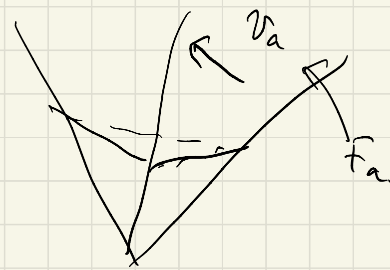
2020-11-10 Kähler geometry

Recall $X = C(S)$

$$C = \mu(X) \subset \mathfrak{t}^* \cong \mathbb{R}^{m+1}$$

$$= \{ p \in \mathfrak{t}^* \cong \mathbb{R}^{m+1} \mid \ell_a(p) \geq 0, \ a = 1, \dots, m+1 \}$$

$$\ell_a(p) = \langle \underset{\mathbb{R}^{m+1}}{\overset{\cap}{p}}, \underset{\mathbb{R}^{m+1}}{\overset{\cap}{v_a}} \rangle \qquad \mathfrak{t} \times \mathfrak{t}^* \cong \mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$$



$$C^* = \{ q \in \mathfrak{t} \cong \mathbb{R}^{m+1} \mid \langle p, q \rangle \geq 0 \ \text{for} \ {}^{\forall} p \in C \}$$

$$= \left\{ \sum_{a=1}^{m+1} \lambda_a v_a \;\middle|\; \lambda_a \geq 0, \ a = 1, \dots, m+1 \right\}$$

$$F_a = \{ p \in C \mid \ell_a(p) = 0 \} \quad \text{"facet"}$$

d = "# of facets"

review

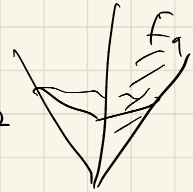
$$L : \mathfrak{t}^* \cong \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^d \qquad \text{injective.}$$
$$p \longmapsto (\ell_1(p), \dots, \ell_d(p))$$

<u>Def</u> (angle cone) ← de Borbon – Legendre.

Angle cone $\mathcal{B} = \{\beta \in (\beta_1, \dots, \beta_d) \in \mathbb{R}^d_{>0}\} \cap \mathrm{Im}(L)$

$= \{\beta = (\underset{\beta_1}{\underset{\|}{l_1(p)}}, \dots, \underset{\beta_d}{\underset{\|}{l_d(p)}}) \mid p \in C\}$

$\mathrm{Int}(C^*) \xrightarrow[L]{\sim} \mathcal{B} \subset \mathbb{R}^d$

$\begin{array}{c} X \\ \cup \\ D_a \end{array} \longrightarrow$ 

Recall

$D_a = \mu^{-1}(F_a)$ invariant divisor

"toric divisor"

We consider T^{m+1}-invariant Kähler metric with cone angle $2\pi\beta_a$ along D_a.

Theorem 1.1 (de Borbon – Legendre, arXiv 2005.03502)

There is an $(m+1)$-dimensional family of T^{m+1}-invariant Calabi–Yau (Ricci flat Kähler) metric with cone angle β_a along D_a. They are described in the following way.

(1) Fix a Reeb vector field $\xi \in \mathrm{Int}(C^*)$.

Then $\exists ! \beta \in \mathcal{B}$ s.t. X has a T^{m+1}-invariant Calabi–Yau metric with cone angle $2\pi\beta_a$

along D_a for $a = 1, \dots, d$.

(2) Fix $\beta \in \mathcal{B}$. Then $\exists !$ $\xi \in \mathrm{Int}(C^*)$ s.t. X has a T^{n+1}-invariant Calabi-Yau metric with cone angle $2\pi\beta_a$ along D_a for $a = 1, \dots, d$.

Outline of Proof

(1) $\xi \longrightarrow P_\xi \longrightarrow p =$ barycenter of P_ξ

$\longrightarrow \beta_a = l_a(p)$

"$\log \mathrm{Fut}_\xi = 0$" and then apply Wang-Zhu, Donaldson's analysis.

(2) $\beta \in \mathcal{B} \longrightarrow \beta_a = l_a(p)$ for some $p \in C$.

$\longrightarrow$ Prop 2 determines ξ.

$\longrightarrow$ p is barycenter P_ξ using Prop 1.

$\longrightarrow$ "$\log \mathrm{Fut} = 0$". Wang-Zhu.

Theorem 1.2 (de Borbon–Legendre)

Let $X = C(S)$ be a toric Kähler manifold. The following statements are equivalent.

(1) $\beta \in \mathcal{B}$ i.e. $\exists\, p \in C = \mu(X)$ s.t. $\beta_a = l_a(p)$, $a = 1, \dots, d$.

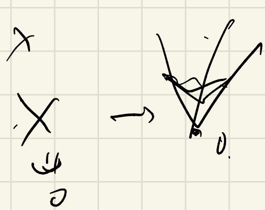
(2) $\exists$ toric Calabi-Yau metric on X with cone angle $2\pi\beta_a$ along D_a for $a = 1, \dots, d$.

(3) $\exists$ a toric Kähler cone metric with cone angle $2\pi\beta_a$ along D_a s.t. $\exists\, h \in C^\infty(X)$ with

$$Ric(\omega) = i\partial\bar\partial h \text{ on } X \setminus \bigcup_a D_a = \mu^{-1}(\mathrm{Int}\, C).$$

(4) $(\bar{X}, \sum_a (1-\beta_a) \bar{D}_a)$ is a klt pair.

$$\left.\begin{array}{l} \bar{X} = X \cup \{0\} \\ \bar{D}_a = D_a \cup \{0\} \end{array}\right) \text{ 0 is the apex}$$



(5) $H = \ker \eta \subset TS$ contact distribution

$$c_1(H) = \sum_a (1-\beta_a)[\Sigma_a]$$

and ⑤

$$c_1^B - \sum_a (1-p_a)\,[\Sigma_a]_B > 0.$$

where $[\Sigma_a] \in H^2(S,\mathbb{R})$ Poincaré dual

of $\Sigma_a := S \cap D_a$ dim $(2m-1)$ in dim $S = 2m+1$

$$S = \{r=1\} \subset C(S) = X.$$

c_1^B the basic first Chern class.

Rem (1) $\Leftrightarrow$ (2) is exactly the same as Theorem 1.1.

(3) appears in the proof of Theorem 1.1.

Hence we concentrate on (4) and (5).

Rem When $p_a = 1$ for $a = 1, \cdots, d$, the result

is due to Martelli-Sparks-Yau (+ F-Ono

- Wang).

In this case $\underbrace{c_1(H) = 0.}$ & $c_1^B > 0.$

$$\Downarrow$$

$$\mathbb{R}^+[d\eta] = c_1^B$$

Rem Special case M : Fano. Contract the zero section of K_M^{-1}, which is our $X = C(S)$

S = the unit circle bundle of K_M^{-1}.

Algebraic point of view ((4) and (5) of Theorem 1.2)

$$\boxed{\text{We write } X \text{ for } \bar{X} = \overline{C(S)}, \quad X^{reg} = \bar{X} - \{0\}}$$

our new notation.

Prop On a toric manifold X,

$$K_X = -\sum_a D_a$$

for toric divisors

(standard result in Toric geometry)

∵ $(z^1, \dots, z^n)$ logarithmic holomorphic coordinates

$w^i = e^{z^i} = e^{x^i + \sqrt{-1}\,\theta^i}$ gives $(\mathbb{C}^*)^{m+1}$-coordinates

$z^i = x^i + \sqrt{-1}\,\theta^i$

$(x^1, \theta^1, \dots, x^n, \theta^n)$ logarithmic holo coord. $\longleftrightarrow$ action-angle coord.

Along $D_{a_1} \cap \dots \cap D_{a_k} = \{w^1 = \dots = w^k = 0\}$

$$dz^1 \wedge \dots \wedge dz^n = \frac{dw^1}{w^1} \wedge \dots \wedge \frac{dw^k}{w^k} \wedge dz^{k+1} \wedge \dots \wedge dz^n$$

☺

More precisely $K_X + \sum D_a$ is a prime divisor defined by a meromorphic function. In particular this is Cartier. (expressed by a single function).

<u>Prop</u> "$\beta_a = l_a(p)$ for $\exists p$" implies

$K_X + \sum (1-\beta_a) D_a$ is $\mathbb{R}$-Cartier, i.e.

$K_X + \sum (1-\beta_a) D_a$ is a $\mathbb{R}$-linear combination of Cartier divisors.

<u>Proof</u> Given $c_a \in \mathbb{R}$, an $\mathbb{R}$-divisor

$$E = \sum_{a=1}^{d} c_a D_a \text{ is } \mathbb{R}\text{-Cartier}$$

$$\iff \exists p \in \mathfrak{t}^* \text{ s.t. } c_a = -\langle p, v_a \rangle$$

for $v_a \in \Lambda \subset \mathfrak{t}$ lattice.

(Cox - Little - Schenk : Toric varieties Theorem 4.2-8.)

In our case $\beta_a = l_a(p) = \langle p, v_a \rangle = -c_a$.

$$K_X + \sum_a (1-\beta_a) D_a = -\sum \beta_a D_a = c_a D_a$$

which is $\mathbb{R}$-Cartier.