Extremal metrics on blowups

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Extremal metrics on blowups

In this final lecture, we will discuss joint work with Ruadhaí Dervan. This is a construction of extremal metrics on the blowup of a manifold in a point. The problem has a long history of study, and we will survey previous results, before embarking on the new method that allows one to deduce stronger results.

The blowup in a point

Let X be a compact Kähler manifold, and let $p \in X$ be a point. We can then define the blowup

$$\pi : \mathrm{Bl}_p X \to X$$

of X in p. This is a manifold satisfying:

- It is isomorphic to $X \setminus \{p\}$ outside the preimage of p;
- The preimage of p via the blowdown map π is a copy E of $\mathbb{P}^{n-1}$, called the exceptional divisor.

We will now define it.

The local model

The local model is the blowup $\mathrm{Bl}_0 \mathbb{C}^n$ of $\mathbb{C}^n$ in the origin. This can be seen as the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^{n-1}$. In other words,

$$\begin{aligned}\mathrm{Bl}_0 \mathbb{C}^n =& \{([z], v) : v = \lambda z \text{ for some } \lambda \in \mathbb{C}\} \\ \subset& \mathbb{P}^{n-1} \times \mathbb{C}^n\end{aligned}$$

The projection to the second factor is a biholomorphism away from the origin. At the origin, the fibre is $\mathbb{P}^{n-1}$. Thus we have replaced the origin with a divisor, the exceptional divisor, describing all the complex directions we can enter the origin in. The name comes from thinking of this as "zooming in" on the origin – we have "blown up" near the point and replaced it with all infinitesimal directions out of the point.

The blowup in a point

For a general complex manifold X, the blowup in p is the complex manifold obtained by replacing a disk about p with the preimage in $\mathrm{Bl}_0 \mathbb{C}^n$ of a disk about the origin in $\mathbb{C}^n$. Since the local model is a biholomorphism away from the origin, we can view this as a gluing of $X \setminus \{p\}$ and the preimage of the disk. The resulting manifold is therefore a complex manifold. As with the local model, we have a map

$$\pi : \mathrm{Bl}_p X \to X$$

and the preimage of p via π is a copy E of $\mathbb{P}^{n-1}$.

The Kähler cone of the blowup

First note that $H^2(\mathrm{Bl}_p X) \cong H^2(X) \oplus \langle [E] \rangle$. Moreover, if Ω is a Kähler class, then $\pi^*(\Omega)$ is on the boundary of the Kähler cone of $\mathrm{Bl}_p X$ – all subvarieties have non-negative volume, but the exceptional divisor has volume 0. So the class cannot be Kähler. However, if we allow the exceptional divisor to get some positive volume, we move into the Kähler cone of $\mathrm{Bl}_p X$. That is, for all $\varepsilon > 0$ sufficiently small, the class

$$\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$$

is a Kähler class on $\mathrm{Bl}_p X$.

The question we want to answer

The question we want to answer is the following: under what conditions on (X, Ω) and p does $(\mathrm{Bl}_p X, \Omega_\varepsilon)$ admit a cscK or extremal metric for all $\varepsilon > 0$ sufficiently small? Note, we are not trying to understand what happens for every value of ε such that Ω_ε is Kähler. We are only trying to understand what happens when ε is very small, i.e. when the volume of the exceptional divisor is very small.

Prior work

Previous results

This question has a rich history of study, through:

- Works of Arezzo–Pacard, Arezzo–Pacard–Singer and Székelyhidi giving sufficient conditions for when the blowup does admit an extremal metric in these classes. In particular, Székelyhidi completely settled the question in the cscK case, in dimension at least 3.
- Stoppa and Stoppa–Székelyhidi, on the other hand, investigated the algebro-geometric counterpart and provided necessary conditions.
- Seyyedali–Székelyhidi have also considered the case of blowing up higher dimensional subvarieties (see also work of Hashimoto).

We will now explain some of the history and known facts before the work of Dervan and the author.

First case – discrete automorphism group

The initial case considered by Arezzo–Pacard is:

- X admits a cscK metric in Ω;
- The automorphism group is discrete.

In this case, any point will do in the construction.

Theorem 1 (Arezzo–Pacard).

Suppose X admits a cscK metric in Ω. Let $p \in X$ be any point. Then there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathrm{Bl}_p X$ admits a cscK metric in the class

$$\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E].$$

First case – discrete automorphism group

On the other hand, Stoppa showed that if one has a test configuration for some (X, L), this induces a test configuration for the blowup of X. He then analysed the Donaldson–Futaki invariant of these test configurations with respect to the polarisations making the exceptional divisor small. The leading order term in the expansion is the Donaldson–Futaki invariant of the original test configuration before the blowup. In particular, if (X, L) is strictly unstable, so will the blowup be, in the polarisations we consider. Moreover, he used the expression for the subleading order term to show that for any strictly K-semistable manifold one can find a point such that the blowup in that point is strictly unstable. Thus in the situation of trivial reduced automorphism group, the only case not fully considered is when X is K-semistable, but does not admit a cscK metric.

What if the automorphism group is non-trivial?

The situation quickly gets more complicated in the presence of automorphisms, however. The simplest case in the presence of automorphisms is when blowing up a fixed point of a maximal torus in the reduced automorphism group of X.
The reduced automorphism group of $\mathrm{Bl}_p X$ can be seen as the subgroup of the reduced automorphism group of X generated by the vector fields that vanish at the blown up point. So, in this case, all the relevant vector fields lift and the extremal equation is unobstructed in this case.

What if the automorphism group is non-trivial?

Theorem 2 (Arezzo–Pacard–Singer/Székelyhidi).
Suppose X admits an extremal metric in Ω. Let $p \in X$ be a point fixed under the action of a maximal torus in the reduced automorphism group of X. Then there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathrm{Bl}_p X$ admits an extremal metric in the class

$$\Omega_\varepsilon = \pi^* \Omega - \varepsilon [E].$$

What if the automorphism group is non-trivial?

The issue comes when we want to blow up a point not fixed by a maximal torus. This is when we see start to see obstructions to solving the extremal equation. The core reason can be understood from the linear theory.

The reduced automorphism group of $\mathrm{Bl}_p X$ is the subgroup of the reduced automorphism group of X generated by the vector fields that vanish at the blown up point. When p is not fixed by the maximal torus, there are then real holomorphic vector fields on X that do not lift to $\mathrm{Bl}_p X$. Hence there will be a *discrepancy* between the mapping properties of the linearised operator on the blowup, and the initial extremal metric. Analytically, this is the core source of the obstruction to solve the extremal equation.

What if the automorphism group is non-trivial?

The works of Arezzo–Pacard and Arezzo–Pacard–Singer used a gluing method where they consider $\mathrm{Bl}_p X$ as two manifolds with boundary glued along the boundary. They produce many extremal metrics on these two manifolds with boundary. They then use a "Cauchy matching" technique, to give sufficient conditions for when these glue to give an extremal metric on the blowup. The conditions appearing in their work come from the conditions needed to perform this matching to subleading order. In fact, they deal with multiple points simultaneously.

Blowing up more than one point

In general, it is important to blow up multiple points simultaneously in the constructions. One does not get the same results when iterating the construction: doing the blow up in one point first, then the other. There are two reasons for this, related to the fact that when we do blowups in points $p_1, \ldots, p_k$, then we have a choice of numbers $a_1, \ldots, a_k > 0$ for the class

$$\Omega_\varepsilon = \pi^*\Omega - \sum_j a_j[E_j].$$

The a_i determine the ratio of the volumes of the exceptional divisors.

Why can't we iterate?

The two reasons are:

- We lose control of the classes when iterating. The possible values of the parameter ϵ that we can apply the construction to is not explicit. If we iterate, we lose control as we have multiple parameter values that become dependent on one another;
- If we are blowing up points that are not all fixed by a maximal torus, it maybe be that we can go in the direction of a certain ratio between the volumes of the exceptional divisors, but we cannot go in the direction of any of them on their own. Thus if we try to iterate, we go into an *unstable* regime, and we lose hope of using the same perturbative techniques to get back into the stable regime.

It is important to note that the work with Dervan only applies to the case of blowing up one point, unlike the other approaches.

What if the automorphism group is non-trivial?

Székelyhidi used a different approach, closer to the general strategy that we have discussed, which allowed him to relate the conditions on the point to K-stability. In the cscK case, when X has dimension at least 3, he showed that the blowup admits a cscK metric if and only if the manifold is K-polystable, and moreover gave a finite dimensional GIT condition that captures precisely what is needed to check K-polystability in this case. As this method forms the core of the method used by Dervan and the author in their approach to the problem as well, we will discuss in greater detail below how this approach works.

What if the automorphism group is non-trivial?

Finally, we also mention that the Donaldson–Futaki invariant and relative K-stability of blowups has been investigated by Stoppa–Székelyhidi. The blowup of a strictly relatively unstable manifold is relatively unstable and, again, the case of a relatively semistable manifold that does not admit an extremal metric is proved to be relatively unstable (strictly) for some choice of point.

What if the automorphism group is non-trivial?

The conclusion is that after the works mentioned above, the cases left that are not solved in complete generality are

- The cscK case in dimension 2;
- The non-cscK extremal case in arbitrary dimension;
- The general strictly semistable case.

In the work with Dervan discussed today, we complete the work on the first two cases, and also prove similar results for *analytically* relatively K-semistable manifolds.

Székelyhidi's approach

Székelyhidi's proof

We will now discuss Székelyhidi's approach. The case that will be most important for us is when blowing up an extremal manifold in a fixed point of the action of a maximal torus. We will mostly consider this case and then remark on why there are obstructions appearing when blowing up a point that is not fixed at the end of the section. The argument is an outline of the main steps only.

Székelyhidi's proof

We will consider $\mathrm{Bl}_p X$ as the union of two non-compact manifolds:

- $X \setminus \{p\}$;
- the preimage $\pi^{-1}(D)$ of a large disk $D \subset \mathbb{C}^n$ via the blowdown map $\mathrm{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$.

We will equip

- $X \ \setminus \{p\}$ with an extremal metric ω from X invariant under the corresponding real maximal torus;
- $\mathrm{Bl}_0 \mathbb{C}^n$ with a certain asymptotically flat, scalar-flat metric η, the *Burns–Simanca metric*.

Over an annular region (whose size depends on ε), we will then interpolate between the two metrics at the level of Kähler potentials to create an initial approximate solution to the equation.

Székelyhidi's proof

This gluing is performed as follows.

- pick holomorphic normal coordinates $(z_1, \ldots, z_n)$ at p in which the torus action becomes a usual linear action;
- Let $(w_1, \ldots, w_n)$ be coordinates on $\mathrm{Bl}_0 \mathbb{C}^n$ away from the exceptional divisor;
- Glue the two coordinate systems via the coordinate change

$$w = \varepsilon^{-1} z.$$

Székelyhidi's proof

Now, locally around p, the extremal metric ω can be written

$$\omega = i\partial\bar{\partial}\left(|z|^2 + \phi(|z|)\right),$$

for a function ϕ which is $O(|z|^4)$, since we have chosen holomorphic *normal* coordinates at p.
Similarly, the Burns–Simanca metric η satisfies that

$$\varepsilon^2\eta = i\partial\bar{\partial}\left(|z|^2 + \varepsilon^2\psi(\varepsilon^{-1}z)\right)$$

under the coordinate change $w = \varepsilon^{-1}z$. This uses that the metric is asymptotically flat.

Székelyhidi's proof

We can then interpolate between the two metrics on the level of potentials over an annular region

$$D_{2r_\varepsilon} \setminus D_{r_\varepsilon} = \{z : r_\varepsilon < |z| \leq 2r_\varepsilon\},$$

where we choose $r_\varepsilon = \varepsilon^{\frac{n-1}{n}}$. This is achieved as follows. Pick a smooth cut-off function $\chi(t)$ which vanishes on $t \leq 1$ and is equal to 1 when $t \geq 2$. Let $\chi_1(z) = \chi(\frac{z}{r_\varepsilon})$ and $\chi_2 = 1 - \chi_1$. Then we define ω_ε to be

- ω on $X \setminus D_{2r_\varepsilon}$;
- $\varepsilon^2\eta$ on $\pi^{-1}(D_{r_\varepsilon})$;
- $i\partial\bar{\partial}\left(|z|^2 + \chi_1(z)\phi(z) + \varepsilon^2\chi_2(z)\psi(\varepsilon^{-1}z)\right)$ on $D_{2r_\varepsilon} \setminus D_{r_\varepsilon}$.

Székelyhidi's proof

One can show that ω_ε is Kähler when ε is sufficiently small. Moreover, we can obtain a uniform bound for the scalar curvature. The analysis all takes place in certain weighted norms $C^{k,\alpha}_\delta$ that measure the blowup or vanishing rate of functions near the exceptional divisor, compared with the parameter ε. To avoid too many technicalities, we will note define these spaces, focusing instead on the main steps.

Székelyhidi's proof

Let $\hat{S}$ be the average scalar curvature of ω and let H_ε be the holomorphy potential of average 0 with respect to ω_ε of the lift of the extremal vector field on X to $\mathrm{Bl}_p X$. One then has the following uniform bound for scalar curvature of ω_ε.

Lemma 3 (Székelyhidi).

For all $\delta < 0$, there exists a $C > 0$ such that for all sufficiently small positive ε,

$$\|S(\omega_\varepsilon) - H_\varepsilon - \hat{S}\|_{C^{k,\alpha}_\delta} \leq C r_\varepsilon^{-\delta}$$

Thus ω_ε is an approximately extremal metric on $\mathrm{Bl}_p X$ and we wish to perturb this to a genuine extremal metric.

Controlling the linearised operator

In order to perturb, we need to control the right inverse to the linearised operator. We blow up at a fixed point of the torus action. So all the holomorphic vector fields on X lift. The (co)-kernel of the Lichnerowicz operator L_ε of ω_ε can therefore be identified with that of the Lichnerowicz operator L of ω on X. We therefore know it has a right inverse, and we want to establish a bound on how the norm of this right inverse depends on ε.

Controlling the linearised operator

Proposition 1 (Székelyhidi).

Assume the dimension is at least 3. Then for $\delta \in (4-2n, 0)$, the operator

$$C_{\delta}^{k+4,\alpha} \times \mathfrak{h} \times \mathbb{R} \to C_{\delta-4}^{k,\alpha}$$

given by

$$(\phi, h, c) \mapsto L_{\varepsilon}(\phi) - h_{\varepsilon} - c$$

is surjective with right inverse Q_{ε} satisfying the uniform estimate

$$\|Q_{\varepsilon}\|_{C_{\delta-4}^{k,\alpha} \to C_{\delta}^{k+4,\alpha}} \leq C$$

for some $C > 0$.

Székelyhidi's proof

When the dimension is 2, the bound for the right inverse blows up with ε. This causes some additional complications in the argument that we will not get further into here (one has to improve the approximate solution by matching closer with the Burns–Simanca metric). Thus for the remainder of the lecture, we will assume the dimension is at least 3, even though the case of dimension 2 also can be dealt with.

Székelyhidi's proof

We now wish to show that we can perturb the approximate solution ω_ε to a genuine solution of the extremal equation. We want to use the contraction mapping theorem. So we need to rephrase the equation as a fixed point problem. Let

$$\Phi : C^{k+4,\alpha}_{\delta} \times \mathfrak{h} \times \mathbb{R} \to C^{k+4,\alpha}_{\delta} \times \mathfrak{h} \times \mathbb{R}$$

be given by

$$(\phi, h, c) \mapsto Q_\varepsilon \left(S(\omega_\varepsilon + i\partial\bar{\partial}\phi) - h_\varepsilon - c - H_\varepsilon - \hat{S} \right).$$

We subtract off $H_\varepsilon + \hat{S}$ to make $(0,0,0)$ an approximate solution to the equation we wish to solve, through Lemma 3.

Székelyhidi's proof

Székelyhidi then shows that there is a constant $c > 0$ such that Φ is a contraction on the set

$$U = \{(\phi, h, c) : \|(\phi, h, c)\|_{C_\delta^{k+4,\alpha}} \leq C\varepsilon^{2-\delta}\}.$$

Since $(r_\varepsilon)^{4-\delta} = \varepsilon^{(4-\delta)\frac{n-1}{n}}$ and $(4-\delta)\frac{n-1}{n} > 2-\delta$ when $\delta < 0$ is sufficiently close to 0, we have that the Φ sends U to itself for a suitably chosen value of the parameter δ, for all sufficiently small ε. Hence the contraction mapping theorem guarantees the existence of an extremal metric on $\mathrm{Bl}_p X$ in $[\omega_\varepsilon]$ when $\varepsilon > 0$ is sufficiently small, which is what we wanted to show.

Székelyhidi's proof

We end the section by briefly explaining what happens when the point is not a fixed point of the maximal torus. In this case, not all vector fields will lift. Székelyhidi defines a lift

$$l : \overline{\mathfrak{h}} \to C^{\infty}(\mathrm{Bl}_p X)$$

of the corresponding holomorphy potentials by using cut-off functions. These are no longer holomorphy potentials on the blowup. When establishing the properties of the linearisation analogous to Proposition 1 in this case, one can only establish a uniform bound orthogonally to $l(\overline{\mathfrak{h}})$. Since not all these functions are holomorphy potentials, this means that we are solving a more general equation than the extremal equation, and the construction is therefore obstructed.

Székelyhidi's proof

Székelyhidi then proceeds to understand this obstruction in terms of K-stability. To get precise information on the relation to K-stability and a finite-dimensional GIT condition, a better approximate solution with higher matching with the Burns–Simanca metric is required. The need to have this higher order matching is why Székelyhidi's results are restricted to the cscK case and dimension at least 3.

The new approach – initial setup and statement of result

The new approach

We now explain the core new idea in the work of Dervan and the author. The idea is to work with a fixed *symplectic* manifold in the gluing process, and let the complex structure vary instead. When blowing up a complex manifold, the complex structure will depend on the point chosen. On the other hand, in the symplectic category, we have no dependence on the point. As a smooth manifold, the blowup is $X \# \overline{\mathbb{CP}^n}$. We will later see that we can make the symplectic form the same independently of the point, at least in an open neighbourhood on X. Of course, the *almost complex structure* will then change when we do this.

The new approach

Our goal is to produce a diffeomorphism $f : X \to X$ sending a point p fixed by the action of a maximal torus to a nearby point q. We want this to happen in such a way that $f^*\omega = \omega$, i.e. such that f is a symplectomorphism. If $X = (M, J)$, where M is the underlying smooth manifold and J is the almost complex structure, we can then define $J_q = f^*J$. The blowup of (M, J_q) in p will then be isomorphic to the blowup of X in q. Since f is a symplectomorphism, we even have that the Kähler manifolds (M, J_q, ω) and $(X, \omega) = (M, J, \omega)$ are isomorphic as Kähler manifolds.

The new approach

The advantage is that in the construction we can now take the point of view that we are blowing up a fixed symplectic manifold (M, ω) in one given point p. The thing that is changing is the almost complex structure, rather than the point. We can then define lifts of holomorphy potentials in X in a more natural way: we simply use the lift coming from the complex structure corresponding to $q = p$, i.e. blowing up at a fixed point of the maximal torus. These will no longer all be holomorphy potentials, though, if $q \neq p$, but are a fixed space of functions, independent of q.

The new approach

In order to achieve this, we need to produce the symplectomorphism f. The key is to use Moser's trick: if β is a 1-form, then by flowing along the vector field dual to β via ω, we can produce a diffeomorphism

$$f : M \to M$$

such that

$$f^*(\omega + d\beta) = \omega.$$

But if β is closed, $\omega + d\beta = \omega$, so then f will be a symplectomorphism. The goal is then to pick a good β so that p is sent to q.

The new approach

To achieve this, we consider the local model of blowing up the origin in $\mathbb{C}^n$ with its Euclidean symplectic form. Suppose q is the point $(\lambda_1, \ldots, \lambda_n)$. The flow generated by the real vector field ν corresponding to

$$\sum_{j=1}^{r} \lambda_j \frac{\partial}{\partial z^j}$$

sends the origin to $(\lambda_1, \ldots, \lambda_n)$. This vector field is dual via the Euclidean symplectic form to $d(\sum_{j=1}^{r} \lambda_j |z^j|^2)$.

The new approach

Using cut-off functions, we can globalise this to an exact form β generating a Hamiltonian symplectomorphism $f : M \to M$. Because the Kähler metric is not exactly the Euclidean metric, f does not necessarily send p to the point q. But it will send p to a point q' such that $\mathrm{Bl}_{q'} X \cong \mathrm{Bl}_q X$, which suffices for our purposes. Thus from the local model we can produce a global symplectomorphism that does what we want.

The new approach

We can also let the complex structure vary even before changing the point, and this allows us to tackle also certain strictly semistable manifolds. Recall from the previous lecture that we say that X is *analytically K-semistable* if it admits a degeneration, invariant with respect to the reduced automorphism group, to a cscK central fibre. Similarly, one says X is analytically relatively K-semistable if the same holds, but where the central fibre is extremal.

The new approach

Thus we have a map

$$\Psi : B \to \mathcal{J}(M, \omega),$$

where B is an open ball in some vector space. This parametrises the isomorphism classes of complex structures near the cscK central fibre X_0 in the Kuranishi model and all nearby points to a point p fixed by a maximal torus in the reduced automorphism group of the cscK central fibre.

The new approach

Ultimately, we prove the following.

Theorem 4 (Dervan–Sektnan '21).

Suppose X is analytically relatively K-semistable. Let $p \in X$. Let $\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$. Then the following are equivalent:

1. $\mathrm{Bl}_p X$ *admits an extremal metric in* Ω_ϵ *for all* $0 < \varepsilon \ll 1$;
2. $(\mathrm{Bl}_p X, \Omega_\epsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$.

Moreover, the relative K-stability criterion is an explicit finite dimensional condition.

Constructing metrics on the blowup

Dividing up the problem

We have explained how we set up our problem initially, before blowing up. We now start our construction on the blowup. On the extremal central fibre in the family

$$\Psi : B \to \mathcal{J}(M, \omega),$$

we will use Székelyhidi's construction that we outlined in the previous section. This corresponds to blowing up an extremal manifold in a fixed point of the action of a maximal torus, and is unobstructed. This gives in particular a one-parameter family ω_ε of symplectic forms on the underlying smooth manifold of the blowup, which we denote $\mathrm{Bl}_p M$.

Dividing up the problem

The next goal is to divide up the problem into two more manageable bits. We first solve a more general equation on the blowup than the extremal equation. Then we analyse when this more general equation actually is a solution to the extremal equation. This is a strategy going back to at least to ideas of Donaldson, which is a very useful general principle in obstructed perturbation problems.

Dividing up the problem

We note that on M before blowing up, when we are changing the point we blow up, we have $S(\omega, J_0) \in \overline{\mathfrak{h}}$, the space of torus-invariant holomorphy potentials on X wrt ω. However, this also holds on the Kuranishi model, by work of Székelyhidi and Brönnle. Adapting their argument, we can show that this holds on the full family where we are parametrising both nearby complex structures in the Kuranishi model, and what point we are blowing up *before the blowup*. I.e. we have

$$S(J_b, \omega) \in \overline{\mathfrak{h}}$$

for all $b \in B$.

The lifts

One advantage of working with the symplectic manifold $(\mathrm{Bl}_p M, \omega_\varepsilon)$ is that even vector fields that do not correspond to holomorphy potentials on the non-zero fibres have a natural lift now – we just use the lift on the central fibre. That is, for any function $h \in \overline{\mathfrak{h}}$ of average 0 with respect to ω, we have a naturally defined lift h_ε which is the Hamiltonian of average 0 with respect to ω_ε of the lift of the corresponding vector field to $\mathrm{Bl}_p X_0$. The only difference in the non-zero fibres is that not all of these functions are holomorphy potentials – they are Hamiltonians on the blowup, but they do not all produce a holomorphic vector field.

Dividing up the problem

Let $\overline{\mathfrak{h}}_\varepsilon$ denote the space of such lifted potentials, with respect to ω_ε. **Goal:** Construct a good lift

$$\Psi_\varepsilon : B \to \mathcal{J}(\mathrm{Bl}_p\, M, \omega_\varepsilon)$$

of the map Ψ to the blowup, so that the scalar curvature of the almost complex structures lands in $\overline{\mathfrak{h}}_\varepsilon$.

Constructing initial AC structures on the blowup

The goal now is to show that we can solve

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon,$$

where $J_{\varepsilon,b} = \Psi_\varepsilon(b)$. It is only after solving this equation that we analyse the relation to K-stability. To do so, we perform the same construction as on the central fibre, on the non-zero fibres. This creates an initial $\omega_{\varepsilon,b}$, which in general is different from ω_ε.

Constructing initial AC structures on the blowup

Since we may have $\omega_{\varepsilon,b} \neq \omega_\varepsilon$, we need to apply Moser's trick again. This is to be able to take the symplectic point of view of having a fixed symplectic manifold (for each ε). This gives a diffeomorphism $f_{\varepsilon,b}$ of M, giving a Kähler isomorphism

$$(\mathrm{Bl}_p\, M, J_b, \omega_{\varepsilon,b}) \cong (\mathrm{Bl}_p\, M, f_{\varepsilon,b}^* J_b, \omega_\varepsilon).$$

Constructing initial AC structures on the blowup

At this stage, we have put ourselves in the symplectic framework, but the Kähler structure $(\mathrm{Bl}_p M, f_{\varepsilon,b}^* J_b, \omega_\varepsilon)$ does not solve any special equation. For the scalar curvature to land in the space of holomorphy potentials on the central fibre, we have to perturb. An important fact in the deformation theory, due to Székelyhidi and Brönnle, is that we can ensure that the initial map $\Psi : B \to \mathcal{J}(M, \omega)$ before the blowup has scalar curvature that lands in $\overline{\mathfrak{h}}$, i.e. we have

$$S(\omega, J_b) \in \overline{\mathfrak{h}}$$

for all $b \in B$. This is what we wish to obtain also on the blowup.

Bounding the approximate solution

Let $H_b = S(\omega, J_b) - \hat{S}$ which lies in $\overline{\mathfrak{h}}$, and let $H_{\varepsilon,b}$ be the corresponding lift. Then similarly to Lemma 3 we have the following.

Lemma 5.

For all $\delta < 0$, there exists a $C > 0$ such that for $b \in B$ and for all sufficiently small positive ε,

$$\|S(\omega_\varepsilon, J_b) - H_{\varepsilon,b} - \hat{S}\|_{C^{k,\alpha}_\delta} \leq C r_\varepsilon^{-\delta}$$

The way the complex structure changes

The next step is to perturb so that the scalar curvature lies in $\bar{\mathfrak{h}}_\varepsilon$. Since for each ε we are keeping the symplectic form fixed, we have to change the almost complex structure. Again, this goes through the Moser trick: if M is a symplectic manifold with almost complex structure J and f is a function, we can flow through the vector field dual to df to produce a symplectomorphism

$$F_f : M \to M.$$

We then define a new almost complex structure $F_f^* J$. The linearisation of the map $f \mapsto F_f^* J$ is the Lichnerowicz operator (with no gradient scalar curvature term).

The linearisation

We can establish bounds for the linearisation in the whole family over B building on the bounds already established in the case of the central fibre in Proposition 1.

Proposition 2.

Assume the dimension is at least 3*. Then for* $\delta \in (4-2n, 0)$*, the operator* $C^{k+4,\alpha}_{\delta} \times \mathfrak{h} \times \mathbb{R} \to C^{k,\alpha}_{\delta-4}$ *given by*

$$(\phi, h, c) \mapsto L_{\varepsilon}(\phi) - h_{\varepsilon} - c$$

is surjective with right inverse Q_{ε} *satisfying the uniform estimate*

$$\| Q_{\varepsilon} \|_{C^{k,\alpha}_{\delta-4} \to C^{k+4,\alpha}_{\delta}} \leq C$$

for some $C > 0$*.*

There are similar bounds in dimension 2, but then they blow up with ε, as in the work of Székelyhidi.

Conclusion

The proof now follows similar steps to that of the proof of Székelyhidi. In the end we end up with a complex structure $J_{\varepsilon,b}$ such that

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon.$$

We recall again that $\overline{\mathfrak{h}}_\varepsilon$ corresponds to holomorphy potentials on the *central fibre*. So we have not necessarily solved the extremal equation on the non-zero fibres yet. We turn to understanding when we have done this now.

Relating the construction to K-stability

Relating the construction to K-stability

So far, we have solved

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon.$$

It is only on the central fibre that all the functions in $\overline{\mathfrak{h}}_\varepsilon$ correspond to holomorphy potentials. So it is only on the central fibre that we have guaranteed to solve the extremal equation. The holomorphy potentials on the non-central fibres lie in a proper subspace of $\overline{\mathfrak{h}}$. The vector space the ball B lies in has a linear action of a torus T, which is a maximal torus of the reduced automorphism group of the cscK/extremal central fibre. We want to understand the condition that we can find, in a given orbit, a point such that the scalar curvature actually lies in the space of holomorphy potentials (or is constant, if we are solving the cscK problem).

Relating the construction to K-stability

The key to achieving this is that we can view this as looking for a zero, or a critical point, of a certain moment map. The reason we can do this is that we can view the map

$$b \mapsto J_{\varepsilon,b}$$

as giving a symplectic embedding into the space $\mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$ of ω_ε-compatible almost complex structures.

Proposition 3.

The image of the map $B \to \mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$ given by

$$b \mapsto J_{\varepsilon,b}$$

is a symplectic submanifold.

Relating the construction to K-stability

The scalar curvature is a moment map for the action of the Hamiltonian symplectomorphism group on $\mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$. Hence its composition with the orthogonal projection to $\bar{\mathfrak{h}}$ is a moment map for the restriction to T of this action. This then also holds on the symplectic submanifold given by the image of the embedding

$$B \to \mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon).$$

Thus we have managed to put ourselves in the position that solving the cscK/extremal equation becomes a finite dimensional moment map problem, which can then be related to a GIT notion of stability/relative stability.

Relating the construction to K-stability

Now, for any element u in the Lie algebra of the torus one can produce a test configuration for X_b, if the element is rational (and a so-called $\mathbb{R}$ test configuration if the element is irrational – we will not go further into this point). This also gives a test configuration for $\mathrm{Bl}_p X_b$. Moreover, the value of the corresponding hamiltonian function at the limit point that b goes to under the action on B generated by u is the Donaldson–Futaki invariant of this test configuration. These ideas were shown by Székelyhidi.

Relating the construction to K-stability

Thus assuming relative K-stability, we get a particular sign for all the Hamiltonian functions at limit points of b. A careful analysis shows that the assumption that all the Hamiltonian functions are negative at the limit point b allows one to produce another point in the same orbit as b where the value of all the Hamiltonians orthogonal to those that are holomorphy potentials vanishes. This uses a rather general framework that Dervan developed in his earlier work. In particular, assuming relative K-stability, we get the scalar curvature lies in the space of holomorphy potentials at complex structure over b. Thus we have produced an extremal metric under the assumption of relative K-stability, completing the proof of the main result.

Relating the construction to K-stability

We end the lecture by noting that we also obtain explicit expansions that give the exact criterion for stability in this case. This follows from an explicit expansion of the Donaldson–Futaki invariant. This uses similar ideas to the analogous statements in Székelyhidi's work on blowups.

Thank you!