Extensions of Holomorphic Forms

Kefeng Liu

UCLA

October 13, 2021

Dedicated to the 110th anniversary of S.-S. Chern.

I first met Professor Chern during the 1985 summer school organized by him in Nankai Institute of Mathematics, since then I have been greatly influenced by Professor Chern both in mathematics and life.

We all belong to the Chern class.

Plan of my lecture

- Motivation and background.
- Results and applications.
- Solving the Beltrami equations.
- Proving the Newlander-Nirenberg theorem.
- Solving the existence equations of local pseudoholomorphic curves.

  A Hodge-theoretic localization method of solving obstruction equations to the extensions of holomorphic forms.

Hodge theory

Let X be a Kähler manifold with a Kähler form γ on which L^2-Hodge theory holds. Let

$$\Box_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

be the $\bar{\partial}$-Laplacian, G and H denote the Green operator and the harmonic projection respectively. On the Hilbert spaces of L^2-forms, we have

$$I = H + \Box_{\bar{\partial}} G.$$

The proof of this theorem was first given by Kodaira and Weyl. Various generalizations by Kohn, Hörmander, Morrey and many others.

Quasi-isometry

We denote by $A^{p,q}(X)$ the space of smooth (p,q)-forms on X. With Rao and Yang, we proved the following quasi-isometry formulas,

Proposition

On compact Kähler manifolds. For any smooth $g \in A^{p,q}(X)$, the L^2-norms satisfy

$$\|\bar{\partial}^* G \partial g\| \leq \|g\|.$$

In particular, if $\bar{\partial}\partial g = 0$ and g is ∂^*-exact, we obtain the isometry

$$\|\bar{\partial}^* G \partial g\| = \|g\|.$$

These formulas have found interesting applications in understanding the structure of deformation spaces of Kähler manifolds.

Kodaira-Spencer-Kuranishi local deformation theory

Let $\pi : \mathcal{X} \to \Delta \subset \mathbb{C}$ be a holomorphic family of compact complex manifolds over a small disc Δ with

$$\pi^{-1}(0) = X_0 = X$$

Kähler. Then the complex structures on $X_t = \pi^{-1}(t)$ are determined by Beltrami differentials

$$\varphi = \varphi(t) = \sum_{i=1}^{\infty} \varphi_i t^i \in A^{0,1}(X, T^{1,0}X)$$

on the central fiber X such that the Maurer-Cartan equation is satisfied,

$$\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]. \quad (1)$$

Variations of complex structures

In general, a Beltrami differential φ is a smooth section of $A^{0,1}(X, T^{1,0}X)$ of the tangent bundle $T^{1,0}X$-valued $(0,1)$ form, such that the integrability condition

$$\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi] \tag{2}$$

is satisfied. By the theorem of Newlander-Nirenberg, such Beltrami differential induces a new complex structure on X.
For simplicity, we denote the corresponding complex manifold by

$$X_\varphi, \ X_{\varphi(t)} \text{ or } X_t.$$

Action of Beltrami differential

Given any differential form ω, let us denote by

$$\varphi \lrcorner \omega = i_\varphi \omega = \varphi \omega$$

the natural contraction morphism, and

$$\rho(\omega) = e^{i_\varphi} \omega = e^\varphi \omega = \omega + i_\varphi \omega + \frac{1}{2!} i_\varphi i_\varphi \omega + \cdots .$$

We will consider the deformed Cauchy-Riemann equation for a differential form f,

$$\bar{\partial} f + \partial(\varphi \lrcorner f) = 0 \tag{3}$$

Equation (3) can be either local or global on manifold. It is the obstruction equation to the extensions of holomorphic forms with respect to the variation of complex structrures.

Solving $\bar{\partial}$-equations

Lemma

Let X be a compact Kähler manifold, then given any $g \in A^{p,q}(X)$ with $\bar{\partial}\partial g = 0$, the equation

$$\bar{\partial} f = \partial g$$

has a solution $f = \bar{\partial}^* G \partial g$ with L^2-estimate

$$\|f\| \leq \|g\|.$$

We will extend and apply this operator $T = \bar{\partial}^* G \partial$ to various situations in studying equations related to complex structures.

Motivation

Chern, Ahlfors and many others introduced integral operators in solving Beltrami equations on discs in $\mathbb{C}$, such as

$$Ph(\zeta) = -\frac{1}{\pi}\int\int h(z)(\frac{1}{z-\zeta} - \frac{1}{z})dxdy$$

$$Th(\zeta) = \lim_{\varepsilon\to 0}\int\int_{|z-\zeta|>\varepsilon} \frac{h(z)}{(z-\zeta)^2}dxdy$$

$$(Ph)_{\bar{z}} = h, \ (Ph)_z = Th.$$

$$L^2 - \text{isometry}: \int\int |Th|^2 dxdy = \int\int |h|^2 dxdy.$$

with $h \in C_0^2$ and $z = x + iy$.

Beltrami equations

Chern: An Elementary Proof of the Existence of Isothermal Parameters on a Surface. (1955).

Beltrami equations have a long history and have important applications in complex and analytic geometry, such as in Teichmüller theory.

Integral methods have been developed for solving such equations by Chern, Morrey, Ahlfors, Bers and many others. The construction by Ahlfors depends on rather deep analysis and estimate of Calderón-Zygmund.

Our method uses L^2-Hodge theory.

Beltrami equations

Recall that the Beltrami equation is to solve for a function on disc D,

$$f: \ D \to \mathbb{C}$$

such that

$$f_{\bar{z}} = \mu_0 f_z.$$

Here z is complex analytic coordinate in D and μ can be considered as a Beltrami differential. Write

$$\mu = \mu_0 \frac{\partial}{\partial z} \otimes d\bar{z}$$

in the standard form of Beltrami differential, we can rewrite the Beltrami equation in the form,

$$\bar{\partial} f = \mu \lrcorner \partial f$$

We will assume μ is measurable, or equivalently μ_0 is a measurable function.

Beltrami equations

Instead we solve the obstruction equation for holomorphic one form in the equation,

$$\bar{\partial} h + \partial(\mu \lrcorner h) = 0$$

where $h = a(z)dz$ is an L^2-one form on D.

This equation is much better behaved, and can be solved by using standard Hodge theory. Its solutions give the solutions of the Beltrami equations.

Motivated by the local integral operators of Chern, Ahlfors et al, we introduce

Definition

$$P = \bar{\partial}^* G, \;\; T = \bar{\partial}^* G \partial.$$

Our previous formula tells us that

$$\text{Quasi-isometry}: \|Tf\| \leq \|f\|,$$

so T has operator norm $\|T\| \leq 1$ in the Hilbert space of L^2-forms.

As a direct corollary we have the following

Corollary

Let φ be a Beltrami differential acting on the Hilbert space of L^2-forms by contraction such that its L_∞ norm $||\varphi||_\infty < 1$, then the operator

$$I + T\varphi$$

is invertible.

This operator appears quite often in studying variations of complex structures.

Generalized Cartan formula

Let us see how such equations are related to variation of complex structures.
Consider the graded Lie derivatives associated to Beltrami differentials,

$$\mathcal{L}_\varphi = -d \circ i_\varphi + i_\varphi \circ d; \quad \mathcal{L}_\varphi^{1,0} = -\partial \circ i_\varphi + i_\varphi \circ \partial.$$

The following formula is a special case of a formula obtained with Rao and Yang.

Generalized Cartan formula

Lemma

For any $\varphi, \psi \in A^{0,1}(X, T^{1,0}X)$, *we have the formula*

$$[\mathcal{L}_\varphi, i_\psi] = i_{[\varphi,\psi]}$$

Given any $\sigma \in A^{p,q}(X)$, taking contraction easily gives a much more general formula than the Tian-Todorov lemma,

$$[\varphi, \psi] \lrcorner \sigma = \varphi \lrcorner \partial(\psi \lrcorner \sigma) + \psi \lrcorner \partial(\varphi \lrcorner \sigma) - \partial(\varphi \lrcorner (\psi \lrcorner \sigma)) - \varphi \lrcorner (\psi \lrcorner \partial \sigma) . \quad (4)$$

Such formulas hold on any complex manifold. It is needed in solving the obstruction equations.

Equation for holomorphic form

The following local formulas are special cases of more general formulas.

Lemma

Let $\varphi \in A^{0,1}(X, T^{1,0}X)$. Then on the space $A^{p,q}(X)$, we have

$$e^{-i_\varphi} \circ d \circ e^{i_\varphi} = d - \mathcal{L}_\varphi^{1,0}.$$

In particular, if $\sigma \in A^{n,q}(X)$, then

$$\left(e^{-i_\varphi} \circ d \circ e^{i_\varphi}\right)(\sigma) = \bar{\partial}\sigma + \partial(\varphi \lrcorner \sigma).$$

Given a Kähler manifold X, and a Beltrami differential φ, as a direct corollary, we get

Corollary

Given a local smooth $(n, 0)$ form σ on X, the $(n, 0)$ form on X_φ given by $\rho(\sigma) = e^{i_\varphi}(\sigma)$ is a holomorphic $(n, 0)$ form on X_φ if and only if

$$e^{-i_\varphi} d\rho(\sigma) = \left(e^{-i_\varphi} \circ d \circ e^{i_\varphi}\right)(\sigma) = \bar{\partial}\sigma + \partial(\varphi \lrcorner \sigma) = 0.$$

Another easy corollary is that when taking f a local smooth function, we get that f is holomorphic on X_φ iff

$$\bar{\partial} f - \varphi \lrcorner \partial f = 0.$$

Solving obstruction equation

To find a $(n,0)$-form η on X such that the $(n,0)$-form

$$\rho(\eta) = e^{i_\varphi}\eta$$

on X_φ is closed, or holomorphic,

$$d\rho(\eta) = d(e^{i_\varphi}\eta) = 0,$$

we only need to find a $(n,0)$-form η on X such that

$$\bar{\partial}\eta + \partial(\varphi \lrcorner \eta) = 0 \tag{5}$$

For general (p,q)-forms, we have a double complex

$$(A^{p,q}(X), \partial, \bar{\partial}_\varphi)$$

where

$$\bar{\partial}_\varphi = \bar{\partial} - \mathcal{L}^{1,0}.$$

First we have the following result:

Lemma

Let φ be a Beltrami differential such that the norm $||\varphi||_\infty < 1$. For a fixed d-closed $(n, 0)$-form η_0 on X, then η is a solution to the equation

$$\eta = \eta_0 - \bar{\partial}^* G\partial\left(\varphi \lrcorner \eta\right) = \eta_0 - T\varphi\eta, \tag{6}$$

if and only if η is the solution to the obstruction equation

$$\bar{\partial}\eta + \partial(\varphi \lrcorner \eta) = 0.$$

Corollary

Given any Beltrami differential φ such that $\|\varphi\|_\infty < 1$, and any closed $(n,0)$-form η_0 on X, there exists a unique closed $(n,0)$ form $e^{i_\varphi}\eta$ on X_φ, where $\eta \in A^{n,0}(X)$, given by

$$\eta = (I + T\varphi)^{-1}\eta_0, \tag{7}$$

satisfies the obstruction equation

$$\bar{\partial}\eta + \partial(\varphi \lrcorner \eta) = 0.$$

To find solution to the obstruction equation

$$\bar{\partial}\eta + \partial(\varphi \lrcorner \eta) = 0,$$

we only need to find a solution η to the equation,

$$\eta = \eta_0 - \bar{\partial}^* G \partial \left(\varphi \lrcorner \eta\right) = \eta_0 - T\varphi\eta \tag{8}$$

which is equivalent to

$$(I + T\varphi)\eta = \eta_0.$$

From this we get

$$\eta = (I + T\varphi)^{-1}\eta_0.$$

Canonical extension formulas

Apply the above corollary to the Beltrami differential $\varphi = \varphi(t)$ from the Kodaira-Spencer-Kuranishi deformaiton theory, we get the canonical extension formulas of holomorphic forms on the local deformation space of compact Kähler manifolds.

Corollary

For any holomorphic $(n,0)$-form $s_0 \in \mathbb{H}^{n,0}(X)$, and the Beltrami differential $\varphi = \varphi(t)$ with $|t| < \varepsilon$ small, there exists a unique holomorphic $(n,0)$ form $s(t)$ on X_t,

$$s(t) = \rho((I + T\varphi)^{-1} s_0).$$

Taylor expansion

The holomorphic $(n, 0)$ form on X_t is given by

$$\rho(s(t)) = e^{i_\varphi} s(t)$$

with

$$\varphi = \varphi(t) = t\varphi_1 + t^2\varphi_2 + \cdots ,$$

its first two terms are

$$\rho(s(t)) = s_0 + t(\varphi_1 \lrcorner s_0 + \partial\overline{\partial}^* G(\varphi_1 \lrcorner s_0)) + O(t^2).$$

We can compute the curvature formulas of the L^2-metric on Hodge bundles, and the curvature formulas for the Weil-Petersson metrics on moduli spaces.

Solving the Beltrami equations

Let us consider the L^2-Hodge theory on the unit disc D with the standard Poincaré metric γ_P of curvature -1. Note that for

$$\mu = \mu_0 d\bar{z} \otimes \frac{\partial}{\partial z},$$

we have

$$||\mu||_\infty = \sup |\mu_0|$$

where the supremum is taken on the unit disc D.
Introduce the operator from the L^2-Hodge theory on D,

$$T = \bar{\partial}^* G \partial.$$

In this case, we can also use L^2-Hodge theory of Kohn with the Euclidean metric γ_E on D.

We have the following,

Proposition

Assume the L_∞-norm of μ, $||\mu||_\infty < 1$. Then given any holomorphic one form $h_0 = a_0(z)dz$ on D, the obstruction equation

$$\bar{\partial} h + \partial(\mu \lrcorner h) = 0 \tag{9}$$

has a unique solution

$$h = (I + T\mu)^{-1} h_0.$$

Solving Beltrami equations

The following result was obtained with Zhu,

Theorem

Given any $\mu = \mu_0 \frac{\partial}{\partial z} \otimes d\bar{z}$ *such that* μ_0 *is a measurable function on the unit disc with* $||\mu||_\infty = \sup|\mu_0| < 1$, *the Beltrami equation*

$$\bar{\partial} f = \mu \lrcorner \partial f$$

has a measurable solution on D.

Solving the Beltrami equations

Take $h_0(z)$ any holomorphic L^2-form on D, we have the corresponding

$$h = (I + T\mu)^{-1} h_0$$

which satisfies the equation

$$\bar{\partial} h + \partial(\mu \lrcorner h) = 0.$$

It follows that

$$e^{-i_\varphi} d(e^{i_\mu} h) = \bar{\partial} h + \partial(\mu \lrcorner h) = 0,$$

Therefore

$$d(e^{i_\mu} h) = 0.$$

Solution of the Beltrami equation

According to the Poincaré lemma on D, there is a measurable function w on D, such that

$$e^{i_\mu} h = dw = \bar{\partial} w + \partial w.$$

Since

$$e^{i_\mu} h = h + \mu \lrcorner h,$$

by comparing types we obtain

$$h = \partial w \quad \text{and} \quad \mu \lrcorner h = \bar{\partial} w.$$

Therefore we have a solution w of the Beltrami equation,

$$\bar{\partial} w = \mu \lrcorner \partial w.$$

Application to moduli

Theorem

Given any point (M, ω_0) in the marked and polarized moduli space $\mathcal{M}$ and a holomorphic n form s_0 on M, there is a canonical section s of the Hodge bundle $\mathcal{H}^{n,0}$, such that for any point M_1 in $\mathcal{M}$, the de Rham cohomology class of s in $H^*(M)$ is represented by

$$s = \rho((I + T\varphi)^{-1} s_0).$$

Here φ is the Beltrami differential associated to M_1, and

$$T = \bar{\partial}^* G \partial$$

is the operator from the Hodge theory on M with Kähler metric ω_0.

Newlander-Nirenberg theorem

Given $\varphi \in A^{0,1}(X, T^{1,0}X)$, if the equation

$$\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$$

is satisfied, then we want to prove the existence of local complex analytic coordinates. This is equivalent to solving equations on a ball $\mathbb{B}^n$ in $\mathbb{C}^n$, as $\|\varphi\|_\infty$ small in Euclidean norm,

$$\bar{\partial} w = \varphi \lrcorner \partial w \tag{10}$$

with $w \in A^{0,0}(\mathbb{B}^n)$ for n linearly independent solutions

$$(w^1, \cdots, w^n).$$

Instead we solve obstruction equation for $f \in A^{1,0}(\mathbb{B}^n)$ such that

$$\bar{\partial} f + \partial(\varphi \lrcorner f) = 0 \tag{11}$$

Kohn's solution to the $\bar{\partial}$-Neumann problem gives us a beautiful L^2-Hodge theory on $\mathbb{B}^n$ with Euclidean metric γ_E, which can be combined with our method to solve equations of such type.

We will denote by G the corresponding Green operator, also called the Neumann operator, in the L^2-Hodge theory on $\mathbb{B}^n$ with Euclidean metric γ_E.

Newlander-Nirenberg theorem

Again introduce the operator

$$T = \bar{\partial}^* G \partial.$$

Here we need to be slightly careful about the boundary value condition related to the adjoint operator $\bar{\partial}^*$. But we can still prove that the operator T extends to a bounded linear operator

$$T : L_2^{p,q}(\mathbb{B}^n, \gamma_E) \to L_2^{p+1,q-1}(\mathbb{B}^n, \gamma_E)$$

with operator norm $\|T\| \leq 1$, so that $I + T\varphi$ is invertible when $\|\varphi\|_\infty < 1$.

Newlander-Nirenberg theorem

Let f_0 be a holomorphic one form on $\mathbb{B}^n$ with $\partial f_0 = 0$. We prove that

$$f = (I + T\varphi)^{-1} f_0$$

satisfies the equation

$$\bar{\partial} f + \partial(\varphi \lrcorner f) = 0$$

as well as $\partial f = 0$. It follows that

$$\begin{aligned} e^{-i_\varphi} d(e^{i_\varphi} f) &= df - \mathcal{L}^{1,0} f \\ &= \bar{\partial} f + \partial(\varphi \lrcorner f) \\ &= 0 \end{aligned}$$

Newlander-Nirenberg theorem

This gives

$$d(e^{i_\varphi} f) = 0.$$

From the Poincaré lemma for currents, there is a function w on $\mathbb{B}^n$ such that

$$e^{i_\varphi} f = dw = \partial w + \bar{\partial} w.$$

On the other hand,

$$e^{i_\varphi} f = f + \varphi \lrcorner f,$$

by comparing types, we obtain

$$f = \partial w, \ \ \varphi \lrcorner f = \bar{\partial} w.$$

Therefore we have a solution w of the equation,

$$\bar{\partial} w = \varphi \lrcorner \partial w.$$

This is the equation of complex analytic coordinates. Take n initial values

$$f_0 = dz^i, \ i = 1, \cdots, n,$$

we get n solutions, which can be written explicitly as,

$$w^i = z^i + \bar{\partial}^* G(\varphi \lrcorner \partial w^i), i = 1, \cdots, n.$$

The regularity of w^i follows from elliptic regularity of the elliptic equation satisfied by w,

$$\Box_{\bar{\partial}} w - \bar{\partial}^*(\varphi \lrcorner \partial w) = 0.$$

Existence of pseudoholomorphic curves

On a disc D with Poincaré metric γ_P or Euclidean metric γ_E, solving the vector-valued equation,

$$w_{\bar{z}} = A(w)w_z$$

where $w = (w_1, \cdots, w_n)^t$ and $A(w)$ is an $n \times n$ matrix of functions in w. Introduce

$$f = udz = (u_1 dz, \cdots, u_n dz)^t.$$

and

$$\varphi(u) = A(u)d\bar{z} \otimes \frac{\partial}{\partial z},$$

We solve the equation for $(1,0)$-form f,

$$\bar{\partial} f + \partial(\varphi \lrcorner f) = 0 \tag{12}$$

Reduce to solving the equation with initial value f_0,

$$f = f_0 + \bar{\partial}^* G \partial(\varphi \lrcorner f) = f_0 + T\varphi f. \tag{13}$$

Existence of pseudoholomorphic curves

When $\|\varphi\|_{C^1} < \varepsilon$, we have the operator norm $\|T\varphi\| \leq \eta < 1$. Banach fixed point theorem gives existence of its solution. Formally

$$f = (I + T\varphi)^{-1} f_0$$

for any given holomorphic one form f_0, is a solution. Again we have

$$d(e^{i_\varphi} f) = 0$$

on D, and Poincáre lemma gives

$$e^{i_\varphi} f = f + \varphi \lrcorner f = dw = \partial w + \bar{\partial} w.$$

Then w satisfies the equation

$$\bar{\partial} w = \varphi \lrcorner \partial w$$

which is equivalent to

$$w_{\bar{z}} = A(w) w_z.$$

Thank You!