2020-11-03 Kähler geometry

About contact manifolds. $(\dim_{\mathbb{R}} S = 2m+1)$ ①

<u>Def</u> An odd dimensional manifold S is called a contact manifold if $\exists$ non-vanishing 1-form η such that $d\eta$ is non-degenerate on

$$D := \operatorname{Ker} \eta.$$

D is called the contact distribution, and

$$\dim_{\mathbb{R}} D = 2m$$

- So, $d\eta$ is a symplectic form on D.
- $d\eta \wedge \cdots \wedge d\eta \wedge \eta$ gives a measure.
- $\exists_1$ vector field ξ such that

$$i(\xi)\eta = 1.$$
$$i(\xi)d\eta = 0.$$

exercise linear algebra.

This ξ is called the <u>Reeb vector field.</u>

Going back to the Sasakian manifold, we defined ξ first by

$$\xi = J\frac{\partial}{\partial r} \quad \text{along } S = \{r = 1\}$$

Then we defined η by

$$\eta(X) = g(X, \xi).$$

Prop (last time)

- $d\eta$ is a contact form.
- By putting $\Phi(X) = \nabla_X \xi$,

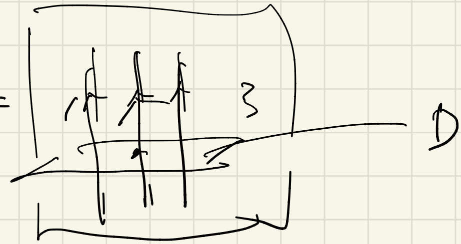
$$d\eta(X, Y) = 2g(\Phi X, Y).$$

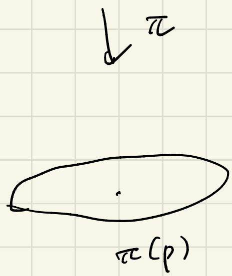
Proof omitted. Tokyo6

Prop $(\Phi \circ \Phi)(Y) = -Y + \eta(Y)\xi$

$$= \begin{cases} -Y & Y \in \ker\eta = D = \xi^{\perp} \\ 0 & Y = \xi \end{cases}$$

Thus this Φ gives a complex str on D.

$S \supset U =$ 



← local orbit space of the flow generated by ξ — V

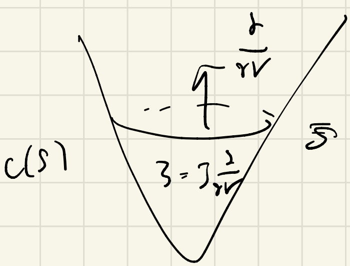
$$D = \pi^* T_{\pi(p)}(\text{orbit space})$$

$\exists\ \omega^T$ 2-form on V s.t. $\pi^* \omega^T = d\eta$.

$\exists\ J^T$ complex str on V s.t. $\pi^* J^T = \Phi$.

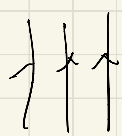
<u>Claim</u> (V, ω^T, J^T) is a Kähler manifold

We call the Kähler geometry on the local orbit spaces "transverse Kähler geometry".

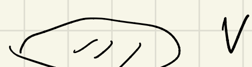


$S \simeq \{r = 1\}$

inside S



Boyer-Galicki called



"Kähler sandwich".

Prop Let Ric^T be the Ricci curvature of the transverse Kähler structure (or on local orbit spaces). Then

$k = 2m+1$

$n-1 = 2m$

$$\mathrm{Ric}^T(X,Y) = \mathrm{Ric}(X,Y) + 2g(X,Y)$$

for $X, Y \in D_p \cong$ "tangent space of local orbit space".

$2m\, g(X,Y)$

$(2m+2)\, g(X,Y)$

In particular, if S is Einstein then

$$\mathrm{Ric}^T = (2m+2)\, g^T \qquad \left(\begin{array}{l} g^T \text{ the transverse} \\ \text{Kähler metric} \end{array}\right).$$

Conclusion

S is Sasaki-Einstein $\quad \mathrm{Ric} = 2m\, g$

$\iff$ $C(S)$ is Ricci-flat Kähler

$\iff$ $\mathrm{Ric}^T = (2m+2)\, g^T$ Kähler-Einstein on local orbit spaces.

$\boxed{\text{Claim}}$ Geometry of a Sasakian manifold is determined only by r.

As long as J on $C(S)$ is fixed, everything is expressed using r.

We see this by the typical example

$$C(S) = \mathbb{C}^{m+1}, \quad S = S^{2m+1} \text{ the standard sphere,}$$

$$\quad\ \ \cup\!\!\!\!\!\shortmid \ \ (z^0, z^1, \dots, z^m)$$

$$r^2 = |z^0|^2 + \cdots + |z^m|^2$$

$$S = \{r = 1\} \subset \mathbb{C}^{m+1}$$

$$z = x + iy$$

$$\partial\bar\partial |z|^2 = dz \wedge d\bar z$$

$$= -2i\, dx \wedge dy$$

$$\bar\omega = \frac{i}{2} \partial\bar\partial r^2$$

$$= dx^0 \wedge dy^0 + \cdots + dx^m \wedge dy^m$$

$$z\frac{\partial}{\partial z} = \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + \frac{i}{2}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{2} r\frac{\partial}{\partial r} + \frac{1}{2i}\frac{\partial}{\partial \theta} \qquad \text{if } z = re^{i\theta}$$

$$r\frac{\partial}{\partial r} = z\frac{\partial}{\partial z} + \bar z\frac{\partial}{\partial \bar z}$$

On $\mathbb{C}^{2n+2}$

$$r\frac{\partial}{\partial r} = \sum_{i=0}^{m} 2\,\mathrm{Re}\left(z^i \frac{\partial}{\partial z^i}\right)$$

$$\tilde{\xi} = J r\frac{\partial}{\partial r} = \sum_i \left(i z^i \frac{\partial}{\partial z^i} - i \bar{z}^i \frac{\partial}{\partial \bar{z}^i} \right) = \sum_i \frac{\partial}{\partial \theta^i}$$

$$d^c := \frac{i}{2}(\bar\partial - \partial) \qquad \text{so} \qquad dd^c = i\partial\bar\partial$$

$$2\, d^c \log r = \frac{1}{r}\, i\,(\bar\partial - \partial)\, r$$

$$= \frac{1}{r}\, i\left(\frac{z\,d\bar z}{2r} - \frac{\bar z\, dz}{2r}\right)$$

$$= \frac{1}{r^2}\left(-y\,dx + x\,dy\right)$$

$$\eta(X) = \langle \xi, X\rangle = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\ a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)$$

$$= -ay + bx$$

$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$

$$\eta = -y\,dx + x\,dy$$

$$= 2\,d^c \log r \,\Big|_{r=1}.$$

$/\!/ \; J\frac{\partial}{\partial r}$

$$\xi = \sum\left(-y^i\frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}\right) = \sum \frac{\partial}{\partial \theta^i} \quad \text{generates}$$

the S^1-action

$$(e^{i\theta}, (z^0, \dots, z^n)) \longmapsto (e^{i\theta} z^0, \dots, e^{i\theta} z^m)$$

The local orbit space of the Reeb flow (7)
is $\mathbb{P}^m(\mathbb{C})$. The transverse Kähler form is
the Fubini-Study metric. Its Kähler form
is

$$\pi^* \omega^T = i\partial\bar\partial \log r \Big|_{r=1}$$
$$= dd^c \log r \Big|_{r=1} = \frac{1}{2} d\eta$$

These computations apply for general Sasakian manifolds, and its cones $C(S)$, and also for transverse Kähler geometry.

$$\boxed{r \text{ determines everything.}}$$

$$\tilde{\xi} = J r \frac{\partial}{\partial r} \quad \text{on } C(S)$$

$$\xi = \tilde{\xi}\Big|_{r=1} = J \frac{\partial}{\partial r} \quad \text{on } S.$$

$$\eta = 2 d^c \log r \Big|_{r=1}. \quad \text{on } S.$$

But we may regard η as a 1-form $C(S)$ by

$$\eta := 2 d^c \log r \quad \text{on } C(S)$$

If a is a positive constant

$$2 d^c \log(ar) = 2 d^c \log r.$$

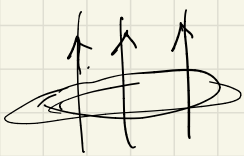
$\therefore$ η is homogeneous of degree 0 on $C(S)$.

<u>Basic cohomology</u> (This is a notion for general contact manifolds.)

<u>Def</u> $\alpha \in \Omega^p(S)$ is said to be basic if

$$i(\xi)\alpha = 0, \quad L_\xi \alpha = 0.$$

This means α does not have ξ-components and constant along ξ-orbit ($\exp t\xi$ -invariant).



[The flow generated by ξ is called the <u>Reeb flow</u>.]

(This simply means, α is lifted from local orbit spaces.

[If α is basic, for each $\lambda \in \Lambda$,

$\exists$ $\alpha_\lambda \in \Omega^p(V_\lambda)$ s.t. $\alpha = \pi_\lambda^* \alpha_\lambda$ on U_λ]

$$\begin{array}{c} U_\lambda \circlearrowleft \upuparrows \\ \downarrow \pi_\lambda \\ V_\lambda \end{array}$$

$\Omega^p_B \overset{\text{def}}{:=}$ the space of basic p-forms on S

$\exists$ well-defined operator ⑨

$$\bar\partial_B : \Omega^p_B \longrightarrow \Omega^{p+1}_B$$

(Φ ...

$$\mathcal{H}^{p,q}_B(S) = \bar\partial_B\text{-cohomology of type } (p,q).$$

$$\pi_\lambda^* \omega_\lambda^T = \underbrace{d\eta}_{\text{basic}}.$$

(we may regard this as a transverse Kähler form. But a bit confusing because $d\eta$ is cohomologous to 0.)

(or $\omega^T = d\eta|_D$).

$$\rho^T = \mathrm{Ric}(\omega^T) = -\partial_B \bar\partial_B \log\det \omega_\lambda^T \qquad \text{basic.}$$

2-forms.

$$[\rho^T] \in H^2_B(S,\mathbb{R})$$

$$\parallel$$

c_1^B the basic first Chern class