \otimes \mathbb{Q}$.
is Hurewicz map.

Consider the natural map
$$j : BTop(2m) \longrightarrow BTop$$

Claim: $j_* : \underset{x}{\pi^{\mathbb{Q}}_{4k+4m} BTop(2m)} \longrightarrow \pi^{\mathbb{Q}}_{4m+4k} BTop$

is nontrivial.

Pf: $\langle \underset{\in H^{4m+4k}(BTop)\otimes\mathbb{Q}}{p_{k+m}}, h j_*(x) \rangle = \langle p_{k+m}, j_* h(x) \rangle$

$= \langle j^* p_{k+m}, h(x) \rangle = \langle \underset{\uparrow\, H^{4m+4k}(BTop(2m))\otimes\mathbb{Q}}{p_{k+m}}, h(x) \rangle$

$$\begin{tikzcd}
\pi^{\mathbb{Q}}_{4m+4k} BTop(2m) \arrow[r, "h"] \arrow[d, "j_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_{4m+4k}(BTop(2m)) \otimes \mathbb{Q} \arrow[d, "j_*"] \\
\pi^{\mathbb{Q}}_{4m+4k} BTop \arrow[r, "h"] & H_{4m+4k}(BTop) \otimes \mathbb{Q}
\end{tikzcd}$$

□

We can take large enough n s.t.
$$j_* : \pi^{\mathbb{Q}}_{4k+4m} BTop(2m) \longrightarrow \pi^{\mathbb{Q}}_{4k+4m} BTop$$
factor through
$$j_{n*} : \pi^{\mathbb{Q}}_{i+n+2} BTop(n) \longrightarrow \pi^{\mathbb{Q}}_{i+n+2} BTop(n+1)$$

claim
$\Rightarrow$ j_{n*} is nontrivial for some i, n

$\Rightarrow$ $\alpha^{\mathbb{Q}}_{n+1*}$ is nontrivial for some i, n.
($i \geq n$
n large enough).

Rk: The same argument shows

Rk: The same argument shows

$$\pi_{i+1}^{\mathbb{Q}} \mathrm{Diff}(D^{n-1},\partial) \xrightarrow{\alpha_{n*}^{\mathbb{Q}}} \pi_i^{\mathbb{Q}} \mathrm{Diff}(D^n,\partial) \xrightarrow{\alpha_{n+1*}^{\mathbb{Q}}} \pi_{i-1}^{\mathbb{Q}} \mathrm{Diff}(D^{n+1},\partial)$$

is nontrivial for some i, n
($i \geq n$, n large enough).

Now show

Th 3.4 Let $i, n \in \mathbb{Z}_{>0}$ and $i \geq n+2$.
Assume

$$\pi_{i+1}^{\mathbb{Q}} \mathrm{Diff}(D^{n-1},\partial) \xrightarrow{\alpha_{n*}^{\mathbb{Q}}} \pi_i^{\mathbb{Q}} \mathrm{Diff}(D^n,\partial) \xrightarrow{\alpha_{n+1*}^{\mathbb{Q}}} \pi_{i-1}^{\mathbb{Q}} \mathrm{Diff}(D^{n+1},\partial)$$

is nontrivial $(*)$.
Then for every closed hyp. mfd M of dim n,
$\exists$ finite sheeted cover $\hat{M}$ of M s.t.

$$F_*: \pi_{i+1}^{\mathbb{Q}} T^{<0}(\hat{M}) \longrightarrow \pi_{i+1}^{\mathbb{Q}} T(\hat{M})$$

is nontrivial.

This is a corollary of Lemmas 3.8, 3.9.

Lemma 3.8 Let N^n be a closed $\boxed{\text{stably parallelizable}}$ mfd admitting nonpositively curved metric and assume $i \geq n+2$. Given a fixed embedding $D^n \overset{\nu}{\hookrightarrow} N$ and let

$\iota = \iota(\nu): \mathrm{Diff}(D^n,\partial) \longrightarrow \mathrm{Diff}_0(N)$ be defined by extending by identity on $N \setminus \nu(D^n)$.

$$\begin{tikzcd}
\pi_i^{\mathbb{Q}} \mathrm{Diff}(D^n,\partial) \arrow[rr, "\alpha_{n+1*}"] \arrow[dd, "\iota_*^{\mathbb{Q}}"'] & & \pi_{i-1}^{\mathbb{Q}} \mathrm{Diff}(D^{n+1},\partial) \\
& x \arrow[r, mapsto, "If"] \arrow[d, mapsto] & \alpha_{n+1*}^{\mathbb{Q}}(x) \neq 0 \\
\pi_i^{\mathbb{Q}} \mathrm{Diff}_0(N) & \iota_*^{\mathbb{Q}} x \neq 0 &
\end{tikzcd}$$

where $\iota: \mathrm{Diff}(D^n,\partial) \longrightarrow \mathrm{Diff}_0(N)$

Lemma 3.9 Let $i \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$.

Lemma 3.9 Let $i\in\mathbb{Z}_{\geq 0}$ and $n\in\mathbb{Z}_{>0}$.

Suppose

$$\begin{tikzcd} \pi_{i+1}^{\mathbb{Q}}\mathrm{Diff}(D^{n-1},\partial) \arrow[r, "\alpha_{n*}\otimes\mathrm{id}_{\mathbb{Q}}"] & \pi_i^{\mathbb{Q}}\mathrm{Diff}(D^n,\partial) \\ \exists\,\cdot \arrow[r, mapsto] & \bar{x} \end{tikzcd}$$

Then $\exists r>0$ with the following property: for any closed real hyp. mfd (N,g) with injectivity radius at least $3r$ at some point, there is an embedding

$$\nu : D^n \longrightarrow N$$

s.t.

$$\begin{tikzcd} & \pi_i^{\mathbb{Q}}\mathrm{Diff}(D^n,\partial) \ni \bar{x} \arrow[d, "\iota(\nu)_*\otimes\mathrm{id}_{\mathbb{Q}}"] \arrow[dd, phantom] & \\ & \pi_i^{\mathbb{Q}}\mathrm{Diff}_0(N) \arrow[d, equal] & \\ \pi_{i+1}^{\mathbb{Q}}T^{<\varepsilon}(N) \arrow[r, "F_*\otimes\mathrm{id}_{\mathbb{Q}}"] & \pi_{i+1}^{\mathbb{Q}}T(N) & \\ \exists\,\cdot \arrow[r, mapsto] & \iota(\nu)_*\bar{x} \arrow[from=1-2, bend left=60, mapsto] & \end{tikzcd}$$

pf Th 3.4 by Lemmas 3.8, 3.9:

$\alpha^{\mathbb{Q}}_{n+1*}\circ\alpha^{\mathbb{Q}}_{n*}$ is nontrivial $\Rightarrow$

$$\begin{tikzcd} \pi_{i+1}^{\mathbb{Q}}\mathrm{Diff}(D^{n-1},\partial) \arrow[r, "\alpha_{n*}^{\mathbb{Q}}"] & \pi_i^{\mathbb{Q}}\mathrm{Diff}(D^n,\partial) \arrow[r, "\alpha_{n+1*}^{\mathbb{Q}}"] & \pi_{i-1}^{\mathbb{Q}}\mathrm{Diff}(D^{n+1},\partial) \\ \cdot \arrow[r, mapsto] & \exists\,\bar{x} \arrow[r, mapsto] & \neq 0. \end{tikzcd}$$

M hyp. $\Rightarrow$ One can take finite sheeted cover $\hat{M}$ of M s.t.

- $\hat{M}$ has a geodesic ball of radius $3r$ (cf. Lemma 3.9).
- $T\hat{M}\oplus\varepsilon^1\cong\varepsilon^{n+1}$.

$\Longrightarrow$ $\exists$ embedding $\nu : D^n \hookrightarrow \hat{M}$ s.t.

$$\pi_i^{\mathbb{Q}}\mathrm{Diff}(D^n,\partial) \ni \bar{x}$$

$$\begin{tikzcd}
 & \pi_i^{\mathbb{Q}}\mathrm{Diff}(D^n,\partial) \arrow[d, "\iota(\nu)_*\otimes\mathrm{id}_{\mathbb{Q}}"] & \ni \bar{x} \arrow[dddl, mapsto, bend left] \\
 & \pi_i^{\mathbb{Q}}\mathrm{Diff}_0(\hat{M}) \arrow[d, equal] & \\
\pi_{i+1}^{\mathbb{Q}}T^{<0}(\hat{M}) \arrow[r, "F_*\otimes\mathrm{id}_{\mathbb{Q}}"] & \pi_{i+1}^{\mathbb{Q}}T(\hat{M}) & \\
\boxed{\exists\ \cdot} & (\iota(\nu)_*\otimes\mathrm{id}_{\mathbb{Q}})(\bar{x}) & \\
\bar{x}\in\mathrm{Im}\,\alpha_{n*}^{\mathbb{Q}} + \hat{M}\supset 3rD^n. \arrow[u, Rightarrow, "\text{Lemma 3.9}"'] & \boxed{\begin{matrix}\nparallel\\ 0\end{matrix}} & \alpha_{n+1*}^{\mathbb{Q}}(\bar{x})\neq 0 + T\hat{M}\oplus\varepsilon^1\cong\varepsilon^{n+1}. \arrow[l, Rightarrow, "\text{Lemma 3.8}"']
\end{tikzcd}$$

Now show Lemma 3.8, it is a corollary following

Lemma 3.10 Let M^n be a closed $\boxed{\text{parallelizable}} \rightarrow TM\cong\varepsilon^n$ (trivial) mfd which admits a nonpositively curved metric and fix an embedding $D^n \overset{\nu}{\hookrightarrow} M$. If $i\geq n\geq 5$.

then $\pi_i\mathrm{Diff}(D^n,\partial)\xrightarrow{\iota_*}\pi_i\mathrm{Diff}_0(M)$. is injective

$(\Rightarrow \iota_*^{\mathbb{Q}}$, too) where $\iota:\mathrm{Diff}(D^n,\partial)\hookrightarrow\mathrm{Diff}_0(M)$

$\wr$ extend by id on $N\setminus\nu(D^n)$.

Pf Lemma 3.8 by 3.10 :

Note: to show $\pi_i^{\mathbb{Q}}\mathrm{Diff}(D^n,\partial)\longrightarrow\pi_i^{\mathbb{Q}}\mathrm{Diff}_0(N)$

$x \longmapsto \neq 0$

it suffices to show $\pi_i^{\mathbb{Q}}\mathrm{Diff}(D^n,\partial)\longrightarrow\pi_i^{\mathbb{Q}}\mathrm{Diff}_0(N)\longrightarrow\pi_i^{\mathbb{Q}}\mathrm{Diff}(N)$

$x \longmapsto \neq 0$

Consider

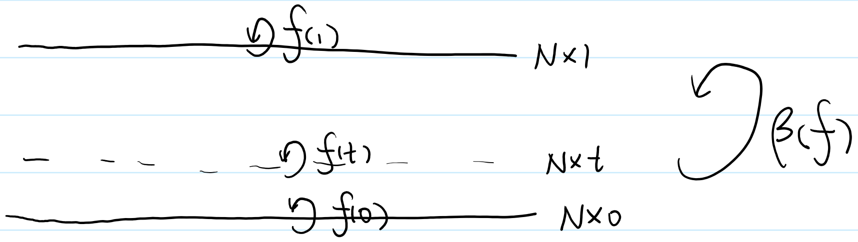
$$\begin{tikzcd}
\pi_i^{Q} \mathrm{Diff}(D^n,\partial) \arrow[r, "\alpha_{n+1*}^{Q}"] \arrow[d, "\iota_*^{Q}"] \arrow[dr, phantom, "\circlearrowright"] & \pi_{i-1}^{Q} \mathrm{Diff}(D^{n+1},\partial) \arrow[d, "\tau_*^{Q}"] \\
\pi_i^{Q} \mathrm{Diff}(N) \arrow[r, "\beta_*^{Q}"] & \pi_{i-1}^{Q} \mathrm{Diff}(N\times[0,1],\partial) \arrow[d, "g_*^{Q}"] \\
 & \pi_{i-1}^{Q} \mathrm{Diff}(N\times S^1)
\end{tikzcd}$$

where

- Analogous to α_{n+1}, we define β:

$$\begin{aligned} \Omega \mathrm{Diff}(N) &\longrightarrow \mathrm{Diff}(N\times[0,1],\partial) \\ f &\longmapsto \beta(f) \end{aligned}$$

$f:[0,1]\to \mathrm{Diff}(N)$, $f(0)=f(1)=id_N$



$$\rightsquigarrow \quad \beta_*: \pi_i \Omega \mathrm{Diff}(N) \longrightarrow \pi_i \mathrm{Diff}(N\times[0,1],\partial)$$
$$\pi_i \Omega \mathrm{Diff}(N) \cong \pi_{i+1} \mathrm{Diff}(N)$$

- $\tau: \mathrm{Diff}(D^{n+1},\partial) \longrightarrow \mathrm{Diff}(N\times[0,1],\partial)$

$\downarrow$ extend by id outside the image of the embedding: $D^{n+1}=D^n\times[0,1] \xrightarrow{\iota\times id_{[0,1]}} N\times[0,1]$

- $g: \mathrm{Diff}(N\times[0,1],\partial) \longrightarrow \mathrm{Diff}(N\times S^1)$

$\wr$ extend by id outside the image of

an embedding $N\times[0,1] \longrightarrow N\times S^1$.

To show $\iota_*^{\mathbb{Q}} x \neq 0$, it suffices to show

$$\ell_*^{\mathbb{Q}} \beta_*^{\mathbb{Q}} \iota_*^{\mathbb{Q}} x \neq 0$$
$$\|$$
$$\ell_*^{\mathbb{Q}} \tau_*^{\mathbb{Q}} \alpha_{n+1*}^{\mathbb{Q}} x$$

Since $\alpha_{n+1*}^{\mathbb{Q}} x \neq 0$, it suffices to show

$\ell_*^{\mathbb{Q}} \tau_*^{\mathbb{Q}}$ is injective.

Since $N\times S^1$ is parallelizable and admits nonpositively curved metric

Lemma 3.10 $\Longrightarrow$ $\ell_*^{\mathbb{Q}} \tau_*^{\mathbb{Q}}$ is injective for $i-1 \geq n+1 \geq 5$

(i.e. $i \geq n+2$, $n \geq 4$.)

$\square$

Rk: Lemma 3.8 is vacuous if $n \leq 3$

since $\mathrm{Diff}(D^n, \partial) \simeq *$ for $n \leq 3$.

cf. [S59] Smale, "Diffeomorphisms of the 2-sphere", 1959

[H83] Hatcher, "A proof of the smale conjecture, $\mathrm{Diff}(S^3) \simeq O(4)$", 1983

Pf Lemma 3.10: The proof is divided into two steps:

S1 $\eta_*: \pi_{i+1} \mathrm{Map}\left(D^n, \partial, \frac{Top(n)}{O(n)}\right) \longrightarrow \pi_{i+1} \mathrm{Map}\left(M, \frac{Top(n)}{O(n)}\right)$

is injective for $i \geq n$.

S2: $\iota_*: \pi_i \mathrm{Diff}(D^n,\partial) \longrightarrow \pi_i \mathrm{Diff}_0(M)$ is injective

for $i \geq n \geq 5$.

Notation:

- $\mathrm{Map}(D^n;\partial, \frac{Top(n)}{O(n)}) \triangleq \left\{ (D^n,\partial) \longrightarrow (\frac{Top(n)}{O(n)}, *) \right\}$.
  with compact-open top.
- $\mathrm{Map}(M, \frac{Top(n)}{O(n)}) \triangleq \left\{ M \xrightarrow{cont.} \frac{Top(n)}{O(n)} \right\}$.
- $\eta: \mathrm{Map}(D^n;\partial, \frac{Top(n)}{O(n)}) \longrightarrow \mathrm{Map}(M, \frac{Top(n)}{O(n)})$ is given by extending each $f: (D^n,\partial) \longrightarrow (\frac{Top(n)}{O(n)}, *)$ via the constant map outside $D^n \overset{\iota}{\subset} M$.

① S1 ⇒ S2:

Consider

$$\begin{tikzcd}
\pi_i \mathrm{Diff}(D^n,\partial) \arrow[d, "\iota_*"] & \pi_{i+1} \frac{Top(D^n,\partial)}{\mathrm{Diff}(D^n,\partial)} \arrow[l, "\partial_{D^n}"', "\cong"] \arrow[r, "\phi(D^n)_*", "\cong"'] \arrow[d, "\bar{\iota}_*"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\boxed{\circlearrowleft}"] & \pi_{i+1} \mathrm{Map}(D^n;\partial, \frac{Top(n)}{O(n)}) \arrow[d, "\eta_*"] \\
\pi_i \mathrm{Diff}_0(M) & \pi_{i+1} \frac{Top_0(M)}{\mathrm{Diff}_0(M)} \arrow[l, "\partial_M"] \arrow[r, "\phi(M)_*", "\cong"'] & \pi_{i+1} \mathrm{Map}(M, \frac{Top(n)}{O(n)}).
\end{tikzcd}$$

(***)

cf. [BL74, Cor. 4.2]

Morlet's isomorphism

where

Let $\mathbb{T}op_0(M) \triangleq$ the singular complex of $Top_0(M)$

$\mathbb{D}iff_0(M) \triangleq$ the singular differentiable complex of $Diff_0(M)$

$\mathbb{D}iff_0(M) \overset{\text{simplicial subset}}{\subset} \mathbb{T}op_0(M)$

$\rightsquigarrow \dfrac{\mathbb{T}op_0(M)}{\mathbb{D}iff_0(M)}$ (simplicial set).

$\dfrac{Top_0(M)}{Diff_0(M)} \triangleq \left| \dfrac{\mathbb{T}op_0(M)}{\mathbb{D}iff_0(M)} \right|$: geometric realization

Similarly define $\dfrac{Top(D^n,\partial)}{Diff(D^n,\partial)}$

- $\iota : \dfrac{Top(D^n,\partial)}{Diff(D^n,\partial)} \longrightarrow \dfrac{Top_0(M)}{Diff_0(M)}$:

$\iota : Diff(D^n,\partial) \longrightarrow Diff_0(M)$ (as a map between sets)

"extends" to $Top(D^n,\partial) \longrightarrow Top_0(M)$

$\rightsquigarrow \dfrac{\mathbb{T}op(D^n,\partial)}{\mathbb{D}iff(D^n,\partial)} \longrightarrow \dfrac{\mathbb{T}op_0(M)}{\mathbb{D}iff_0(M)}$

$\rightsquigarrow \dfrac{Top(D^n,\partial)}{Diff(D^n,\partial)} \xrightarrow{\iota} \dfrac{Top_0(M)}{Diff_0(M)}$

- $\phi(M)_*$:

Th (BL74) M^n: closed, smooth, parallelizable mfd of dim $n \neq 4$. Then $\exists$ cont. map mfd of dim $n \neq 4$. Then $\exists$ cont. map

$$\phi(M): \frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)} \longrightarrow \mathrm{Map}\left(M, \frac{\mathrm{Top}(n)}{O(n)}\right).$$

inducing $\cong$ on π_i for all $i>0$.

- $\partial_{D^n}, \partial_M$: Consider homotopy exact sequence for fibrations:

$$|\mathrm{Top}(D^n,\partial)| \longrightarrow \left|\frac{\mathrm{Top}(D^n,\partial)}{\mathrm{Diff}(D^n,\partial)}\right| = \frac{\mathrm{Top}(D^n,\partial)}{\mathrm{Diff}(D^n,\partial)}$$

$$|\mathrm{Top}_0(M)| \longrightarrow \left|\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)}\right| = \frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)}$$

$$\Rightarrow \begin{tikzcd}
\pi_{i+1}\frac{\mathrm{Top}(D^n,\partial)}{\mathrm{Diff}(D^n,\partial)} \arrow[r, "\bar{\partial}_{D^n}"] \arrow[d, "\iota_*"'] \arrow[rr, bend left, "\partial_{D^n}"] \arrow[dr, phantom, "\circlearrowright"] & \pi_i|\mathrm{Diff}(D^n,\partial)| \arrow[r, "\cong"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \pi_i\mathrm{Diff}(D^n,\partial). \arrow[d, "\iota_*"] \\
\pi_{i+1}\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)} \arrow[r, "\bar{\partial}_M"] \arrow[rr, bend right, "\partial_M"'] & \pi_i|\mathrm{Diff}_0(M)| \arrow[r, "\cong"'] & \pi_i\mathrm{Diff}_0(M)
\end{tikzcd}$$

- ∂_M is injective: it suffices to show $\bar{\partial}_M$ is injective.

<u>Th</u> M^n: closed, smooth mfd admitting nonpos. curved metric, $n \geq 5$. then $\mathrm{Diff}(M) \longrightarrow \mathrm{Top}(M)$ induces surjective homomorphisms on π_j for all $j \geq 2$.

(cf. [FW91, Cor. 4.3])

$$\begin{tikzcd}
\pi_{i+1}|\mathrm{Diff}_0(M)| \arrow[r, two heads, "\text{for } i\geq 1"] \arrow[d, "\cong"] & \pi_{i+1}|\mathrm{Top}_0(M)| \arrow[r, "0"] \arrow[d, "\cong"] & \pi_{i+1}\frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)} \arrow[r, tail, "\bar{\partial}_M"] & \pi_i|\mathrm{Diff}_0(M)| \\
\pi_{i+1}\mathrm{Diff}_0(M) \arrow[r, two heads] \arrow[d, "\cong"] & \pi_{i+1}\mathrm{Top}_0(M) \arrow[d, "\cong"] & & \\
\pi\,\mathrm{Diff}(M) \arrow[r, two heads] & \pi\,\mathrm{Top}(M) & &
\end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[d, "\cong"] & {} \arrow[d, "\cong"] \\ \pi_{i+1}\mathrm{Diff}(M) \arrow[r, two heads, "\text{for } i\geq 1"'] & \pi_{i+1}\mathrm{Top}(M) \end{tikzcd}$$

Consequently, (***) $\Rightarrow$ "S1 $\Rightarrow$ S2".