Today is the last class in this semester. We will continue in the end of February → exact date will be posted.

L_∞ algebra from BV perspective

Idea of the BV proof of homotopic transfer theorem.

1. Definition of L_∞ is natural from BV point of view
2. Homotopic transfer is not a trick but a natural procedure from BV integral point of view.

S - functions on $T^*[1]X$ solving equation (classical master eqn)

$$\frac{\partial S}{\partial \varphi^i}\frac{\partial S}{\partial \varphi_i^*}=0$$

φ^i are coord on X

φ_i^* are coord on the fiber

Let us take $X=\mathfrak{g}[1]$ - a vector space

$\mathfrak{g}$ ↑ Lie algebra shifted by [1] in parity.

Coord on $\mathfrak{g}[1]$ would be $\underline{\underline{\text{odd}}}$ variables c^a. $c^a c^b = -c^b c^a$

$T^*[1](\mathfrak{g}[1]) = T^*_{\mathfrak{g}}\mathfrak{g}$. coord on $T^*_{\mathfrak{g}}\mathfrak{g}$ would be denoted as $\boxed{c_a^* \leftarrow \text{are even}}$

$$S_{\mathfrak{g}} = f_{ab}^{e}\, c^a c^b c_e^*$$

(note, that S is even)

f_{ab}^e – standard st. constants; $c^a c^b$ – odd

inf. about Lie algebra is in $S_{\mathfrak{g}}$

Remark. $\{S, S\}_{BV} = 0 \longleftrightarrow \{S, \cdot\}_{BV}$

homological vector field.

For S_{YM} corresp $\{S, \cdot\}$ can be computed on functions of c only

$$\{S, P(c)\} = f^e_{ab}\, c^a c^b \frac{\partial P(c)}{\partial c^e}$$

in this way we obtain Ch.-Eil. differential

$\{S, S\} = 0$ correspond to Jacobi identit. on f.

New idea – generalize, namely consider supervector space W

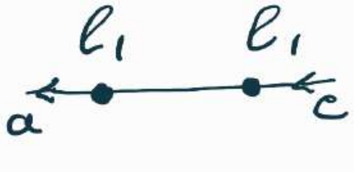
$W^* \times W[1]$ (S – linear in w^*, w – coord. on $W[1]$, w^* – coord. on W^*)

$$S = w^*_a\, l^a_{1\,b}\, w^b + w^*_a\, l^a_{2,\,b_1 b_2}\, w^{b_1} w^{b_2} + \ldots + w^*_a\, l^a_{k,\,b_1\ldots b_k}\, w^{b_1} \ldots w^{b_k}$$

$$\frac{\partial S}{\partial w^*_a} \frac{\partial S}{\partial w^a} = 0 \longrightarrow \text{quadratic equations on } l^a_{k,\,b_1\ldots b_k}$$

1) Linear order in w:

$$l^a_{1\,b}\, l^b_{1\,e} = 0$$



2) Quadratic order in w

1) l_1 – is a differential
2) multiplication l_2 sat. Leibnitz rule w.r.t the differential l_1

Assume that there is no l_3, then in the order ω^3 we will get only

one can sheck that in is a Jacobi quadric $J(l_2, l_2) = 0$

Definition: if $l_3, l_4, \ldots = 0$, then the algebraic structure with only l_1 and l_2 is called DGL

Differential graded Lie algebra.

Example of DGL:

Consider the supercommutative associative DGA (like algebra of dif. forms)

$$\omega, \quad \omega_1 \wedge \omega_2 \to \text{multiplication},$$

$$d: \omega \to \underset{\uparrow \text{D.R dif.}}{d\omega} \to \text{differential}$$

Consider the Lie algebra $\mathfrak{g}$

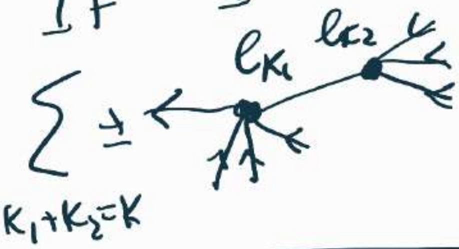
Then on the product $\mathfrak{g} \otimes \Omega_X$

there is a dGL structure

$$l_1: \quad e \otimes \omega \mapsto e \otimes d\omega$$

$$l_2: \quad (e_1 \otimes \omega_1, e_2 \otimes \omega_2) \longmapsto \underbrace{[e_1, e_2]}_{\mathfrak{g}} \otimes \omega_1 \wedge \omega_2$$

If I allow higher operations

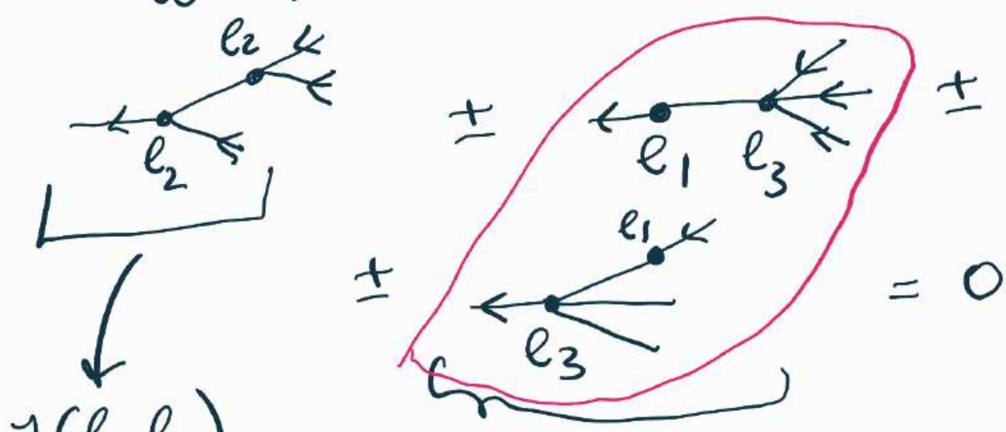


$= 0$

$l_3 \neq 0$ $\quad \omega$: l_1 is a differential

ω^2: l_1 and l_2 satisfy Leibnitz

w^3 :



$\mathcal{J}(l_2, l_2)$

Definition: Let l_1 be a differential $l_1^2 = 0$,

Let $\mathcal{O}$ be an operation

Then l_1 acts on operations as follows

$l_1(\mathcal{O}) =$

And

$l_1(l_1(\mathcal{O})) = 0$, i.e.

l_1 provides a differential on operations

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w^3 relation means just

$Jac(l_2, l_2) = l_1(l_3)$, i.e.

Jac relation on l_2 is satisfied only homotopically, in the sence that right hand side is l_1-exact! And l_3 is just what stands here.

If we apply l_1 twice

$= 0 \qquad l_1^2 = 0$

such term appears twice

Algebraic explanation

l_1 corresponds to dif. operator $l^b_{1a} w^a \frac{\partial}{\partial w^b}$ "$\hat{l}$"

operation O corresponds to

$$\hat{O} = O^b_{a_1 \ldots a_k} w^{a_1} \ldots w^{a_k} \frac{\partial}{\partial w^b}$$

Geometric definition of l_1 action may be rewritten as an algebraic

$\hat{l_1(O)} = [\hat{l}, \hat{O}]$, and $\hat{l}^2 = 0$

break

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dGL to L_∞, actually, from L_∞ to L_∞. by BV integral

$$W[1] = W_0[1] \oplus W_{AC}[1]$$

$$W^* = W_0^* \oplus W_{AC}^*$$

bicomplex

the following data $l_1 = Q + \tilde{l}_1$

I want $Q^2 = 0$, I will also demand for simplicity $\tilde{l}_1^2 = 0$ $\{Q, \tilde{l}_1\} = 0$

$$W[1] = \underset{\circlearrowleft Q}{W_0[1]} \oplus \underset{\circlearrowleft Q,\ \circlearrowleft h}{W_{AC}[1]}$$ as Q-complex:

moreover, I want $W_{AC}[1]$ to be an acyclic subcomplex: h - homotopy, $h^2 = 0$

$h\, W_0[1] = 0$ $\qquad Qh + hQ = Pr_{W_{AC}}$

Example: $W = \mathcal{J} \otimes \dot{\Omega}_X$, then we may take W_0 as $\mathfrak{g} \otimes H^\bullet_X$

W_{AC} - orthogonal complement

Today I will explain not the general case but only

$$\Omega Q$$

$$W[1] = H_Q[1] \oplus W_{AC}[1]$$

$$L \cap h$$

$$W^* \times W[1] = \underbrace{W^*_{AC} \times W_{AC}[1]}_{\text{BV-space}} \times \underbrace{H^*_Q \times H_Q[1]}_{\Pi T^* Y}$$

find a nice Lagrangian submanifold

It would be Linear; I will integrate against this Lagrangian submanifold

$$T^*[1]X \times T^*[1]Y$$

$$\cup$$

$$\mathcal{L}$$

$$\int e^{\frac{1}{\hbar}S} \mu =$$

$$= e^{\frac{1}{\hbar}S^{ind} + \hbar\text{-corrections}} \cdot \tilde{\mu}$$

For solutions to classical master equation I may ignore $\hbar$ corrections

I will take $\mathcal{L}$ in $W^*_{AC} \times W_{AC}[1]$ as follows:

$$\begin{cases} h\, W_{AC}[1] = 0 \\ h^*\, W^*_{AC} = 0 \end{cases}$$

Simplectic form is $dw^*_i \wedge dw^i$

If $hw^i = 0 \Rightarrow w^i = hu^i$

really, $w^i = (Q\underbrace{h}_{\downarrow 0}+hQ)\underline{w}^i = h\underbrace{(Qw^i)}_{u^i}$

$dw_i^* \wedge dhu^i =$

$= d\underbrace{h^* w_i^*}_{=0} \wedge du^i = 0$

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choice of homotopy gives a clever choice of $\mathcal{L}$ in $W_{AC}^* \times W_{AC}[1]$

$$w = w_0 + w_{AC}$$

$$S_{BV}(w^*, w) =$$

$$= w^*(Q+\tilde{l}_1)(w) + w^* l_2(w,w) =$$

$$= w_0^* \tilde{l}_1 w_0 + w_0^* \tilde{l}_1 w_{AC} + w_{AC}^* \tilde{l}_1 w_0$$

$$+ w_{AC}^* Q w_{AC} + w_{AC}^* \tilde{l}_1 w_{AC}$$

$$+ w_0^* l_2(w_0, w_{AC}) + \ldots + w_{AC}^* l_2(w_{AC}, w_{AC})$$

Interesting integral over w_{AC}^* w_{AC}. This integral contains quadratic term and linear terms and highly nonlinear terms.

First consider the simplest case when $l_2 = 0$

In this case we have a gaussian integral

We try to make a complete square $w_{AC} = hu_{AC}$

$$w_0^* \tilde{l}_1 h u + w_{AC}^* \tilde{l}_1 w_0 + w_{AC}^* \tilde{l}_1 h u$$

$$+ w_{AC}^* Q h u \to$$

$$\underline{\underline{w_{AC}^* u}} + \underline{\underline{w_{AC}^*}} \tilde{l}_1 w_0 + w_0^* \tilde{l}_1 \underline{\underline{h u}} +$$

$$+ w_{AC}^* \tilde{l}_1 h u.$$

Quadratic term is

$$w_{AC}^* (1 + \tilde{l}_1 h) u +$$

$$+ w_{AC}^* l_1 w_0 + w_0^* \tilde{l}_1 h u$$ ← linear terms

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Such gaussian integrals are known in the theory of formal Feynman integrals.

You may consider the result of integration as a sum over Feynman diagrams with

a root - corresponding to projector of

$W \xrightarrow{\pi} H_Q(W)$ 

a leaf corresponding to 

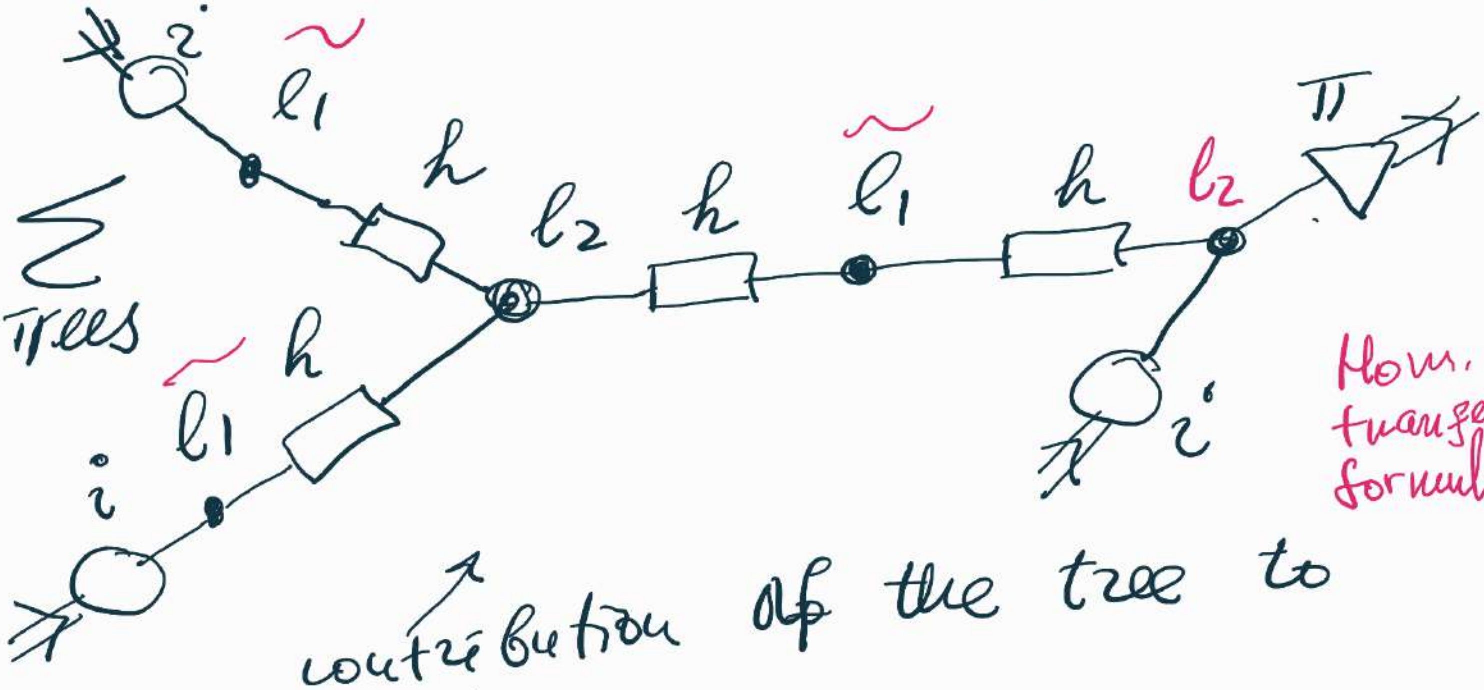
$i: H_Q(W) \to W$

Vertices corresponding to $\tilde{l}_1$ and l_2

And the answer would be a sum over trees like this:

$h:$ 

$\sum_{\text{Trees}}$



Hom. transfer formula

contribution of the tree to l_3^{ind}

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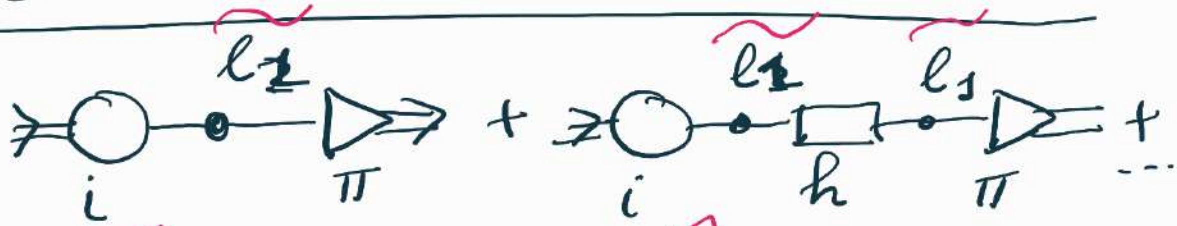
l_1^{ind} 

$\omega_0^* \, \tilde{l}_1 \, \omega_0$

To see this term let us ignore for a moment $\tilde{l}_1 h$ and in making of the complete square we get:

$$\omega_{AC}^{*}\left(1+\cancel{\tilde{\ell}_1 h}\right)u +$$

$$+\ \omega_{AC}^{*}\,\ell_1\,\omega_0 + \omega_0^{*}\,\tilde{\ell}_1 h\,u$$

$\leftarrow$ linear terms

$$\omega_{AC}^{*} u + \omega_{AC}^{*}\ell_1\omega_0 + \omega_0^{*}\tilde{\ell}_1 h u$$

$$z^{*}z + z^{*}a + b^{*}z \longrightarrow (z^{*}+b)(z+a) - b^{*}a$$

$$z^{*} = \omega_{AC}^{*} \qquad z = u$$

$$a = \tilde{\ell}_1\omega_0 \qquad b^{*} = \omega_0^{*}\tilde{\ell}_1 h$$

$$b^{*}a = \omega_0^{*}\tilde{\ell}_1 h\tilde{\ell}_1\omega_0$$

other terms come from $(1+\tilde{\ell}_1 h)^{-1} =$

$$= 1 - \tilde{\ell}_1 h + (\tilde{\ell}_1 h)^2 + \ldots$$