Hodge Laplacian and geometry of Kuranishi family of Fano manifolds

Akito Futaki
清華大学, 丘成桐数学科学中心

Current Developments in Mathematics and Physics
Celebrating the 15th Anniversary of
Yau Mathematical Sciences Center
2024年4月4日　丘成桐先生75歳生日快樂

Kodaira-Spencer deformation theory of compact complex manifolds

If $H^2(M, \mathcal{O}(T'M)) = 0$ e.g. when M is Fano,

then $\exists \varpi : \mathcal{M} \to B \subset \mathbb{C}^n \cong H^1(M, \mathcal{O}(T'M))$ such that for $M_t := \varpi^{-1}(t)$

$$\left\{ \frac{\partial M_t}{\partial t}\bigg|_{t=0} \right\} \cong H^1(M, \mathcal{O}(T'M))$$

where $\frac{\partial M_t}{\partial t}$ is the infinitesimal generator, or the Cech coholomogy class of the derivative of the coordinate changes.

Study the Kähler geometry of deformations of Kähler manifolds

H.D. Cao, X.F. Sun, S.T. Yau, Y.Y. Zhang (2022, Math.Ann.) and earlier works by Sun, Zhang :
The case when $M_0 = M$ is a Kähler-Einstein manifold.

F-Sun-Zhang (to appear in Kyoto J. Math.):
The case when $M_0 = M$ is a Fano manifold.

F (in press, Pure Applied Math Quaterly):
The case when $M = M_0$ is a Fano manifold with a weighted soliton (generalized Kähler-Ricci soliton).

Preparatory results on eigenvalues of Hodge Laplacian

Let us recall the Kodaira vanishing.

Let (M, g) be a compact Kähler manifold,

(L, h) be an Hermitian line bundle over M.

For the $\overline{\partial}$-Laplacian $\Delta^L_{\overline{\partial}} = \overline{\partial}^*_L \overline{\partial} + \overline{\partial} \overline{\partial}^*_L$
acting on an L-valued $(0, q)$-form η we have

$$\begin{aligned}(\Delta^L_{\overline{\partial}}\,\eta)_{\overline{j}_1 \cdots \overline{j}_q} &= -g^{i\overline{j}} \nabla^L_i \nabla_{\overline{j}} \eta_{\overline{j}_1 \cdots \overline{j}_q} \\ &\quad - \sum_{\beta=1}^{q} (-1)^\beta g^{i\overline{j}} [\nabla^L_i, \nabla_{\overline{j}_\beta}] \eta_{\overline{j}\,\overline{j}_1 \cdots \widehat{\overline{j}_\beta} \cdots \overline{j}_q}.\end{aligned}$$

Using the Ricci identity we obtain the following *Bochner-Kodaira formula*

$$
\begin{aligned}
(\Delta^L_{\bar\partial}\eta)_{I\bar j_1\cdots\bar j_q} &= -g^{i\bar j}\nabla^L_i\nabla_{\bar j}\eta_{\bar j_1\cdots\bar j_q} \\
&\quad + \sum_{\beta=1}^{q} g^{i\bar j}(R_{i\bar j_\beta}+\psi_{i\bar j_\beta})\eta_{\bar j_1\cdots\bar j_{\beta-1}\bar j\bar j_{\beta+1}\cdots\bar j_q}.
\end{aligned}
$$

Hence, if $-K_M + L$ is positive then

$$H^q(M,\mathcal{O}(L)) = 0 \quad \text{for} \quad q > 0.$$

This is the proof of Kodaira vanishing.

Let M be a Fano manifold of dimension m,

i.e. $2\pi c_1(M)$ is represented by a Kähler form.

Let ω be a Kähler form in $2\pi c_1(M)$, and

$$\mathrm{Ric} - \omega = \sqrt{-1}\partial\bar{\partial} f;$$

f is called the **Ricci potential**.

Let $L = \mathcal{O}$ be the trivial line bundle with the Hermitian metric e^f.

We write $\Delta_f := \Delta^L_{\bar\partial}$ for our choice of the Hermitian metric e^f on L.

This is the same as considering the weighted volume form $e^f \omega^m$ for $(0,q)$-forms,

and considering the weighted Hodge Laplacian $\Delta_f = \bar\partial^*_f \bar\partial + \bar\partial\, \bar\partial^*_f$

acting on differential forms of type $(0,q)$ where

$\bar\partial^*_f$ is the formal adjoint of $\bar\partial$ with respect to

the weighted L^2-inner product $\int_M (\cdot,\cdot) e^f \omega^m$.

For $\eta \in A^{0,q}(L) = A^{0,q}(M)$, and then, by the Bochner-Kodaira formula reads

$$
\begin{aligned}
(\Delta_f \eta)_{\bar{j}_1 \cdots \bar{j}_q} &= -g^{i\bar{j}} \nabla_{i,f} \nabla_{\bar{j}} \eta_{\bar{j}_1 \cdots \bar{j}_q} \\
&\quad + \sum_{\beta=1}^{q} g^{i\bar{j}} (R_{i\bar{j}_\beta} - f_{i\bar{j}_\beta}) \eta_{\bar{j}_1 \cdots \bar{j}_{\beta-1} \bar{j} \bar{j}_{\beta+1} \cdots \bar{j}_q} \\
&= -g^{i\bar{j}} \nabla_{i,f} \nabla_{\bar{j}} \eta_{\bar{j}_1 \cdots \bar{j}_q} + q\ \eta_{\bar{j}_1 \cdots \bar{j}_q}
\end{aligned}
$$

where

$$\nabla_{i,f} = \nabla_i + f_i.$$

Hence if $\Delta_f \eta = \lambda \eta$ then

$$\lambda(\eta, \eta)_f = (\Delta_f \eta, \eta)_f = (\nabla'' \eta, \nabla'' \eta)_f + q(\eta, \eta)_f$$

and

$$\lambda \geq q$$

.

If $\lambda = q$ then $\nabla'' \eta = 0$.

Since $\overline{\partial} \eta$ is the skew-symmetrization of $\nabla'' \eta$ it follows that $\overline{\partial} \eta = 0$.

Moreover, since $H^{0,q}_{\overline{\partial}}(M) = 0$ for $q \geq 1$ on the Fano manifold M, η is exact.

Thus we have proved

Theorem A(FSZ)

Let M be a Fano manifold and Δ_f be the weighted Hodge Laplacian as above.

(1) If $\Delta_f \eta = \lambda\eta$ and $\eta \neq 0$ for a $(0, q)$-form η then $\lambda \geq q$.

(2) If, in (1), $\lambda = q$ and $\eta \neq 0$ then $\nabla''\eta = 0$.

In particular η is closed, and for $q \geq 1$, η is exact,

and indeed, it is expressed as $\eta = \frac{1}{q}\overline{\partial}(\overline{\partial}^*_f \eta)$.

Theorem B(FSZ)

Let M be a Fano manifold and Δ_f be the weighted Hodge Laplacian as above.

(1) If $\Delta_f \eta = \lambda \eta$ for a (0, q)-form η and $\overline{\partial}\eta \neq 0$ then $\lambda \geq q + 1$.

(2) If, in (1), $\lambda = q + 1$ then

$$(\overline{\partial}\eta)^\sharp := g^{i\bar{j}} g^{i_1\bar{j}_1} \cdots g^{i_q\bar{j}_q} \nabla_{\bar{j}} \eta_{\bar{j}_1 \cdots \bar{j}_q} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial z^{i_q}} \quad (1)$$

is a holomorphic section of $\wedge^{q+1} T'M$.

Corollary C [F, around 1980]

(1) If $\Delta_f u = \lambda u$ for a non-constant complex-valued smooth function u then $\lambda \geq 1$.

(2) If, in (1), $\lambda = 1$ then $(\overline{\partial} u)^\sharp$ is a holomorphic vector field.

Kuranishi family

Let $\varpi : \mathcal{M} \to B$ be a deformation of $M_0 := \varpi^{-1}(0) = M$.
Considering at $t = 0$, for $t \in B$ small,

The small deformation is described by

$$\varphi(t) = \varphi^i_{\bar{j}}(t)\frac{\partial}{\partial z^i} \otimes dz^{\bar{j}} \in \Omega^{0,1}(T'M)$$

satisfying

$$\begin{cases} \overline{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0; \\ \varphi(0) = 0; \\ \frac{\partial \varphi(t)}{\partial t}|_{t=0} =: \eta. \end{cases}$$

where $\overline{\partial}\eta = 0$ and $[\eta] \in H^{0,1}_{\overline{\partial}}(T'M) \cong H^1(M, \mathcal{O}(T'M))$.

Choose a Kähler form ω in $2\pi c_1(M)$ with the Ricci potential f as before.

In this case, we have the Kuranishi family described by

$$\varphi(t) = \sum_{|I|=1} t^I \varphi_I + \sum_{|I|\geq 2} t^I \varphi_I$$

$$\begin{cases} \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]; \\ \\ \overline{\partial}_f^*\varphi(t) = 0; \\ \\ \text{For } |I| = 1, \varphi_I \text{ is } \Delta_f\text{-harmonic, so } \eta = \sum_{|I|=1} t^I \varphi_I \end{cases}$$

Theorem D(FSZ)

Let M be a Fano manifold, ω a Kähler form in $2\pi c_1(M)$,

and $\{M_t\}$ be the Kuranishi family of the deformation of complex structures described as above.

Then ω is a Kähler form on M_t for any t.

Proof of Theorem D: Consider

$$\begin{aligned}\varphi\lrcorner\omega &:= \varphi^i{}_{\overline{k}} dz^{\overline{k}} \wedge \sqrt{-1} g_{i\overline{j}} dz^{\overline{j}} + \varphi^i{}_{\overline{j}} dz^{\overline{j}} \wedge \sqrt{-1} g_{i\overline{k}} dz^{\overline{k}} \\ &= -\sqrt{-1}(\varphi_{\overline{j}\overline{k}} - \varphi_{\overline{k}\overline{j}}) dz^{\overline{j}} \wedge dz^{\overline{k}} \\ &= -\sqrt{-1}\ \psi_{\overline{j}\,\overline{k}}\, dz^{\overline{j}} \wedge dz^{\overline{k}},\end{aligned}$$

For the Kuranishi family above, we can show

$$\Delta_f(\varphi\lrcorner\omega) = \frac{1}{2}\varphi\lrcorner\omega.$$

Combining this with Theorem A, we obtain $\varphi\lrcorner\omega = 0$.

Since $d\omega = 0$ it is sufficient to show ω is J_t invariant.

But
ω is J_t invariant
$\Longleftrightarrow$
$\varphi_{\overline{k}\overline{j}} := g_{i\overline{k}}\varphi^i{}_{\overline{j}}$ is symmetric in j and k
(since $T''M_t$ is spanned by $\frac{\partial}{\partial z^{\overline{j}}} - \varphi^i{}_{\overline{j}}(t)\frac{\partial}{\partial z^i}$).
$\Longleftrightarrow$
$\varphi \lrcorner \omega = 0$.

This completes the proof of Theorem D.

Theorem E(FSZ) In Theorem D, the Ricci potential of (M_t, ω) is given by

$$f + \log\det(1 - \varphi(t)\overline{\varphi}(t))$$

up to an additive constant, more precisely,

$$\mathrm{Ric}(M_t, \omega) = \omega + \sqrt{-1}\partial_t\overline{\partial}_t(f + \log\det(1 - \varphi(t)\overline{\varphi}(t)))$$

where $\mathrm{Ric}(M_t, \omega)$ denotes the Ricci form with respect to the complex structure J_t on M_t.

Remark: In the case when M_0 is a Kähler-Einstein manifold, Theorem E has been obtained by Cao-Sun-Yau-Zhang who gave a necessary and sufficient condition for the existence of Kähler-Einstein metrics on small deformations of a Fano Kähler-Einstein manifolds.

Application to weighted solitons on Fano manifolds

Let M be a Fano manifold. i.e. $c_1(M) > 0$, so a Kähler class.

The Kähler form ω is expressed as

$$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}.$$

Let $T \subset \mathrm{Aut}(M)$ be a toral subgroup,

and assume that ω is T-invariant.

Since M is Fano and simply connected, the T-action is Hamiltonian with respect to ω, and we have a canonically normalized moment map $\mu_\omega : M \to \mathfrak{t}^*$.

Let $\Delta := \mu_\omega(M)$ be the moment polytope.

Then Δ is independent of $\omega \in 2\pi c_1(M)$.

Let v be a positive smooth function on Δ.

Regarding μ as coordinates on Δ using the action angle coordinates, we may sometimes write $v(\mu)$ instead of v.

The pull-back $\mu_\omega^* v$ is a smooth function on M, and for this we write $v(\mu_\omega) = \mu_\omega^* v = v \circ \mu_\omega$.

We say that a Kähler metric ω in $2\pi c_1(M)$ a **weighted v-soliton** or simply v-**soliton** if

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\overline{\partial}\log v(\mu_\omega)$$

where $\mathrm{Ric}(\omega) = -i\partial\overline{\partial}\log \omega^m$ is the Ricci form.

We also call ω simply a **weighted soliton** when it is v-soliton for some v, or when v is obvious from the context.

Examples of weighted solitons:

(1) $v(\mu) = e^{\langle \mu, \xi \rangle}$ for some $\xi \in \mathfrak{t}$ induces a **Kähler-Ricci soliton**,

(2) $v(\mu) = \langle \mu, \xi \rangle + a$ for some positive constant a induces a **Mabuchi soliton**,

(3) $v(\mu) = (\langle \mu, \xi \rangle + a)^{-m-2}$, $m = \dim_{\mathbb{C}} M$, induces a **Sasaki-Einstein metric** on the $U(1)$-bundle of K_M.

The Kuranishi family we consider in this talk is described by a family of vector valued 1-forms parametrized by $t \in B$

$$\varphi(t) = \sum_{i=1}^{k} t^i \varphi_i + \sum_{|I|\geq 2} t^I \varphi_I \ \in \ A^{0,1}(T'M)$$

such that

$$\begin{cases} \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]; \\ \overline{\partial}^*_f \varphi(t) = 0; \\ \varphi_1, \cdots, \varphi_k \text{ form a basis of the space of all} \\ \qquad\qquad T'M\text{-valued } \Delta_f\text{-harmonic } (0,1)\text{-forms} \end{cases}$$

where $\Delta_f = \overline{\partial}^*_f \overline{\partial} + \overline{\partial}\, \overline{\partial}^*_f$ is the weighted Hodge Laplacian with $\overline{\partial}^*_f$ the formal adjoint of $\overline{\partial}$ with respect to the weighted L^2-inner product $\int_M (\cdot,\cdot) e^f \omega^m$.

Recall that Futaki-Sun-Zhang showed that the Kähler form ω on $M_0 = M$ remains to be a Kähler form on M_t.

Theorem F

Suppose that M_0 has a weighted v-soliton. Consider the Kuranishi family with $f = \log v(\mu_\omega)$ as above.

Then, shrinking B if necessary, the following statements are equivalent.

(1) M_t has a weighted v-soliton for all $t \in B$.

(2) T is included in $\mathrm{Aut}(M_t)$, and for the centralizer $\mathrm{Aut}^T(M_t)$ of T in $\mathrm{Aut}(M_t)$, $\dim \mathrm{Aut}^T(M_t) = \dim \mathrm{Aut}^T(M_0)$ for all $t \in B$.

(3) T is included in $\mathrm{Aut}(M_t)$, and the identity component $\mathrm{Aut}_0^T(M_t)$ of $\mathrm{Aut}^T(M_t)$ is isomorphic to $\mathrm{Aut}_0^T(M_0)$ for all $t \in B$.

Happy 75th birthday of Prof. Yau.
Thank you for your attention.