Rigid Local Systems: Arithmetic Properties

Hélène Esnault
Freie Universität Berlin

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Rigid Local Systems (RLS)

• X topological space, for us *complex algebraic variety*;
• irreducible *local system* (LS) $\stackrel{\text{dfn}}{=}$ irreducible
$\rho : \pi_1^{\mathrm{top}}(X, x) \to GL_r(\mathbb{C})$ up to gauge transformation;
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• ρ *rigid*: deformation $\rho_t : \pi_1^{\rm top}(X, x) \to GL_r(\mathbb{C}[[t]])$ gauge equivalent to ρ_0, i.e. $\exists g_t \in GL_r(\mathbb{C}[[t]])$ with $\rho_t = g_t \rho_0 g_t^{-1}$, i.e. moduli point $[\mathcal{V}_\rho]$ isolated.
• May fix determinant and conjugacy classes of local monodromy at ∞ for X not proper, and pose the moduli problem with them.

Occurence

$\dim(X) = 1$ Katz: $\exists$ RLS $\Longrightarrow$
• $X = \mathbb{P}^1 \setminus \{\text{finitely many points}\}$;
• moduli points have no multiplicity;
• all RLS *come from geometry*, i.e. 'like' (summand of)
$\mathcal{V}_{\rho,\tau\in\mathbb{P}^1\setminus\{\infty,\tau_1,\ldots,\tau_n\}} = H^1_c(Y_\tau), Y_\tau \subset \mathbb{A}^2 : \; y^N = \prod_1^n (x-\tau)(x-\tau_i)$.

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comes from geometry: $\exists f : Y \to U$ smooth projective with $U \subset X$ open dense so $\mathcal{V}_\rho$ is a summand of $\oplus R^i f_*\mathbb{Z}$.

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Geometricity conjecture inacessible, except in dim. 1 (Katz).
Instead study consequences.

Consequences of Simpson's geometricity conjecture

Integrality conjecture (Simpson 1990)

RLS are integral, i.e. $\rho : \pi_1(X, x) \to GL_r(\mathcal{O})$, $\mathcal{O}$ number ring.

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Crystalline conjecture (E-Groechenig 2018)

connection corresponding via the Riemann-Hilbert correspondence to a RLS $/X_{\mathbb{Q}_q}$ for a.a. p is
i) an isocrystal with a Frobenius structure;
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i) Gauß-Manin connections are so by Deligne;

ii) if $f : Y \to X/K$ K p-adic field is projective and has good reduction mod p, then $R^i f_* \mathbb{Q}_p$ is crystalline.

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Remark

- Not a single RLS known with moduli point with higher multiplicity.
- Crystallinity applies for all proper or rank ≥ 2 Shimura varieties.

Method: integrality

- $\rho : \pi_1(X, x) \to GL_r(\mathcal{O}_E[1/N])$
- λ place of $E \nmid N, p : \pi_1^{\mathrm{top}}(X, x) \to \pi_1(X_{\bar{\mathbb{F}}_p}, x) \to GL_r(E_\lambda)$ for p large (Grothendieck)
- rigid $\Longrightarrow$ descends to $\pi_1(X_{\mathbb{F}_q}, x) \to GL_r(E_\lambda)$ (Simpson)
- companions for $\lambda'|N$: $\rho' : \pi_1^{\mathrm{top}} \to \pi_1(X_{\bar{\mathbb{F}}_p}, x) \to GL_r(E_{\lambda'})$ also rigid (assumption no multiplicity or all LS are RLS)
- bijection $\rho \leftrightsquigarrow \rho' \Longrightarrow$ integrality.

Method: crystallinity

- $(\hat{E}, \hat{\nabla})$ on $X_W, \ W = W(\mathbb{F}_q) \rightsquigarrow$ Higgs-de Rham flow
- Higgs-de Rham flow $\rightsquigarrow$ Fontaine-Lafaille module (Lan-Sheng-Zuo)
- $\implies$ (fully faithful) crystalline representation $\pi_1(X_{\mathrm{Frac}\,W}) \to GL_r(W(\mathbb{F}_{p^f}))$ (Faltings)
- when all LS are RLS: again counting $\implies$ all crystalline.

One consequence of our crystallinity theorem

Pila-Ananth Shankar-Tsimerman (posted on Sept. 21 2021)

André-Oort conjecture for all Shimura varieties.
[i.e: Zariski closure of special points is special $\subset$ Shimura variety.]

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Proof of Theorem in the hands of experts now for checking.