2020-10-6 Kähler geometry

Next time is Sunday 10月11日 於上海

First Bianchi identity

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

Second Bianchi identity

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.$$

Riemannian

Kähler case

$$\nabla_i R_{j\bar k l\bar m} + \nabla_j R_{\bar k i l\bar m} + \nabla_{\bar k} \underbrace{R_{ijl\bar m}}_{=0} = 0$$

$$\nabla_i R_{j\bar k l\bar m} = \nabla_j R_{i\bar k l\bar m}$$

$$\nabla_i R_{j\bar k}{}^{p}{}_{p} = \nabla_j R_{i\bar k}{}^{p}{}_{p}$$

$$\nabla_i R_{j\bar k} = \nabla_j R_{i\bar k}$$

Ricci

scalar curvature

$$\nabla_i R_j{}^{i} = \nabla_j R_i{}^{i} = \nabla_j S$$

Cor $S = \text{const} \iff \underbrace{\operatorname{div} R}_{=\partial^* \rho} = 0.$

$$\rho = i R_{i\bar{j}} dz^i \wedge d\bar{z}^j$$ Ricci form.

Recall $\bar{\partial}\rho = 0$ $\quad 0 = d\rho = (\partial + \bar{\partial})\rho$

Cor Scal $=$ const $\Longleftrightarrow$ ρ is a harmonic form.

Recall Hodge theory

$$\Delta_d = d^* d + d d^* \qquad , \qquad \Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial}\bar{\partial}^*$$

$$(= -\nabla^i \nabla_i f) \qquad \Delta_\partial = \partial^* \partial + \partial \partial^*$$

Kähler case

$$\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$$

$$\ker \Delta_d = \ker \Delta_{\bar{\partial}} = \ker \partial.$$

$\alpha \in \ker \Delta_{\bar{\partial}}$ is called a harmonic form

$\Longleftrightarrow$ $\bar{\partial}\alpha = 0$, $\bar{\partial}^*\alpha = 0$.

(e) harmonic form is unique in each cohomology class.

Suppose M is Fano, $c_1(M) > 0$.

$$\omega \in c_1(M)$$

ω is harmonic, $\omega \in c_1(M)$

$\rho \in c_1(M)$

$s = \text{const} \iff \rho$ is harmonic

$\iff \rho = \omega \iff \omega$ K-E.

K-E problem $=$ <u>cscK</u> problem in $c_1(M)$

constant scalar curvature Kähler metric. $i = 1, \cdots, 2m$ TM

<u>Remark</u> Riem $g^{ij} R_{kiej}$ $\sum_i R(X, e_i, Y, e_i)$

Kähler $g^{i\bar{j}} R_{p\bar{j}q\bar{i}} = R_{p\bar{q}}$ $\leftarrow$ <u>Our Ricci</u>

$R(X, \bar{f}_i, Y, f_i)$ <u>TM</u>

$$f_i = \frac{1}{\sqrt{2}}(e_i - iJe_i)$$

<u>$i = 1, \cdots, m.$</u>

Note <u>Our Ricci</u> $= \frac{1}{2}$ (Riem Ricci)

Our Scal $= \frac{1}{4}$ Riem Scal

2 Δ. Rien Mri

- $$\Delta f = \lambda f \implies \lambda \le 0$$
$$\Delta f = \nabla^i \nabla_i f$$

$$\int \Delta f \cdot f = -\int \nabla^i f \cdot \nabla_i f$$
$$\lambda \|f\|_{L^2} \qquad\qquad -\|\nabla f\|_{L^2}$$

$$\lambda = 0 \iff f \text{ const.}$$

- Hodge decomp.

$$\alpha = H\alpha + \Delta G\alpha$$
$$= H\alpha + dd^* G\alpha + d^* dG\alpha.$$
$$= \text{harmonic part} + d\text{-exact part}$$
$$+ d^*(\qquad)$$

Lichnerowicz - Matsushima theorem (1956) ⑥

cscK. KE

Theorem Let M be a compact Kähler manifold with constant scalar curvature. Then the Lie algebra $\underset{\sim}{h}(M)$ of all holomorphic vector fields is reductive, i.e. $\exists$ real Lie alg k of a compact Lie group K such that

German h

$$h(M) = k \otimes \mathbb{C}.$$

abeli semi-simple part

$$\Longleftrightarrow \quad h = \underline{\underline{n}} + [h, h]$$

Proof For simplicity we assume $\underline{H^1(M) = 0}$.

$X = X^i \frac{\partial}{\partial z^i} \in h(M)$ holo vector field

$$\Longleftrightarrow \quad \underline{\nabla_{\bar{j}} X^i = 0} \qquad \underline{\nabla_{\bar{j}} X^i} = \underline{\frac{\partial X^i}{\partial \bar{z}_j}} + \underline{\Gamma^{i}_{\bar{j}\ldots} A^{\ldots}}$$

$$\Longleftrightarrow \quad \underline{\nabla_{\bar{j}} X_{\bar{i}} = 0} \qquad \nabla_{\bar{j}} \left(g_{\bar{i}k} X^k \right)$$

Put $\alpha = g_{i\bar{k}} X^i d\bar{z}^k = X_{\bar{k}} d\bar{z}^k$

Then $\bar\partial\alpha = \nabla_{\bar j} X_{\bar k}\, d\bar z^j \wedge d\bar z^k$

$$\frac{\partial X_{\bar k}}{\partial \bar z^j} + \cancel{\Gamma^{\bar l}_{\bar j \bar k} X_{\bar l}}$$

$= 0$

Since we assume $H^1(M) = 0$, we have

$\alpha = \bar\partial u$, $u \in C^\infty(M) \otimes \mathbb{C}$

$(u \in \mathbb{C})$

$$\mathfrak{f}(M) \cong \left\{ u \in C^\infty(M) \otimes \mathbb{C} \;\middle|\; \nabla_{\bar j} \nabla_{\bar k} u = 0, \int_M u\, \omega^n = 0 \right\}$$

Lemma $\nabla_{\bar j} \nabla_{\bar k} u = 0 \iff \nabla^{\bar j} \nabla^{\bar k} \nabla_{\bar j} \nabla_{\bar k} u = 0$.

$\because$ $(\Leftarrow)$

$$0 = \int \bar u \cdot \nabla^{\bar j} \nabla^{\bar k} \nabla_{\bar j} \nabla_{\bar k} u \ \omega^n$$

$$= -\int \nabla^{\bar j} \bar u \cdot \nabla^{\bar k} \nabla_{\bar j} \nabla_{\bar k} u \ \omega^n$$

integration by parts

$$= +\int \nabla^{\bar k} \nabla^{\bar j} \bar u \cdot \nabla_{\bar j} \nabla_{\bar k} u \cdot \omega^n$$

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$$= \int g^{l\bar{j}} g^{m\bar{k}} \overline{\nabla_{\bar{l}} \nabla_{\bar{m}} u} \cdot \nabla_{\bar{j}} \nabla_{\bar{k}} u \, \omega^n$$

$$= \| \nabla'' \nabla'' u \|^2_{L^2} \qquad \underset{(1,0)}{\nabla = \nabla'} + \underset{(0,1)}{\nabla''}$$

$$\therefore \nabla'' \nabla'' u = 0$$

$$\therefore \nabla_{\bar{i}} \nabla_{\bar{j}} u = 0$$

$(\Longrightarrow)$ Trivial. ;)

Put $Du = \nabla^{\bar{j}} \nabla^{\bar{k}} \nabla_{\bar{j}} \nabla_{\bar{k}} u$

"Lichnerowicz operator"

$$\therefore \mathfrak{f}(M) = \{ u \in C^\infty(M) \otimes \mathbb{C} \mid Du = 0, \ \int u \, \omega^n = 0 \}$$

$$Du = \nabla^{\bar{i}} \left(\nabla^{\bar{k}} \nabla_{\bar{j}} (\nabla_{\bar{k}} u) \right)$$

Ricci formula

$$= \nabla^{\bar{i}} \left(\nabla_{\bar{j}} \nabla^{\bar{k}} \nabla_{\bar{k}} u - R^{\bar{k}}{}_{\bar{j}}{}^{\bar{l}}{}_{\bar{k}} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + \nabla^{\bar{i}} \left(R^{\bar{l}}{}_{\bar{k}\bar{j}}{}^{\bar{k}} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + \nabla^{\bar{i}} \left(R^{\bar{l}k}{}_{\bar{j}k} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + \nabla^{\bar{i}} \left(R^{\bar{l}}{}_{\bar{i}} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + R^{\bar{l}}{}_{\bar{j}}\, \nabla^{j} \nabla_{\bar{l}} u + \nabla^{j} R^{\bar{l}}{}_{\bar{j}}\, \nabla_{\bar{l}} u$$

$$= \underbrace{\Delta^2 u + R^{i\bar{j}} \nabla_i \nabla_{\bar{j}} u}_{Du} + \cancel{\nabla_i S \, \partial^i u}$$

$$\left(\Delta u = g^{i\bar{j}} \left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} + \underbrace{\Gamma^A_{i\bar{j}}}_{0} \frac{\partial u}{\partial z^A} \right) \right)$$

$$\overline{\Delta u} = \Delta \bar{u} \qquad \left(\Delta_{\bar{\partial}} = \frac{1}{2} \Delta_d \right)$$

$\therefore$ Δ is a real operator ".

$$\overline{R^{i\bar{j}} \nabla_i \nabla_{\bar{j}} u} = R^{\bar{i}j} \nabla_{\bar{i}} \nabla_j \bar{u}$$

$$= R^{j\bar{i}} \nabla_j \nabla_{\bar{i}} \bar{u}$$

$$= R^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \bar{u}$$

By assumption $S = \text{const}$ $\therefore \nabla_i S = 0$

So $\overline{Du} = D\bar{u}$

D is a real operator.

$$\begin{cases} u = f + ig & \text{real part + imaginary part.} \\ Du = 0 \end{cases}$$

$$\Rightarrow \quad 0 = \overline{Du} = D\bar{u} = Df - iDg$$

$$0 = Du = Df + iDg$$

$$\therefore \quad Df = Dg = 0$$

$$D(ig) = 0 \iff \nabla'' \nabla'' ig = 0$$

Well known fact.

For a purely imaginary function ig, if the gradient of ig generates a holomorphic vector field it generates a Killing vector field.

(Kobayashi's book
Transformation groups in differential geometry)

$\therefore \; \mathfrak{h}(M) = \{ X + iY \mid X, Y \text{ are Killing} \}$

$\mathfrak{h}(M) = \mathfrak{k} \otimes \mathbb{C}$

$\mathfrak{k}$ = Killing vector fields.

= Lie algebra of the Lie group of all isometries. K.

$K = \mathrm{Isom}_0(M)$ = compact if M is compact.

well known in Riem geom.

<u>Def</u> A Kähler metric g is an extremal Kähler metric if

$$g^{i\bar j} \frac{\partial S}{\partial \bar z^{j}} \frac{\partial}{\partial z^{i}} \quad \left(= \mathrm{grad}' S \right)$$

is a holomorphic vector field.

Fact (Calabi) Ω Kähler class

$$\Phi : \Omega_+ \to \mathbb{R}, \qquad \Phi(g) = \int_M \mathrm{Scal}(g)^2 \, \omega_g^m$$

↑ Kähler forms

Φ is called the Calabi energy.

g is extremal $\Longleftrightarrow$ g is a critical point of Φ.

Example

$$\{KE\} \subset \{\underline{\text{csck}}\} \subset \{\underline{\text{extremal}}\}.$$

Matsushima Lichnerowicz Calabi

Th (Calabi) 1980's $\quad \underline{\mathfrak{h}(M)} = \sum \mathfrak{h}_\lambda(M)$

$$\mathfrak{h}_\lambda = \{ X \in \mathfrak{h}(M) \mid [\underline{\text{grad}' s}, X] = \lambda X \}$$

$\boxed{\mathfrak{h}_0 = \mathfrak{k} \otimes \mathbb{C}}$ reductive.