Logarithmic vanishing theorems on compact Kähler manifolds

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Current Developments in Mathematics and Physics
Tsinghua University, 2024/04/03-06

Motivations

- Various vanishing theorems play the fundamental role in algebraic geometry and complex geometry. There are many important works in this field.

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- For instance,

  (1) Kodaira's vanishing theorem;
  (2) Griffiths' vanishing theorem;
  (3) Le Potier's vanishing theorem;
  (4) Mumford's vanishing theorem;
  (5) Ramanujam's vanishing theorem;
  (6) Kawamata-Viehweg's vanishing theorem;
  (7) Nadel's vanishing theorem;
  (8) Demailly's vanishing theorem; and
  (9) Guan-Zhou's vanishing theorem.

- The first notable vanishing theorem is due to Kodaira.

Theorem (Kodaira)

Let (X^n, ω) be a compact Kähler manifold and (L, h) be a positive line bundle over X. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0,$$

where $K_X = \Omega_X^n = \Lambda^n T^*X$ is the canonical line bundle.

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Theorem (Akizuki-Nakano)

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- There are many important application, for instances, Kodaira's embedding theorem, Kodaira's classification of compact complex surfaces, Kodaira-Spencer deformation theory and etc..

- The "Kawamata-Viehweg vanishing theorem" is a significant generalization on Kodaira's vanishing theorem.

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If L is a nef and big line bundle over a projective manifold X, then for $q \geq 1$,

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- There is also another version of Kawamata-Viehweg vanishing theorem

Theorem

Let X be a projective manifold. If F is a line bundle such that some multiple of L has a decomposition $mF = L + D$ where

(1) *L is an ample line bundle, and*

(2) *D an effective divisor,*

then for $q \geq 1$

$$H^q(X, K_X \otimes F \otimes \mathcal{J}(m^{-1}D)) = 0.$$

- The following is Demailly's refined version of Nadel's vanishing theorem.

Theorem (Nadel)

Let (X,ω) be a compact Kähler manifold. If (L,φ) is a big line bundle, then for $q \geq 1$,

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- Based on Qi'an Guan and Xiangyu Zhou's solution to Demailly's strong openness conjecture, they obtained the following vanishing theorem which generalizes Junyan Cao's vanishing theorem.

Theorem (Guan-Zhou, Ann. of Math. 2015)

Let (X, ω) be a compact Kähler manifold. If (L, φ) is a pseudo-effective line bundle with numerically dimension $nd(L, \varphi) = k$, then for any $q \geq n - k + 1$,

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- This vanishing theorem works for many cases since the curvature of a pseudo-effective line bundle is merely semi-positive in the sense of current.

- Deligne and Illusie algebraic presented a nice algebraic proof for Kodaira's vanishing in 1980's (see also the work of Esnault-Viehweg by using spectral sequences and the cyclic covering trick over projective manifolds), which also generalizes the classical work:

Theorem (Norimatsu)

Let X be a projective manifold of dimension n and D be a simple normal crossing divisor. Suppose L is an ample line bundle. Then for any $p+q \geq n+1$

$$H^q(X, \Omega_X^p(\log D) \otimes L) = 0.$$

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- In this talk, we present a uniform analytical approach for such vanishing theorems on compact Kähler manifolds.

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- $E \to X$ is an ample (resp. nef) vector bundle if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is an ample (resp. nef) line bundle.

Simple normal crossing divisors

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- The sheaf of germs of differential p-forms on X with at most logarithmic poles along D, denoted $\Omega_X^p(\log D)$ (Defined by Deligne, IHES 1969) is the sheaf whose sections on an open subset V of X are

$$\Gamma(V, \Omega_X^p(\log D)) := \{\alpha \in \Gamma(V, \Omega_X^p \otimes \mathcal{O}_X(D)) \text{and } d\alpha \in \Gamma(V, \Omega_X^{p+1} \otimes \mathcal{O}_X(D))\}$$

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- Locally integrability and bases: $\frac{dz^i}{z^i} = d \log z^i$, dz^i.

- Our main theorem is

Theorem (Huang-L.-Wan-Yang)

Let X be a compact Kähler manifold of dimension n. Suppose that $D = \sum_{i=1}^{s} D_i$ is a SNC divisor and N is a line bundle such that

$$N \otimes \mathcal{O}_X([\Delta]) \quad \text{is a } k\text{-positive } \mathbb{R}\text{-line bundle}$$

where $\Delta = \sum_{i=1}^{s} a_i D_i$ is an $\mathbb{R}$-divisor for some $a_i \in [0,1]$. Then

$$H^q(X, \Omega_X^p(\log D) \otimes N) = 0 \quad \text{for any } p + q \geq n + k + 1.$$

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- One of the key ingredients and major advantages in complex geometry: the use of curvature $\mathbb{R}$-line bundles. For instance, (L, h) is a positive line bundle, and $(\sqrt{3}L, h^{\sqrt{3}})$ is also a positive $\mathbb{R}$-line bundle. That is

$$-\sqrt{-1}\partial\overline{\partial}\log h > 0 \Longrightarrow -\sqrt{-1}\partial\overline{\partial}\log h^{\sqrt{3}} = -\sqrt{3}\sqrt{-1}\partial\overline{\partial}\log h > 0.$$

- We can "write" the previous theorem into the following version

Theorem

Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a SNC divisor in X. Let $\Delta = \sum_{i=1}^{s} a_i D_i$ be an $\mathbb{R}$-divisor with $a_i \in [0,1]$ and F is a k-positive $\mathbb{R}$-line bundle such that $F \otimes \mathcal{O}_X(-[\Delta])$ is a line bundle. Then

$$H^q(X, \Omega_X^p(\log D) \otimes F \otimes \mathcal{O}(-[\Delta])) = 0 \quad \text{for any } p + q \geq n + k + 1.$$

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- Note that when $p = n = \dim X$,

$$\Omega_X^n(\log D) = K_X \otimes \mathcal{O}([D])$$

and $\mathcal{O}([D]) \otimes \mathcal{O}(-[\Delta]) = \mathcal{O}(\sum_i (1-a_i)[D_i])$ is still effective.

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- Morally, for general p, $\Omega_X^p(\log D) \sim \Omega_X^p + [D]$.

We obtain several applications:

- log type Gibrau's vanishing theorem and Norimatsu-Deligne-Illusie's vanishing theorem on compact Kähler manifolds (by taking $\Delta = 0$)

Corollary

Let X be a compact Kähler manifold of dimension n and D be a SNC divisor in X. If L is a k-positive line bundle over X, then

$$H^q(X, \Omega_X^p(\log D) \otimes L) = 0 \quad \text{for any } p + q \geq n + k + 1.$$

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- log type of Le Potier's vanishing theorem for ample vector bundles.

Corollary

Let X be a compact Kähler manifold of dimension n and D be a SNC divisor. Suppose that $E \to X$ is an ample vector bundle of rank r. Then

$$H^q(X, \Omega_X^p(\log D) \otimes E) = 0 \quad \text{for any } p + q \geq n + r.$$

- log type of Akizuki-Kodaira-Nakano log type vanishing theorem

Theorem (Huang-L.-Wan-Yang)

Let X be a compact Kähler manifold of dimension n and $D = \sum_{i=1}^{s} D_i$ be a SNC divisor. Suppose F is a line bundle over X and m is a positive real number such that $mF = L + D'$, where $D' = \sum_{i=1}^{s} \nu_i D_i$ is an effective normal crossing $\mathbb{R}$-divisor and L is a k-positive $\mathbb{R}$-line bundle. Then

$$H^q\left(X, \Omega^p(\log D) \otimes F \otimes \mathcal{O}_X\left(-\sum_{i=1}^{s}\left(1 + \left[\frac{\nu_i}{m}\right]\right) D_i\right)\right) = 0 \qquad (0.1)$$

for $p + q \geq n + k + 1$.

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for $p + q \geq n + k + 1$.

- In particular, when L is ample, $D' = D$ and $p = n$, we obtain the Kawamata-Viehweg vanishing theorem.

Recall that, our main theorem is

Theorem

Let X be a compact Kähler manifold of dimension n. Suppose that $D = \sum_{i=1}^{s} D_i$ is a SNC divisor and N is a line bundle such that

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$$H^q(X, \Omega_X^p(\log D) \otimes N) = 0 \quad \text{for any } p + q \geq n + k + 1.$$

Step 1. An L^2-type Dolbeault isomorphism.

Theorem

Let (X,ω) be a compact Kähler manifold of dimension n and D be a simple normal crossing divisor in X. Let ω_P be a smooth Kähler metric on $Y = X - D$ which is of Poincaré type along D. Then there exists a smooth Hermitian metric h_Y^L on $L|_Y$ such that the sheaf $\Omega^p(\log D) \otimes \mathcal{O}(L)$ over X enjoys a fine resolution given by the L^2 Dolbeault complex $(\Omega_{(2)}^{p,*}(X, L, \omega_P, h_Y^L), \bar{\partial})$, that is, we have an exact sequence of sheaves over X

$$0 \to \Omega^p(\log D) \otimes \mathcal{O}(L) \to \Omega_{(2)}^{p,*}(X, L, \omega_P, h_Y^L) \tag{0.2}$$

such that $\Omega_{(2)}^{p,q}(X, L, \omega_P, h_Y^L)$ is a fine sheaf for any $0 \leq p, q \leq n$. In particular,

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}(L)) \cong H_{(2)}^{p,q}(Y, L, \omega_P, h_Y^L) \cong \mathbb{H}_{(2)}^{p,q}(Y, L, \omega_P, h_Y^L). \tag{0.3}$$

Step 2. Prove the L^2 vanishing theorem by using Hormander's $\overline{\partial}$-estimate:

Lemma

Let (M, ω) be a complete Kähler manifold. Let (E, h^E) be an Hermitian vector bundle over M. Assume that $A = [i\Theta(E, h^E), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q} T^* M \otimes E$, $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q} T^* M \otimes E)$ satisfying $\overline{\partial} g = 0$ and $\int_M (A^{-1} g, g) dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1} T^* M \otimes E)$ such that $\overline{\partial} f = g$ and

$$\int_M |f|^2 dV_\omega \leq \int_M (A^{-1} g, g) dV_\omega.$$

Indeed, in Step 1, we also need this lemma to prove the exactness of the sequence:

$$0 \to \Omega^p(\log D) \otimes \mathcal{O}(L) \to \Omega^{p,*}_{(2)}(X, L, \omega_P, h^L_Y) \tag{0.4}$$

Step 3. Construct various Hermitian metrics on $Y = X - D$ and $L|_Y$ such that

$$A = [i\Theta(E, h^E), \Lambda_\omega]$$

is (positive) uniformly bounded from below when acting on certain bundle valued (p, q) forms. Hence, we obtain the desired vanishing theorem

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}(L)) \cong \mathbb{H}^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y) = 0.$$

Thank you!