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Calabi-Yau geometry and beyond

Caucher Birkar (Tsinghua)

Celebration of 73rd birthday of S.-T. Yau, 2 April 2022

We work over $\mathbb{C}$.

Elliptic curves

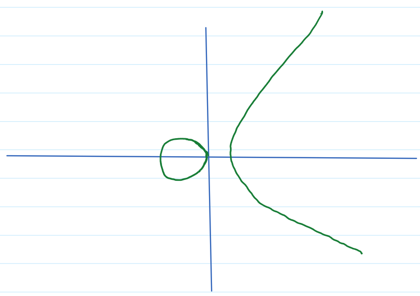
Arc length of an ellipse $\xrightarrow{\text{e.g. Euler}}$ elliptic integrals

$$\int \frac{g(x)}{\sqrt{f(x)}}\,dx$$

where $g(x), f(x)$ are polynomials, $\deg f(x) = 3$.

This naturally led to study of elliptic curves

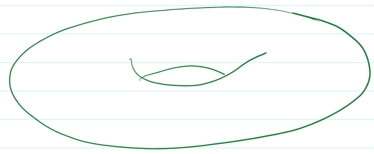
$$y^2 = f(x) \qquad \subseteq \mathbb{A}^2 = \text{2-dim affine space.}$$



Geometry of elliptic curves is very different from lines and conics.

Their group law makes them very special.

Topologically elliptic curves over $\mathbb{C}$ are simple:



They are very important in number theory, e.g. the Birch-Swinnerton-Dyer conjecture: E an elliptic curve over $\mathbb{Q}$, then

$$\operatorname{rank} E(\mathbb{Q}) = \text{order of zero of } L(s) \text{ at } s=1$$

where $L(s)$ is the associated L-series.

Abelian varieties are higher dimensional analogues of elliptic curves. They are among the best behaved varieties.

If X is smooth projective with $H^1(X, \mathcal{O}_X) \neq 0$, then the albanese map

$$X \longrightarrow Y \quad \text{abelian variety}$$

is a very useful tool to study X.

Kummer and K3 surfaces

X abelian surface.

$$\delta : \begin{array}{c} X \longrightarrow X \\ x \longmapsto -x \end{array} \qquad \text{involution}, \qquad G = \langle \delta \rangle \subseteq \operatorname{Aut}(X).$$

$Y = X/G$, $\quad \pi : X \longrightarrow Y$ quotient map.

The points $x = -x$ give singularities $\pi(x) \in Y$, Du Val sings.

Y has 16 singular points.

$K_X = \pi^* K_Y$, so $K_Y \equiv 0$.

Y is a Kummer surface. They appeared in study of light through crystals.

$g : W \longrightarrow Y$ minimal resolution, then $K_W \equiv 0$ & W is K3.

Calabi-Yau manifolds

A Calabi-Yau manifold is a smooth projective variety X with $K_X \sim 0$.

Example: abelian varieties, $K3$ surfaces.

Example: hypersurface $X \subseteq \mathbb{P}^n$ of deg $n+1$.

Yau (1978): On a Calabi-Yau manifold, $\exists$ Kähler metric with vanishing Ricci curvature.

Calabi-Yau manifolds play an important role in mathematical physics.

Math physicists noticed Calabi-Yau manifolds come in mirror pairs.

Here usually one assumes $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Example (Greene-Plesser): the quintic 3-fold.

Consider $X = V(x_0^5 + \cdots + x_4^5 - 5\lambda x_0 \cdots x_4) \subseteq \mathbb{P}^4$, $\lambda^5 \neq 1$.

which is Calabi-Yau.

$$G = \{ (a_0, \ldots, a_4) \in \mathbb{Z}_5^5 \mid \sum a_i = 0 \} / \{ (a, \ldots, a) \} \simeq \mathbb{Z}_5^3$$

G acts on X via $(x_0 : \cdots : x_4) \mapsto (\alpha^{a_0} x_0 : \cdots : \alpha^{a_4} x_4)$, $\alpha = e^{2\pi i/5}$.

We have

$$\begin{tikzcd} & & \check{X} \arrow[dl, "\text{resolution}"] \\ X \arrow[r, "\text{quotient}"'] & X/G & \end{tikzcd}, \quad \check{X} \text{ Calabi-Yau}$$

It turns out the Hodge numbers are mirror: $h^{i,j}(X) = h^{3-i,j}(\check{X})$.

Batyrev gave a generalisation using toric geometry.

<u>Conjecture</u> (Strominger-Yau-Zaslow)

A Calabi-Yau manifold has a mirror $\check{X}$ s.t.

$X \qquad \check{X}$

A Calabi-Yau manifold has a mirror $\wedge$ s.t.

- We have special Lagrangian torus fibrations

$$\begin{tikzcd} X \arrow[dr, "f"'] & & \check{X} \arrow[dl, "\check{f}"] \\ & B & \end{tikzcd}$$

- over each $b \in B$, smooth fibres are torus dual,
- certain transforms on fibres interchange complex and symplectic data.

<u>Conjecture:</u> $X \dashrightarrow Y$ birational Calabi-Yau manifolds.

Then $$D^b(X) \simeq D^b(Y).$$

Calabi-Yau varieties

A Calabi-Yau variety is a projective variety X s.t. $\begin{cases} K_X \equiv 0 \\ X \text{ has good singularities.} \end{cases}$

Good singularity means Klt and more generally log canonical singularities.

Taking a resolution $\varphi : W \longrightarrow X$ and writing $K_W + B_W = \varphi^* K_X$,

X is Klt $\Leftrightarrow$ coeffs of $B_W < 1$

X is log can. $\Leftrightarrow$ coeffs of $B_W \leq 1$.

Example: abelian varieties, K_3 surfaces, Calabi-Yau manifolds.

Example: V Calabi-Yau manifold, $G \subseteq Aut(V)$ finite group,

$V \xrightarrow{\pi} X = V/_G$ quotient map.

We have $K_V = \pi^*(K_X + B)$ for some $B \geq 0$.

This gives a log Calabi-Yau. When $B = 0$, we get a Calabi-Yau X.

Calabi-Yau varieties have a special place in many areas of maths, e.g.

algebraic geometry, differential geometry, arithmetic geometry, mathematical physics, computer science (cryptography).

They are one of the building blocks of varieties along with Fano and general type varieties.

Conjecture (Morrison-Kawamata)

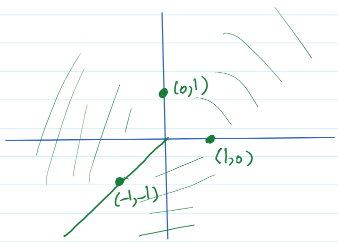
X a Calabi-Yau variety, $A^e(X) =$ cone of nef effective divisors $\subseteq N^1(X)$.
Then $\exists$ rational polyhedral cone $\Pi \subseteq A^e(X)$ which is a fundamental domain for the action of $\mathrm{Aut}(X)$.

Toric geometry

In toric geometry: convex geometry $\longrightarrow$ algebraic geometry

fan $\Delta \subseteq \mathbb{R}^d$
collection of convex cones $\longrightarrow$ $X(\Delta)$ toric variety

Example:



$\longrightarrow$ $X(\Delta) = \mathbb{P}^2$

Toric geometry is a rich source of examples.

$X(\Delta)$ toric variety, $B =$ sum of toric divisors on $X(\Delta)$,

then $K_{X(\Delta)} + B \sim 0$ & $(X(\Delta), B)$ has nice singularities.

Log Calabi-Yau varieties

A log Calabi-Yau variety is a pair (X, B) where

$$\begin{cases} X \text{ is normal, proj} \\ B = \sum b_i B_i \text{ divisor}, \quad b_i \in [0,1] \\ (X,B) \text{ have "nice" singularities} \end{cases}$$

Singularities are defined similar to varieties.

This is a large and important class of spaces.

Example: X Calabi–Yau variety, $B=0$.

Example: X toric variety, B toric boundary divisor.

Example: X Fano variety, $B \equiv -K_X$,

e.g. $X = \mathbb{P}^2$, B = smooth cubic or nodal cubic curve.

(X,B) log Calabi-Yau variety:

$$\begin{array}{ccl} (X,B) \overset{\text{birational}}{\dashrightarrow} & Y = Y_1 & \\ & \downarrow & \text{tower of Fano fibrations} \\ & Y_2 & \\ & \downarrow & \\ & \vdots & \\ & \downarrow & \\ & Z = Y_r & \text{Calabi-Yau variety} \end{array}$$

Log Calabi-Yau fibration

A log Calabi-Yau fibration $(X,B) \xrightarrow{f} Z$ consists of

$$\begin{cases} (X,B) \text{ pair with good singularities} \\ f \quad \text{projective morphism} \\ K_X + B \equiv 0 \text{ over } Z \end{cases}$$

X, Z may not be projective.

Example: (X,B) log Calabi-Yau variety, $Z = pt$.

Example: $X \longrightarrow Z$ minimal elliptic surface.

Example: $X \longrightarrow Z$ Fano fibration, $B \equiv -K_X / Z$.

Example: Z smooth variety, $\mathcal{E}$ coherent locally free sheaf on Z,

$X = \mathbb{P}(\mathcal{E}) \longrightarrow Z$ is a Fano fibration, so can choose $B \equiv -K_X / Z$

to get a log Calabi-Yau fibration $(X, B) \to Z$.

Example: $X \longrightarrow Z$ divisorial contraction, or flipping contraction.

Example: $X = Z$, (X, B) germ of singularity.

Conjecture: W smooth projective variety, with Kodaira dimension $\kappa(X) < \dim X$.

Then $\exists$

$$\begin{tikzcd} W \arrow[r, dashed, "\text{birational}"] & X \arrow[d, "\text{Calabi-Yau fibration}"] \\ & Z = \operatorname{Proj} \bigoplus_{m \geq 0} H^0(W, mK_W) \end{tikzcd}$$

A similar statement applies to pairs (W, C).

- There are many problems about log Calabi-Yau fibrations, e.g.
    - How the singularities of X, Z, and fibres are related?
    - A ample divisor on Z, how the linear systems

$$| n f^* A - m K_X |$$

behave, for $n, m \in \mathbb{N}$?

Moduli spaces

Under what conditions log Calabi-Yau varieties have compactified moduli spaces?

Special cases, e.g. K3 surfaces, abelian varieties, are very well-studied.

One problem with constructing moduli spaces is to understand limits.

Birkar (2020): $d \in \mathbb{N}$, Φ finite set of rational numbers, $v \in \mathbb{Q}^{>0}$. Then

$$\left\{ (X, B), A \;\middle|\; \begin{array}{l} (X,B) \text{ log Calabi-Yau of } \dim = d, \\ \text{good singularities, } \operatorname{coeff}(B) \subset \Phi \end{array} \right\}$$

$$\left\{ (X,B), A \;\middle|\; \begin{array}{l} (X,B) \text{ log Calabi-Yau of } \dim = d, \\ \text{good singularities, } \operatorname{coeff}(B) \subseteq \Phi \\ A \geq 0 \text{ ample with volume } A^d = v \end{array} \right\}$$

admits a projective moduli space.

This is a very general result.

It can be applied to usual Calabi-Yau varieties and much more.

Generalised Calabi-Yau varieties

X projective variety with $-K_X$ nef, i.e. $-K_X \cdot C \geq 0$ $\forall$ curve $C \subseteq X$.

This is a big and important class of varieties.

Example: X Calabi-Yau.

Example: X Fano.

Example: X rational minimal elliptic surface.

It has not been easy to study such varieties because $-K_X$ nef is not stable under many transformations.

Put $M = -K_X$.

Then $K_X + M = 0$.

Now we can view (X, M) as a generalised Calabi-Yau variety and apply machinery of generalised pairs:

$$(X, M) \overset{\varphi}{\dashrightarrow} (X', M')$$

φ birational

φ^{-1} contracts no divisor

(X', M') still generalised Calabi-Yau although $-K_{X'}$ usually not nef.

Arithmetic

k number field,

X Calabi-Yau or (X, B) log Calabi-Yau over k.

$X(k) = \{ k\text{-rational point} \}$.

$X(k)$ can be empty.

Conjecture: $\exists$ finite extension $k \subseteq k'$ s.t. $X(k')$ is dense.

This is known only in special cases & some low dimension cases.