The Seiberg-Witten moduli space

$\Omega^1 \xrightarrow{d^* \oplus d^+} \Omega^0 \oplus \Omega^2_+$

$\ker = H^1$ $\quad$ $\mathrm{coker} = H^0 \oplus H^2_+$

Assume $\pi_1(X) = 1$.

The S.W. equation $\begin{cases} F^+_{A^e} = \rho^{-1}(\phi\phi^*)_0 & \text{curvature eq.} \\ D_A \phi = 0 & \text{Dirac eq.} \end{cases}$

$\phi \in \Gamma(S^+)$ $\quad$ A: spin^c connection on S^+/S

$\leftarrow$ spin^c structure

$\mathcal{M}_{SW}(X, \mathfrak{s}) = \{\text{solutions } (A, \phi)\}/\mathcal{G}$ $\leftarrow$ gauge group

$\mathcal{G} = C^\infty(X, U(1))$

Theorem: $\mathcal{M}_{SW}$ is compact

(This is called "the Seiberg-Witten miracle")

$\boxed{\text{Lemma}}$: $\forall (A, \phi)$, we have

$$\Delta|\phi|^2 = 2\mathrm{Re}\langle \nabla_A^* \nabla_A \phi, \phi\rangle - 2\langle \nabla_A \phi, \nabla_A \phi\rangle.$$

(This is just a fancy version of $-(f^2)'' = -2f''\cdot f - 2f'^2$)

proof: A is unitary $\Rightarrow$ $d\langle \phi, \psi\rangle = \langle \nabla_A \phi, \psi\rangle + \langle \phi, \nabla_A \psi\rangle$

$$d^*\langle \phi, \alpha\rangle = -\langle \nabla_A \phi, \alpha\rangle + \langle \phi, \nabla_A^* \alpha\rangle$$

$\forall \phi, \psi \in \Omega^0(S^+)$ $\quad$ $\alpha \in \Omega^1(S^+)$

$$\begin{aligned} \Delta|\phi|^2 &= d^* d\langle \phi, \phi\rangle \\ &= d^*\langle \nabla_A \phi, \phi\rangle + \langle \phi, \nabla_A \phi\rangle \\ &= 2d^* \mathrm{Re}\langle \phi, \nabla_A \phi\rangle \\ &= 2\,\mathrm{Re}(\langle \phi, \nabla_A^* \nabla_A \phi\rangle - \langle \nabla_A \phi, \nabla_A \phi\rangle) \end{aligned}$$ $\square$

$-(f^2)'' \leq -2f''f$

$\boxed{\text{Corollary}}$: $\Delta|\phi|^2 \leq 2\mathrm{Re}\langle \nabla_A^* \nabla_A \phi, \phi\rangle$

$\boxed{\text{Proposition}}$: Let (A, ϕ) be a solution to S.W. eq. Then

$$|\phi|^2_{C^0} := \max_{p \in X} |\phi(p)|^2 \leq \max\{0, -\min_{p \in X} s(p)\}$$

$\leftarrow$ scalar curvature

Proof: Assume $|\phi(x)|^2$ achieves max at $p \in X$.

Then $\Delta|\phi|^2(p) \geq 0$

So $\mathrm{Re}\langle \nabla_A^* \nabla_A \phi, \phi\rangle \geq 0$ $\quad$ $D_A \phi = 0$

$$\nabla_A^* \nabla_A \phi + \frac{1}{2}\rho(F^+_{A^e})\phi + \frac{s}{4}\phi = D_A^* D_A \phi = 0$$

so $\quad \mathrm{Re}\langle \rho(F^+_{A^e})\phi, \phi\rangle + \frac{s}{2}\langle \phi, \phi\rangle \leq 0$

$\mathrm{Re}\langle \rho(F^+_{A^t})\phi,\phi\rangle + \frac{s}{2}\langle\phi,\phi\rangle \le 0$ at p

$\rho(F^+_{A^t}) = (\phi\phi^*)_0 \Rightarrow \langle \underset{(\phi\phi^*)_0}{\rho(F^+_{A^t})}\phi,\phi\rangle = \langle \frac{|\phi|^2}{2}\phi,\phi\rangle = \frac{|\phi|^4}{2}$

so $\frac{|\phi(p)|^4}{2} + \frac{s(p)}{2}|\phi(p)|^2 \le 0$ $\quad |\phi(p)|^2\cdot(|\phi(p)|^2 + s(p)) \le 0$

so either $\phi(p)=0$ or $|\phi(p)|^2 \le -s(p)$

Recall $|\phi(p)| = \max_{x\in X}|\phi(x)|$. so either $\phi = 0$ or

$|\phi(x)|^2 \le |\phi(p)|^2 \le -\min_{x\in X} s(x)$. $\square$

Corollary: Suppose X has a metric with $s(x) \ge 0$ $\forall x\in X$.

Then (A,ϕ) is a solution $\Rightarrow \phi = 0$.

$(A,0)$ reducible solution $\qquad \mathcal{G}_h = \underset{\sim}{H^1(X;\mathbb{Z})} \times S^1$

Theorem: $\mathcal{M}_{SW}$ is compact.

Proof: $\qquad F^+_{A_i^t} + \rho^{-1}(\phi_i\phi_i^*)_0 = 0$

Suppose $\{(A_i,\phi_i)\}_{i\ge 1}$ is a sequence of solutions.

After gauge transformation, may assume $\quad P_{H_1}\alpha_i$ bounded

$A_i = A_0 + \alpha_i \quad \alpha_i \in \Omega^1(X; i\mathbb{R}) \quad \boxed{d^*\alpha_i = 0}$

$d^+ = \frac{d + *d}{2}$ base connection $\qquad$ Coulomb gauge fixing condition

Then we have $\quad F^+_{A_i^t} = F^+_{A_0^t} + 2d^+\alpha_i \quad D_{A_i} = D_{A_0} + \rho(\alpha_i)$

$$\begin{cases} 2d^+\alpha_i = -F^+_{A_0^t} + \rho^{-1}(\phi_i\phi_i^*)_0 & \text{--- ①} \\ d^*\alpha_i = 0 \quad (d^* = *d*) & \text{--- ②} \\ D_{A_0}\phi_i = -\rho(\alpha_i)\cdot\phi_i & \text{--- ③} \end{cases}$$

$d: \Omega^1 \to \Omega^2$

$*d: \Omega^1 \to \Omega^2$

LHS: elliptic operator

$|\phi_i|_{C^0}$ bounded

elliptic bootstrapping

$\rightsquigarrow$

$|(\alpha_i,\phi_i)|_{C^k}$ bounded $\forall k$

so $\exists$ converging subsequence. $\square$

Now we want to count # of solutions/gauge.

We write the S.W. eqs as a map $\qquad \int d^*\alpha \, dvol = 0$

$$\widetilde{SW}: \{\text{spin}^c\text{-connections}\} \oplus \Gamma(S^+) \longrightarrow \Omega^2_+(X; i\mathbb{R}) \oplus \Omega^0(X; i\mathbb{R})/\mathbb{R} \oplus \Gamma(S^-)$$

$$(A, \phi) \longmapsto (F^+_{A^c} - \rho^{-1}(\phi^*\phi)_0, \; d^*(A - A_0), \; D_A\phi)$$

$b_1 = 0$

$$M_{SW} = \widetilde{SW}^{-1}(0)/G_n = \widetilde{SW}^{-1}(0)/S^1 \qquad \Rightarrow G_n = \{\text{constant } X \to S^1\}$$

This space may not be a manifold. for 2 reasons

- $\widetilde{SW}^{-1}(0)$ may not be a manifold
- S^1-action may not be free.

$$\begin{cases} F^+_{A^c} + \rho^{-1}(\phi\phi^*)_0 = \eta \\ D_A\phi = 0 \end{cases}$$

$e^{i\theta}(A, \phi) = (A, e^{i\theta}\phi)$

Idea: consider $\widetilde{SW}^{-1}(\eta, 0, 0)/S^1$ for a suitable $\eta \in \Omega^2_+$.

Theorem: For a generic choice of $\eta \in \Omega^2_+(X; i\mathbb{R})$, $\widetilde{SW}^{-1}(\eta, 0, 0)$ is a compact manifold of dimension $\dfrac{c_1(S)^2 - \sigma(X)}{4} + b^1(X) - b^2_+(X)$

proof: We consider the differential of $\widetilde{SW}$

$$d\widetilde{SW}_{(A,\phi)}: \Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+) \longrightarrow \Omega^0(X; i\mathbb{R})/\mathbb{R} \oplus \Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-)$$

$$(\alpha, \psi) \longmapsto \underbrace{(d^*, d^+, \not{D}_A)}_{\text{elliptic}} + \text{0-th order term}$$

We let $H_1 = L^2_k(\Omega^1 \oplus \Gamma(S^+))$

$\qquad H_2 = L^2_{k-1}(\Omega^0/\mathbb{R} \oplus \Omega^2_+ \oplus \Gamma(S^-))$

$$\langle f, g \rangle_{L^2_k} := \sum_{i=0}^{k} \int \langle \nabla^{(i)} f, \nabla^{(i)} g \rangle \, dvol \quad \text{sobolev norm}$$

Then $\widetilde{SW}: H_1 \to H_2$ is a C^∞- Fredholm map

(i.e. $d\widetilde{SW}_{(A,\phi)}$ is Fredholm).

$d\widetilde{SW}_{(A,\phi)}$ is a Fredholm operator whose index is

$$\operatorname{index}(D_A) + \operatorname{index}(\Omega^1 \xrightarrow{d^* \oplus d^+} \Omega^0/\mathbb{R} \oplus \Omega^2_+)$$

Atiyah-Singer index theorem $\Rightarrow$

$c_1(\$) := c_1(S^\pm)$

$$\operatorname{ind}_{\mathbb{C}}(D_A) = \frac{c_1(\$)^2 - \sigma(X)}{8}$$

$$\operatorname{ind}_{\mathbb{R}}(\Omega^1 \xrightarrow{d^* \oplus d^+} \Omega^0/\mathbb{R} \oplus \Omega^2_+) = b_1(X) - b_2^+(X)$$

$\ker = \mathcal{H}^1$ $\quad$ $\operatorname{coker} \mathcal{H}^2_+$

so $\widetilde{SW}: H_1 \to H_2$ is Fredholm map whose index is

$$\frac{c_1(\$)^2 - \sigma(X)}{4} + b^1(X) - b_2^+(X).$$

Sard-Smale theorem: Let $f: B_1 \to B_2$ be a C^q-Fredholm map between two Banach manifolds. Suppose $q > \max\{0, \operatorname{index}(f)\}$. Then the set of regular value of f is residue.

residue: intersection of countably many open dense set. (generic)

b is a regular value: $\forall x \in f^{-1}(b)$

$df(x): T_x B_1 \to T_x B_2$ is surjective.

(If $\operatorname{index}(f) < 0$, this simply means $f^{-1}(b) = \phi$.)

b is regular value $\Rightarrow f^{-1}(b)$ is a manifold of $\dim = \operatorname{index}(f)$.

so we see for generic η, ψ, β.

$\widetilde{SW}^{-1}(\eta, \psi, \beta)$ is a manifold of dimension $\frac{c_1^2 - \sigma}{4} + b^1 - b^+$.

some more work $\Rightarrow \psi, \beta$ can be chosen to be 0. $\square$.

So we pick generic η s.t. $\widetilde{SW}^{-1}(\eta,0,0)$ is a manifold of dimension $\frac{c_1^2-\sigma}{4}+b_1-b_2^+$.

Let's assume $b_1(X)=0$. Then $\mathcal{G}_n=$ constant map $X\to U(1)$

$$\mathcal{M}_{SW}^{\eta}=\widetilde{SW}^{-1}(\eta,0,0)/\mathcal{G}_n=\widetilde{SW}^{-1}(\eta,0,0)/S^1$$

$u\cdot(A,\phi)=(A-u^{-1}du,\ u\phi)$ $\quad u: X\to S^1$

Note $e^{i\theta}(A,\phi)=(A,e^{i\theta}\phi)$. So if $\phi\neq 0$ $\forall (A,\phi)\in\widetilde{SW}^{-1}(\eta)$

Then the S^1-action is free.

We say a solution $(A,\phi)\in\widetilde{SW}^{-1}(\eta)$ is irreducible if $\phi\neq 0$. Otherwise, we say it is reducible.

Theorem: Suppose $b^+(X)>0$. Then for a generic η, $\widetilde{SW}^{-1}(\eta)$ has no reducible point. Furthermore, if $b^+(X)>1$. Then any such η_0,η_1 can be connected by a generic path $\{\eta_t\}$ s.t. $\widetilde{SW}^{-1}(\eta_t)$ has no reducible point $\forall t$. $pr_{H^+}F^+_{A^c}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

proof: $(A,0)\in\widetilde{SW}^{-1}(\eta)\Rightarrow F^+_{A^c}=\eta$ --- ① $pr_{H^+}F_{A^c}$

Recall: $\Omega^2=\text{image}\, d\oplus\mathcal{H}^+\oplus\mathcal{H}^-\oplus\text{image}\, d^*$

$c_1(\mathcal{S})=[\frac{i}{2\pi}F_{A^c}]$ so $pr_{\mathcal{H}^+}F_{A^c}=pr_{\mathcal{H}^+}(-2\pi i c_1(\mathcal{S}))$

$c_1(S^+)$ $\quad$ treat $c_1(\mathcal{S})$ as in $\mathcal{H}^+\oplus\mathcal{H}^-$

① $\Rightarrow pr_{\mathcal{H}^+}\eta=pr_{\mathcal{H}^+}F^+_{A^c}=pr_{\mathcal{H}^+}F_{A^c}$

$=pr_{\mathcal{H}^+}(-2\pi i c_1)$

So if we pick any $\eta \in \Omega^2_+(X; i\mathbb{R})$ s.t.

$P_{\mathcal{H}^+}\eta \neq P_{\mathcal{H}^+}(-2\pi i c_1(\mathfrak{S}))$. Then $\widetilde{SW}^{-1}(\eta)$ has no reducible.

Note $\dim \mathcal{H}^+ = b_2^+(X)$ so if $b_2^+(X) > 0$, we can also choose such η.

- when $b_2^+ > 1$, any two choices of η can be connected by $\{\eta_t\}$ s.t. $\widetilde{SW}^{-1}(\eta_t)$ has no reducible.
- when $b_2^+ = 1$. $\mathcal{H}^+ \setminus P_{\mathcal{H}^+}(-2\pi i c_1(\mathfrak{S})) = \mathbb{R} \setminus \{a\}$ has two components. So upto homotopy, $\exists$ two choices of η.

If $b^+ > 0$

So we choose generic η s.t. $\widetilde{SW}^{-1}(\eta)$ is a manifold with free S^1-action.

So $\mathcal{M}_{SW} = \widetilde{SW}^{-1}(\eta) / \mathcal{G}_n$ is a compact manifold.

It's dimension equals

$$d(\mathfrak{S}) = \frac{c_1^2 - \sigma}{4} + b_1 - b^+ - 1 = \frac{c_1(\mathfrak{S})^2 - (3\sigma(X) + 2\chi(X))}{4}$$

In particular, when $d(\mathfrak{S}) = 0$, $\mathcal{M}_{SW}$ is a finite collection of points.

Actually, $\mathcal{M}^{\eta}_{SW}$ is an orientable manifold, and an orientation is specified by an orientation of $\Lambda^{\max} \ker d\widetilde{SW}$. We can deform $d\widetilde{SW}$ to $D_{A_0} \oplus (d^* \oplus d^+)$

So there is natural bijection

orientation of $\Lambda^{\max} d\widetilde{SW}_{(A,\phi)} \longleftrightarrow$ orientation of $\Lambda^{\max}\ker(D\oplus d^*\oplus d^+) \otimes \Lambda^{\max}\operatorname{coker}(D\oplus d^*\oplus d^+)$

We have proved

Proposition: A orientation of $\mathcal{M}_{SW}$ is specified by

by an orientation of $H^2_+(X)\oplus H^1(X) \longleftarrow$ homology orientation of X.

The Seiberg-Witten invariants of X

X: Smooth 4-mfd $b^+(X)>1$

g: Riemannian metric on X

$\mathfrak{s}$: spin^c structure on X

η: a generic element in $\Omega^2_+(X; i\mathbb{R})$

S^1 acts freely on $\widetilde{SW}^{-1}(\eta)$, give us P:

$$\begin{array}{ccc} S^1 & \hookrightarrow & \widetilde{SW}^{-1}(\eta) \\ & & \downarrow \\ & & \mathcal{M}_{SW} \end{array}$$

Definition: The Seiberg-Witten invariant

$$SW(X,\mathfrak{s}) := c_1(P)^{\frac{d(\mathfrak{s})}{2}} \cdot [\mathcal{M}_{SW}] \in \mathbb{Z}$$

When $d(\mathfrak{s})=0$, this is just signed count of points in $\mathcal{M}_{SW}$

simple type conjecture: If $d(\mathfrak{s})>0$, $b^+>1$ then $SW(X,\mathfrak{s})=0$.

Theorem (Taubes) The simple type conjecture holds for symplectic manifolds.

We still have to prove that $SW(X, \mathfrak{S})$ is actually an invariant.

Proposition: Given (g_j, η_j) $j=0,1$ s.t.

$$P_{H^+_{g_j}}(\eta_j + 2\pi i c_1(\mathfrak{S})) \neq 0 \quad \text{--- } ①$$

admissibility condition

If $b^+(X) > 1$, then they give the same Seiberg-Witten invariant.

Proof: We connect (g_j, η_j) by a path (g_t, η_t) s.t. ① is satisfied for any t. Then $\cup \widetilde{SW}^{-1}_{g_t}(\eta_t)$ is a cobordism between $\widetilde{SW}^{-1}_{g_0}(\eta_0)$ and $\widetilde{SW}^{-1}_{g_1}(\eta_1)$.

$$S^1 \hookrightarrow \bigcup_t \widetilde{SW}^{-1}_{g_t}(\eta_t) \longrightarrow \bigcup_t \mathcal{M}_{sw}(g_t, \eta_t).$$

so $[\mathcal{M}_{sw}(g_0, \eta_0)] = [\mathcal{M}_{sw}(g_1, \eta_1)] \in H_*(\bigcup_t \mathcal{M}_{sw}(g_t, \eta_t))$ $\square$

When $b_2^+(X) = 1$, $\{(g, \eta) \mid ① \text{ is satisfied}\}$ has two components called "chambers". Since we have oriented H^+, it makes sence to talk about "+" chamber, "−" chamber

so we have $SW^+(X, \mathfrak{S})$, $SW^-(X, \mathfrak{S})$

(Wall-crossing formula) If $b_1(X) = 1$,

$$SW^+(X, \mathfrak{S}) - SW^-(X, \mathfrak{S}) = \pm 1$$

Let's assume $b^+(X) > 1$.

$$SW_X : \operatorname{Spin}^{\mathbb{C}}(X) \longrightarrow \mathbb{Z}$$

Theorem: $\exists$ only finitely many $\mathfrak{s}$ s.t. $SW_X(\mathfrak{s}) \neq 0$.

Alternatively, we can define

$$SW_X : H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$\alpha \longmapsto \sum_{\substack{\mathfrak{s} \in \operatorname{Spin}^{\mathbb{C}}(X) \\ c_1(\mathfrak{s}) = \alpha}} SW_X(\mathfrak{s})$$

Theorem (Witten) If $b_2^+(X) > 1$, X has psc metric, then $SW_X = 0$.

$$S_0 = \min_p S(p) > 0$$

proof: Let $(A,\phi) \in \widetilde{SW}^{-1}(\eta)$. Let $C = \max_p |\phi(p)|$

Then repeat our proof in the compactness theorem.

We get $C^4 + C^2(-|\eta|_{C^0} + S_0) \leq 0$

If we choose $|\eta|_{C^0} < S_0$, and (η, g) is admissible.

Then $C = 0$, i.e. (A,ϕ) is a reducible solution. which is not possible. So $\widetilde{SW}^{-1}(\eta) = \phi$. So $SW_X(\mathfrak{s}) = 0$. $\square$