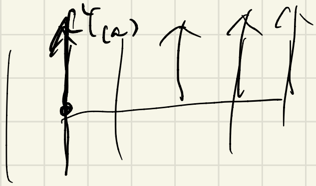
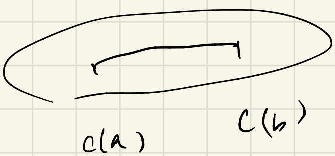
2021-10-11 Kähler geometry

Parallel transport and holonomy groups. ①

Let $E \to M$ be a vector bundle with a connection ∇.

Let $c : [a,b] \to M$ be a smooth curve, and $Y_{c(a)} \in E_{c(a)}$ be a vector in the fiber $E_{c(a)}$ over $c(a) \in M$.





Solve the ordinary differential equation

$$\nabla_{c_*\left(\frac{d}{dt}\right)} Y_t = 0 \quad \text{along } c(t)$$

$$Y_t \in E_{c(t)}, \qquad Y_0 = Y_{c(a)}$$

If $e_1(t), \dots, e_r(t)$ are frame over $c(t)$

$$Y_t = Y^1(t)\, e_1(t) + \dots + Y^r(t)\, e_r(t).$$

$$\nabla_{c_*(\frac{d}{dt})} Y_t = \left(\frac{dY^i}{dt} + \theta^i_j(\dot{c}) Y^j \right) e_i .$$ ②

So $\nabla_{\dot c} Y_t = 0 \iff \frac{dY^i}{dt} + \theta^i_j(\dot c) Y^j = 0.$

Def Y_t is called the parallel transport of $Y_{c(a)}$ along $c(t)$.

Let M be a Riemannian manifold, and take $E = TM$ with Levi-Civita connection.

Choose $p \in M$ and let

$$\mathcal{C} = \{ c : [0,1] \to M, \text{ piece-wise smooth} \mid c(0) = c(1) = p,\ c \text{ is contractible} \}$$

For $X_p, Y_p \in T_pM$ and $c \in \mathcal{C}$, we consider the parallel transport of X_p, Y_p along c. Then

$$\frac{d}{dt} g(X_t, Y_t) = g(\nabla_{\dot c} X, Y) + g(X, \nabla_{\dot c} Y) = 0.$$

So the parallel transport $P_c : T_pM \to T_pM$ preserves the inner product, thus $P_c \in SO(T_pM)$.

Put $\mathrm{Hol}(M) = \{ P_c : T_pM \to T_pM \mid c \in \mathcal{C} \} \subset SO(n)$

and call it the holonomy group.

(exercise: this is independent of p).

Theorem (Berger, (Simons, Olmos))

Let M^n be an irreducible Riemannian manifold of dim n.

If M is not a locally symmetric space, the the holonomy group is one of the following.

- $SO(n)$ general one.
- $U(m)$, $n=2m$ general Kähler manifold
- $SU(m)$, $n=2m$ Calabi-Yau mfd (Ricci-flat Kähler)
- $Sp(k)\cdot Sp(1)$, $n=4k$, quaternionic Kähler
- $Sp(k)$, $n=4k$, hyperkähler manifold.
- G_2 $n=7$
- $Spin(7)$ $n=8$

$\left.\begin{matrix} G_2 & n=7 \\ Spin(7) & n=8 \end{matrix}\right\}$ exceptional holonomy.

At the beginning these were just possibity. Many later works show these actually occur as a holonomy of a Riemannian manifold.

For example, Yau's work shows that there are compact examples of $SU(m)$-holonomy.

Exceptional holonomy: Bryant (locally), Salamon (complete), Joyce (compact).

Rem

$Sp(k) \subset SU(2k)$ Ricci-flat Kähler

$Sp(k)\cdot Sp(1)$ non-Ricci-flat, non-Kähler, Einstein.

"quaternionic Kähler" is not Kähler

G_2, $Spin(7)$ Ricci-flat. (not Kähler).

Rem Some people call Ricci-flat Kähler manifolds "Calabi-Yau". In this case holonomy is $SU(m)$ or $Sp(k)$.

But some other people (just as in the list above), Riemannian manifolds with $SU(n)$ holonomy "Calabi-Yau" and $Sp(k)$ holonomy "hyperkähler".

Rem Simons's explanation is as follows.
Except for symmetric spaces, the holonomy group acts on the unit sphere of the tangent space T_pM, $p \in M$, transitively. So, Berger's list is the list of subgroups in $SO(n)$ acting on the unit sphere transitively.

Basics of integration in Riemannian and Kähler manifolds.

Let (M,g) be a compact oriented manifold without boundary. $\dim_{\mathbb{R}} M = n$.

$$dV_g = \sqrt{\det(g_{ij})} \; dx^1 \wedge \cdots \wedge dx^n$$

in coordinate nbds.

excercise : This is independent of local coordinates.

dV_g gives a measure on (M,g) "volume element".

Given p there is a coord system s.t. $dg_{ij}=0$, $\Gamma_{ij}^k = 0$ at p, called the normal coordinates

$$X = X^i \frac{\partial}{\partial x^i}$$

$$\underbrace{d\big(\overbrace{i(X)\, dV_g}^{(n-1)\text{ form}}\big)}_{n\text{-form}} =: \operatorname{div} X \cdot dV_g$$

$$\operatorname{div} X = \nabla_i X^i$$

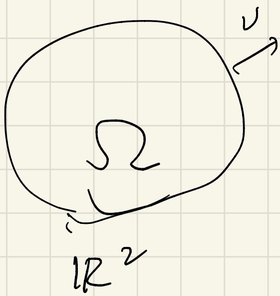
(excercise : use normal coordinates).

By Stokes theorem "div X" is called the divergence of X. ⑥

$$\int_M \operatorname{div}(X)\, dV_g = 0.$$

$$\int_M \operatorname{div}(X)\, dV_g = \int_M d\big(i(X)\, dV_g\big) = 0$$

" divergence theorem "



$$\left(\int_\Omega \sum_i \frac{\partial X^i}{\partial x^i} = \int_{\partial\Omega} \langle \nu, X \rangle\, d\sigma \right)$$

Application

$\langle df, T \rangle$ ← vector field

(i) $$\int_M \underline{T^i \nabla_i f}\, dV_g \qquad \operatorname{div}(fT)$$

$$= \underbrace{\int_M \nabla_i (T^i f)\, dV_g}_{=0} - \int_M (\nabla_i T^i) \cdot f\, dV_g$$

$$= - \int_M (\nabla_i T^i) \cdot f\, dV_g.$$

"integration by parts"

(2) $\Delta f = \nabla^i \nabla_i f = \nabla_i \nabla^i f$

$$\underbrace{\nabla^i \nabla_i f}_{\text{''}} \quad g^{ij} \nabla_i \nabla_j f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k}\right)$$

$$\frac{\partial f}{\partial x^j}$$

$f \in C^\infty(M)$

$\nabla_i \nabla_j f = \nabla_j \nabla_i f$

$\therefore \quad \Delta f = \operatorname{div}(\# df)$

$\therefore \int_M \Delta f \, dV_g = 0.$

(3) Conversely if $\int_M u \, dV_g = 0$

then $\exists f \in C^\infty(M)$ s.t. $\Delta f = u$.

(Exercise : Use Hodge ~~theory~~)

<u>Kähler case:</u>

$$\frac{\omega^m}{m!} = \det(g_{i\bar{j}})\, i^m dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m$$

$$= dV_g$$