- p prime number
- $\mathbb{F}$: algebraic closure of $\mathbb{F}_p$
- $\mathbb{F} = \bigcup_{d \geqslant 1} \mathbb{F}_{p^d}$
- "Variety" = quasi-projective algebraic variety over $\mathbb{F}$ (not necessarily irreducible $\neq$ Hartshorne)
- $GL_n(\mathbb{F}) = \bigcup_{d \geqslant 1} GL_n(\mathbb{F}_{p^d})$

$\Rightarrow$ All elements of $GL_n(\mathbb{F})$ have finite order.

PART II. Deligne - Lusztig theory for finite reductive groups

Chapter 5. Review of algebraic groups

Chapter 6. $\mathbb{F}_q$ - structures on alg. groups

Chapter 7. Harish - Chandra theory

Chapter 8. Deligne - Lusztig theory

Chapter 9. Application, open problems

Chapter 5. Review on alg. groups.

Ref. Borel or Humphrey or Springer, "Linear alg. groups"

5.A. Definition, examples

Definition 5.1. An algebraic group $\mathbb{G}$ is a variety and a group such that the multiplication $\mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$ and the inverse $\mathbb{G} \longrightarrow \mathbb{G}$ are morphisms of varieties.

A morphism of alg. groups is a map which is both a morphism of varieties and a morphism of groups.

Examples 5.2. (0) A finite group is an algebraic group of dim. 0.

(1) $\mathbb{F}^+ = \mathbb{G}_a$: $\mathbb{F}$ endowed with addition

(1') $\mathbb{F}^\times = \mathbb{G}_m$: $\mathbb{F}^\times$ endowed with multiplication

(1'') $\mathcal{E}$ smooth elliptic curve (projective)

Theorem 5.3. Any connected alg. group of dim. 1 is of the form (1), (1') or (1'').

(2) $GL_n(\mathbb{F})$; the map $\det : GL_n(\mathbb{F}) \longrightarrow \mathbb{F}^\times$ is a morphism of alg. groups.

(3) Any closed subgroup of an alg. group is an alg. group : e.g. $SL_n(\mathbb{F})$, $Sp_{2n}(\mathbb{F})$, ... or

$$\mathbb{B}_n = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset GL_n(\mathbb{F})$$

$$\mathbb{U}_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \subset \mathbb{B}_n \subset GL_n(\mathbb{F})$$

$$\mathbb{T}_n = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset GL_n(\mathbb{F})$$

Theorem 5.4. If $\mathbb{G}$ is an affine alg. group, then there exists $n \geqslant 0$ and a closed immersion $\mathbb{G} \hookrightarrow GL_n(\mathbb{F})$ which is a morphism of alg. groups ("embedding").

"Linear alg. group" = "affine alg. group".

(4) A product of alg. groups is an alg. group
$\left(\mathbb{T}_n \simeq (\mathbb{F}^\times)^n \right)$.

Theorem 5.5. Any alg. group is smooth.
In particular, $\mathbb{G}$ is connected if and only if $\mathbb{G}$ is irreducible.

Theorem 5.6. Let $\pi : \mathbb{G} \to \mathbb{G}'$ be a morphism of alg. groups. Then $\pi(\mathbb{G})$ is closed.

Proposition 5.7. The connected component of 1 is a normal subgroup of $\mathbb{G}$ (denoted by $\mathbb{G}^\circ$) and $\mathbb{G}/\mathbb{G}^\circ$ is finite.

Theorem 5.8. A projective connected alg. group is abelian.

"Abelian varieties" = "Proj. alg. groups"

> From now on, all alg. groups are assumed to be linear (i.e. affine)

We fix an alg. group $\mathbb{G}$.

5.B. Jordan decomposition.

Definition 5.9. An element of $\mathbb{G}$ is called semisimple (resp. unipotent) if its action on $\mathcal{O}(\mathbb{G})$ is locally semisimple (resp. unipotent).

locally → restriction to any g-stable fin. dim. subspace

semisimple → i.e. diagonalizable

Theorem 5.10. Let $g \in \mathbb{G}$. Then:

(a) g is semisimple (resp. unipotent) if and only if $\rho(g)$ is semisimple (resp. unipotent) for some (or any) closed embedding $\rho : \mathbb{G} \hookrightarrow GL_n$.

(b) If $\rho : \mathbb{G} \longrightarrow \mathbb{G}'$ is a morphism and if g is semisimple (resp. unipotent), then $\rho(g)$ is semisimple (resp. unipotent).

(c) There exists a unique decomposition $g = su$ with $su = us$, s semisimple and u unipotent

(Jordan decomposition)

Remark 5.11 (specific to char. p). Let $g \in GL_n$.

Then g is unipotent $\iff \exists k \geq 1$ s.t. $(g - I_n)^k = 0$

$\iff \exists k \geq 1$ s.t. $(g - I_n)^{p^k} = 0$

$\iff \exists k \geq 1$ s.t $g^{p^k} = I_n$.

$\iff g$ is a p-element.

Also, g is semisimple $\iff g \sim \mathrm{diag}(\xi_1, \dots, \xi_n)$, with $\xi_1, \dots, \xi_n \in \mathbb{F}^\times$

$\Rightarrow \mathrm{Gcd}(p, o(g)) = 1$

$\iff g$ is a p'-element.

Conversely, if $g^m = I_n$ for some m not divisible by p, then g is semisimple since the polynomial $X^m - 1$ has only simple roots. ∎

Theorem 5.12. If $G \subset GL_n(\mathbb{F})$ and if all its elements are unipotent, then there exists $g \in GL_n(\mathbb{F})$ such that $gGg^{-1} \subset U_n$.

G is called unipotent if all its element are unipotent; G is called a torus if $G \simeq (\mathbb{F}^\times)^n$, for some $n \geq 0$.

5.C. Quotients of varieties

Let X be a variety on which G acts regularly (i.e. $G \times X \longrightarrow X$, $(g, x) \longmapsto g \cdot x$ is a morphism of varieties).

Proposition 5.13. Any G-orbit is locally closed in X.

Definition 5.14. Let $\pi : X \longrightarrow Y$ be a morphism of varieties.

(a) π is called an orbit map if $\pi^{-1}(y)$ is a G-orbit for any $y \in Y$.

(b) π is called a (geometric) quotient if π is an orbit map such that

(b1) π is open

(b2) For any U open in Y, the map $\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\pi^{-1}(U))^G$ is an isomorphism.

If $\pi : X \longrightarrow Y$ is a quotient, then we write $Y = X/G$: by (a), as a set, we have $Y \leftrightarrow X/G$.

If π is an orbit map, then all orbits are closed.

Proposition 5.15. Let $\pi : X \to Y$ be a quotient. If $\varphi : X \to X'$ is a morphism which is constant on G-orbits, then there exists a unique morphism $\bar{\varphi} : Y \to X'$ making the following diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & Y \arrow[d, dashed, "\bar{\varphi}"] \\ & X' \end{tikzcd}$$

commutative.

$\Rightarrow$ If there exists a quotient, it is unique.

Theorem 5.16. Let $\pi : X \to Y$ be an orbit map. Assume that:

(1) Y is normal (for instance smooth)

(2) The irreducible components of X are its connected components

(3) There exists a smooth point $x \in X$ such that $\pi(x)$ is smooth and the map $d_x \pi : T_x X \longrightarrow T_{\pi(x)} Y$ is surjective.

Then π is a quotient.

Examples 5.17.

(1)
$$\begin{aligned} \mathbb{A}^n(\mathbb{F}) \setminus \{0\} &\longrightarrow \mathbb{P}^{n-1}(\mathbb{F}) \\ (x_1, \dots, x_n) &\longmapsto [x_1, \dots, x_n] \end{aligned}$$
is a/the quotient of $\mathbb{A}^n(\mathbb{F}) \setminus \{0\}$ by $\mathbb{F}^\times$ (acting by homotheties).

(2)
$$\begin{aligned} GL_2(\mathbb{F}) &\longrightarrow \mathbb{P}^1(\mathbb{F}) \\ g &\longmapsto g \cdot [1, 0] \end{aligned}$$
is the quotient of $GL_2(\mathbb{F})$ by B_2.

(3)
$$\begin{aligned} SL_2(\mathbb{F}) &\longrightarrow \mathbb{A}^2(\mathbb{F}) \setminus \{0\} \\ g &\longmapsto g \cdot (1, 0) \end{aligned}$$
is the quotient of $SL_2(\mathbb{F})$ by U_2.

Theorem 5.18. If $\mathbb{H}$ is a closed subgroup of $\mathbb{G}$, then the quotient $\mathbb{G}/\mathbb{H}$ exists.

If $\mathbb{H}$ is normal, then $\mathbb{G}/\mathbb{H}$ a linear algebraic group.

If $\mathbb{H} \triangleleft \mathbb{G}$ and $\mathbb{H} \subset \mathbb{H}' \subset \mathbb{G}$, then

$$\mathbb{G}/\mathbb{H}' \simeq (\mathbb{G}/\mathbb{H}) / (\mathbb{H}'/\mathbb{H}).$$

5.D. Particular subgroups.

Definition 5.19. (1) The unipotent radical of $\mathbb{G}$ is the maximal normal connected unipotent alg. subgroup of $\mathbb{G}$. It is denoted by $R_u(\mathbb{G})$.

(2) A Levi complement in $\mathbb{G}$ is an alg. subgroup $\mathbb{L}$ such that $\mathbb{G} = \mathbb{L} \ltimes R_u(\mathbb{G})$.

(3) A Borel subgroup of $\mathbb{G}$ is a maximal connected solvable alg. subgroup of $\mathbb{G}$.

(4) A parabolic subgroup of $\mathbb{G}$ is an alg. subgroup $\mathbb{P}$ such that $\mathbb{G}/\mathbb{P}$ is a projective variety.

Theorem 5.20. (a) All Borel subgroups are conjugate in $\mathbb{G}$.

(b) All maximal tori are conjugate in $\mathbb{G}$.

(c) A subgroup of $\mathbb{G}$ is parabolic if and only if it contains a Borel subgroup.

(d) If $\mathbb{G}$ is connected and solvable and if $\mathbb{T}$ is a maximal torus, then $\mathbb{G} = \mathbb{T} \ltimes R_u(\mathbb{G})$ and $R_u(\mathbb{G}) = \{g \in \mathbb{G} \mid g \text{ is unipotent}\}$

Example 5.21. $\mathbb{B}_n$ is solvable. Indeed,

$$D(\mathbb{B}_n) = \mathbb{U}_n \qquad (\text{exercise})$$

and $\mathbb{U}_n$ is nilpotent (exercise).

Moreover

$$\mathbb{B}_n = \mathbb{T}_n \ltimes \mathbb{U}_n \quad (\text{exercise}). \blacksquare$$

If $\mathbb{B}$ is a Borel subgroup of $\mathbb{G}$, then $\mathbb{G}/\mathbb{B}$ is called the flag variety of $\mathbb{G}$ (it is projective).

5.E. Reductive groups.

Definition 5.22. The group $\mathbb{G}$ is called reductive if $R_u(\mathbb{G}) = 1$.

Examples 5.23. (a) $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$, $PGL_n(\mathbb{F}) = GL_n(\mathbb{F})/\mathbb{F}^\times$, $Sp_{2n}(\mathbb{F})$, $SO_n(\mathbb{F})$, $O_n(\mathbb{F})$, ...

(b) $\mathbb{B}_n$ is not reductive if $n \geqslant 2$.

(c) A torus is reductive.

(d) A finite group is reductive.

(e) A Levi complement is reductive. $\blacksquare$

Let $\mathbb{B}$ be a Borel subgroup of $\mathbb{G}$.
Let $\mathbb{T}$ be a maximal torus of $\mathbb{B}$.
Then $\mathbb{T}$ is a maximal torus of $\mathbb{G}$ (exercise).

<u>Theorem 5.24.</u> Assume that $\mathbb{G}$ is connected reductive.
Let $W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$. Then:

(a) W is finite and $C_{\mathbb{G}}(\mathbb{T}) = \mathbb{T}$

(b) $\mathbb{G} = \dot{\bigcup}_{w \in W} \mathbb{B} w \mathbb{B}$ (Bruhat decomposition)

(c) Let $S = \{ s \in W \mid \dim \mathbb{B}s\mathbb{B} = \dim \mathbb{B} + 1 \}$
Then $s^2 = 1$ for all $s \in S$ and $W = \langle S \rangle$.
Moreover (W, S) is a Coxeter group.

W is called the Weyl group of $\mathbb{G}$ (relative to $\mathbb{T}$).

(d) If $\mathbb{P}$ is a parabolic subgroup of $\mathbb{G}$, then $\mathbb{P}$ admits a Levi complement.

A Levi complement of a parabolic subgroup of $\mathbb{G}$ is called a Levi subgroup of $\mathbb{G}$.

Moreover, $N_{\mathbb{G}}(\mathbb{P}) = \mathbb{P}$.

If $\mathbb{T}$ is a maximal torus of $\mathbb{P}$, then there exists a unique Levi complement containing $\mathbb{T}$

(e) If $I \subset S$, let $W_I = \langle I \rangle$ and let $\mathbb{P}_I = \dot{\bigcup}_{w \in W_I} \mathbb{B} w \mathbb{B}$. Then $\mathbb{P}_I$ is a (parabolic) subgroup of $\mathbb{G}$

Note that $\mathbb{P}_\emptyset = \mathbb{B}$ and $\mathbb{P}_S = \mathbb{G}$.

(f) If $\mathbb{P}$ is a parabolic subgroup of $\mathbb{G}$ containing $\mathbb{B}$, then there exists a unique $I \subset S$ such that $\mathbb{P} = \mathbb{P}_I$.

(g) Let $I, J \subset S$ be such that $\mathbb{P}_I$ and $\mathbb{P}_J$ are conjugate in $\mathbb{G}$.
Then $I = J$.

(h) Let $\mathbb{L}_I$ be the Levi complement of $\mathbb{P}_I$ containing $\mathbb{T}$. Then $W_I = N_{\mathbb{L}_I}(\mathbb{T})/\mathbb{T}$ (so it is the Weyl group of $\mathbb{L}_I$) and $\mathbb{B} \cap \mathbb{L}_I$ is a Borel subgroup of $\mathbb{L}_I$.
Moreover

$$\mathbb{B} = (\mathbb{B} \cap \mathbb{L}_I) \ltimes R_u(\mathbb{P}_I).$$

5.F. Example : $\mathbb{G}_n = GL_n(\mathbb{F})$

Proposition 5.25. $\mathbb{T}_n$ is a maximal torus.

Proof. Let $\mathbb{T}$ be a torus in $\mathbb{G}_n$. Then all elements of $\mathbb{T}$ are semisimple (i.e. diagonalizable) and they commute: an easy exercise in linear algebra tells you that they are diagonalizable in a common basis. In other words, there exists $g \in \mathbb{G}_n$ such that $g\mathbb{T}g^{-1} \subset \mathbb{T}_n$. ∎

Proposition 5.26. $W = N_{\mathbb{G}_n}(\mathbb{T}_n)/\mathbb{T}_n \simeq S_n$

Proof. $\mathbb{G}_n$ acts on $\mathbb{P}^{n-1}(\mathbb{F})$ and

$$\mathbb{P}^{n-1}(\mathbb{F})^{\mathbb{T}_n} = \{p_1, \dots, p_n\}$$

where $p_i = [0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0]$

Therefore $N_{\mathbb{G}_n}(\mathbb{T}_n)$ permutes the p_i's.

$\Rightarrow N_{\mathbb{G}_n}(\mathbb{T}_n)$ is the group of monomial matrices (i.e. exactly one non-zero coef on each row and on each line)

$\Rightarrow S_n \simeq \{\text{permutation matrices}\} \subset N_{\mathbb{G}_n}(\mathbb{T}_n)$

and $N_{\mathbb{G}_n}(\mathbb{T}_n) = S_n \ltimes \mathbb{T}_n$. ∎

Exercise 5.27. Show that $N_{\mathbb{G}_n}(\mathbb{U}_n) = \mathbb{B}_n$

Proposition 5.28. $\mathbb{B}_n$ is a Borel subgroup.

Proof. The group $\mathbb{B}_n$ is connected and solvable. So it is contained in some Borel subgroup $\mathbb{B}$.

Then $\mathbb{T}_n \subset \mathbb{B}$ so it is a maximal torus of $\mathbb{B}$: so

$$\mathbb{B} = \mathbb{T}_n \ltimes R_u(\mathbb{B})$$

and $R_u(\mathbb{B}) = \{g \in \mathbb{B} \mid g \text{ is unipotent}\} \supset \mathbb{U}_n$.

By theorem 5.12, there exists $g \in \mathbb{G}_n$ such that

$$g\mathbb{U}_n g^{-1} \subset g R_u(\mathbb{B}) g^{-1} \subset \mathbb{U}_n$$

For dimension reason, we have equality everywhere and so $R_u(\mathbb{B}) = \mathbb{U}_n$.

So

$$\mathbb{B} \subset N_{\mathbb{G}_n}(\mathbb{U}_n) = \mathbb{B}_n. \ ∎$$

Remark 5.29. The Bruhat decomposition

$$\mathbb{G}_n = \dot{\bigcup_{w \in S_n}} \mathbb{B}_n \, w \, \mathbb{B}_n$$

is just a consequence of Gauss elimination method. ■

Proposition 5.30. $S = \{s_1, \dots, s_{n-1}\}$

where $s_i = (i, i+1)$ (transposition).

Proof. First

$$\mathbb{B}_n s_i \mathbb{B}_n = \left\{ \begin{pmatrix} * & & & & & \\ & \ddots & & & * & \\ & & * & * & & \\ 0 & & x & * & & \\ & & & * & & \\ & & & & \ddots & \\ 0 & & & 0 & & * \end{pmatrix} \;\middle|\; x \neq 0 \right\}$$

↑ i-th column

so $s_i \in S = \{ w \in S_n \mid \dim \mathbb{B}_n w \mathbb{B}_n = \dim \mathbb{B}_n + 1 \}$

So $\{s_1, \dots, s_{n-1}\} \subset S$.

The converse (not easy) is left as an exercise ☺ . ■

What is $\mathbb{G}_n / \mathbb{B}_n$?

Let $\mathcal{F}_n = \{ (\overset{\circ}{\overset{\shortparallel}{V_0}} \subset V_1 \subset \dots \subset V_n) \mid \forall i,\ V_i \text{ is a dim. } i \text{ subspace of } V_n = \mathbb{F}^n \}$

$\mathcal{F}_n$ is a projective variety

$$\subset Gr(1,n) \times Gr(2,n) \times \dots \times Gr(n-1,n)$$

Let $(e_1, \dots, e_n)$ denote the canonical basis of $\mathbb{F}^n$. Let

$$F_{can} = (0 \subset \mathbb{F}e_1 \subset \mathbb{F}e_1 \oplus \mathbb{F}e_2 \subset \dots \subset \mathbb{F}^n) \in \mathcal{F}_n.$$

Then
- $\mathbb{G}_n$ acts transitively on $\mathcal{F}_n$
- $\mathbb{B}_n = \mathrm{Stab}_{\mathbb{G}_n}(F_{can})$

So the map

$$\begin{aligned} \mathbb{G}_n &\longrightarrow \mathcal{F}_n \\ g &\longmapsto g \cdot F_{can} \end{aligned}$$

is an orbit map.

Theorem 5.31. This orbit map induces an isomorphism

$$\mathbb{G}_n / \mathbb{B}_n \simeq \mathcal{F}_n.$$

Parabolic subgroups?

$I \subset S$; $I = S \setminus \{ s_{k_1}, s_{k_1+k_2}, \dots, s_{k_1+\dots+k_{n-1}} \}$

and we set $k_n = n - (k_1 + \dots + k_{n-1})$.

Then $W_I = S_{k_1} \times S_{k_2} \times \dots \times S_{k_n} \subset S_n$

$$\mathbb{P}_I = \begin{array}{c} k_1 \, (\\ k_2 \, (\\ \vdots \\ k_n \, (\end{array} \begin{pmatrix} * & * & * & * \\ & * & * & * \\ 0 & & \ddots & * \\ & & & * \end{pmatrix}$$

$$R_u(\mathbb{P}_I) = \begin{pmatrix} I_{k_1} & & * \\ & I_{k_2} & \\ & & \ddots & \\ 0 & & & I_{k_n} \end{pmatrix}$$

$$\mathbb{L}_I = \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

$\simeq \mathbb{G}_{k_1} \times \dots \times \mathbb{G}_{k_n}$

(5.31) $\mathbb{G}_n / \mathbb{P}_I \simeq \mathcal{F}\ell(k_1, \dots, k_n)$

$= \{ (V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n) \mid V_i \in Gr(k_1 + \dots + k_i, n) \}$