Construction of exotic smooth structures

- Exotic $\mathbb{R}^4$

Theorem (Stallings) $\mathbb{R}^n$ has a unique smooth structure if $n \geq 5$.

Theorem (Taubes) $\mathbb{R}^4$ has uncountably many smooth structures.

We first construct a single exotic $\mathbb{R}^4$.

Take $X = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$. $\quad Q_X \cong (-E_8) \oplus (-1) \oplus (1)$

so $\exists\, \alpha \in H_2(X)$ s.t. $\cdot\, \alpha^2 = 1$

- let $\alpha^\perp = \{\beta \mid \alpha\cdot\beta = 0\}$. Then $Q_X|_{\alpha^\perp} \cong (-E_8) \oplus (-1)$.

Claim: α can not be represented smoothly embedded 2-sphere Σ,

Otherwise $\nu(\Sigma) = \mathbb{C}P^2 \setminus D^4$ and $X = X' \# \mathbb{C}P^2$ with

$$Q_{X'} = Q|_{\alpha^\perp} = (-E_8) \oplus (-1)$$

negative definite, not diagonalizable. Contradiction to Donaldson's theorem

However, Freedman's work $\Rightarrow$

- α can be represented by a locally flat, topological embedding

$$\Sigma \hookrightarrow X$$

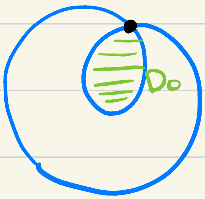
locally flat: $\forall\, x \in \Sigma$, $\exists$ open nbd V of x in X and homeomorphism $(V, V \cap \Sigma) \xrightarrow{\cong_{top}} (\mathbb{R}^4, \mathbb{R}^2)$

Furthermore, Σ has an open neighborhood U s.t. U can be smoothly embedded into $\mathbb{CP}^2$.

Rough idea of proof: $\pi_1(X)=1 \Rightarrow H_2(X)\cong\pi_2(X)$

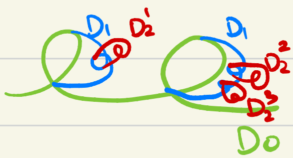
so α can be reprented by immersed $\Sigma_0 \looparrowright X$

It has some self intersection points, e.g.



Want to find an embedded Whitney disk D_0 so we can isotope Σ_0 along D_0 remove the self intersection points.

But D_0 may have self intersection, to remove them, we need second-level Whitney disks D_1^1, D_1^2



Third level, D_2^1, D_2^2, D_2^3 - - -

Let's go crazy and construct an infinite tower of whitney disks. $D_\infty := \bigcup_{i=1}^{\infty} \bigcup_j D_i^j$

Freedman: $\nu(D_\infty) \cong_{top} \operatorname{int}(D^2\times D^2)$

so $\nu(\Sigma_0 \cup D_\infty) \cong_{top} \operatorname{int}(D^4 \cup \underset{\substack{1\text{ framed}\\\text{unknot}}}{2\text{-handle}}) = \mathbb{CP}^2 \setminus \mathring{D}^4$

Furthermore, $\nu(\Sigma_0 \cup D_\infty)$ can be smoothly embedded into $\mathbb{CP}^2$.

So we have:

- $\Sigma \xrightarrow[\text{locally flat}]{} X = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2 \qquad g(\Sigma)=0, \ \Sigma^2=1$

- $Q_X|_{[\Sigma]^\perp} \cong (-E_8) \oplus (-1)$

- an open neighborhood U of Σ in X s.t. $\exists\ U \xhookrightarrow{\text{smooth embedding}} \mathbb{CP}^2$

Let $R = \mathbb{CP}^2 \setminus \Sigma$. Then R satisfies the following properties

(i) R is contractible (M.V. sequence)

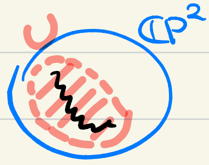
(ii) R is simply-connected at ∞. I.e.

$\forall$ compact $Z \subset R$ $\exists$ compact $C \subset R$ s.t. $\pi_1(R \setminus C) = 0$

(iii) $\exists$ compact subsets $K \subset R$ $\quad K' \subset X \setminus \Sigma$

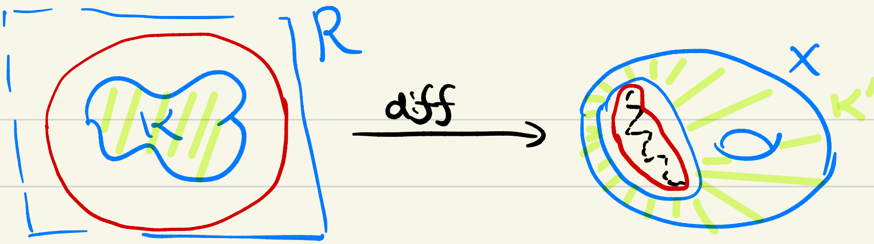
s.t. $R \setminus K \cong_{\text{diff}} (X \setminus \Sigma) \setminus K'$

Actually, we can set $K = \mathbb{CP}^2 \setminus U$, $K' = X \setminus U$



Freedman's theorem for open 4-mfds: Any contractible 4-mfds that is simply connected at ∞ is homeomorphic to $\mathbb{R}^4$.

So $R \cong_{\text{top}} \mathbb{R}^4$



Claim: $R\setminus K \cong_{diff} (X\setminus\Sigma)\setminus K' \Rightarrow R \not\cong_{diff} \mathbb{R}^4$

Otherwise, we have diffeomorphism $f: \mathbb{R}^4\setminus K \to (X\setminus\Sigma)\setminus K'$

Take $a \gg 0$ s.t. $K\subset B_a$ ← ball of radius a

Then $X' = (X\setminus\Sigma)\setminus f(\mathbb{R}^4\setminus B_a)$ is a manifold

with $\partial X' = \partial B_a = S^3$ $\quad (X' \cong (B_a\setminus K)\cup K')$

$Q_{X'} = Q_X|_{[\Sigma]^\perp} = (-E_8)\oplus(-1)$.

Let $X'' = X' \cup D^4$. Then $Q_{X''} = (-E_8)\oplus(-1)$.

contradict to Donaldson's theorem.

Uncountably many exotic $\mathbb{R}^4$'s

Let R be as above. Then $\exists$ homeomorphism $h: \mathbb{R}^4 \to R$

$\exists\, a_0 \gg 0$ s.t. $K \subset h(B_{a_0})$

$\forall\, r > a_0$, set $R_r := h(\mathring{B}_r)$ Then $R_r \cong_{top} \mathbb{R}^4$

Theorem (Taubes) $\forall\, r > s > a_0$, $R_r \not\cong_{diff} R_s$.

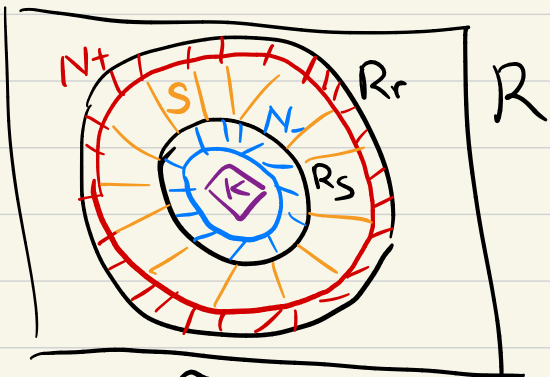
Suppose $\exists f: R_s \xrightarrow{\cong_{\text{diff}}} R_r$ with $r > s > a_0$

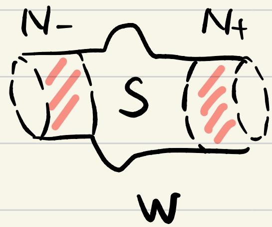
Take small $\varepsilon > 0$, consider $W = R_r \setminus \overline{R_{s-\varepsilon}}$ $\quad (W \cong_{\text{top}} (0,1)\times S^3)$

$W = N_- \cup S \cup N_+$ where

$N_- = R_s \setminus \overline{R_{s-\varepsilon}}$ $\qquad N_+ = f(N_-) = R_r \setminus \overline{f(R_{s-\varepsilon})}$

$S = \overline{f(R_{s-\varepsilon})} \setminus R_s$



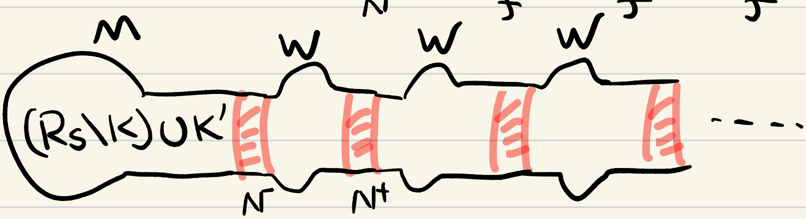


Note $f: N_- \xrightarrow{\cong} N_+$

Recall $(R\setminus K)\cup K' \cong X \setminus \Sigma$ where $Q_X|_{[\Sigma]^\perp} = (-E_8)\oplus H$

We form the noncompact manifold

$$\tilde{X} = \overbrace{\left((R_s \setminus K)\cup K'\right)}^{M} \cup_{N^-} W \cup_f W \cup_f W \cup_f W \cdots$$



This is called a manifold with periodic end modeled on W.

Note $Q_M \cong Q_X|_{[\Sigma]^\perp} = (-E_8)\oplus(-1)$. This contradicts with the following theorem:

(Taubes)
Theorem: Let $\tilde{X} = M \cup_N W \cup_N W \cup \cdots$ be an end periodic manifold. Suppose:
- $\pi_1(W)=1$
- $H_1(N;\mathbb{R}) = H_2(N;\mathbb{R}) = 0$
- Q_M is negative definit.
- Let $Y = W/_{N_- \sim N_+}$. Then Q_Y is negative definite.

Then Q_M is diagonalizable.

Idea of proof: Consider the ASD Yang-Mills equation $F_A^+ = 0$ over $\tilde{X}$.

$\pi_1(W)=0 \Rightarrow$ no nontrivial flat $SU(2)$-connection on W.

$\Rightarrow \tilde{X}$ looks like a manifold with cylindrical end

$$M \cup_{S^3} (S^3 \times [0,\infty))$$

so the Yang-Mills proof of Donaldson's theorem can be adapted to show Q_M is diagonalizable.

Remark: So far, we still don't have a Seiberg-Witten proof of this. (Hard to find a condition to replace $\pi_1(W)=1$)

Fintushel-Stern knot surgery

Recall that the Seiberg-Witten invariant of $X = X_1 \cup_{S^3} X_2$ can be understood via neck-stretching argument.
(e.g. if $b^+(X_1), b^+(X_2) > 0$, then $SW_X = 0$).
We essentially use the fact that S^3 has a positive scalar metric.
So $\nexists$ irreducible solution of the Seiberg-Witten equation.

Now consider $X = X_1 \cup_{T^3} X_2$. Since T^3 has a flat metric, we could expect a similar neck-stretching argument works.

This turn out to be the case, we will mention one particularly useful construction: $\boxed{\text{Fintushel-Stern knot surgery}}$

Take $T^2 \hookrightarrow X$ with $T^2 \cdot T^2 = 0$. Then $\overline{\nu(T^2)} = T^2 \times D^2 = S^1 \times (S^1 \times D^2)$
Take a knot K, consider $S^1 \times (S^3 \setminus \nu(K))$
The Fintushel-Stern knot surgery

$$X_K := (X \setminus \nu(T^2)) \cup_{T^3} (S^1 \times (S^3 \setminus \nu(K))) \quad (\text{Note } X_U = X)$$

The gluing map $\partial(S^1 \times (S^3 \setminus \nu(K))) \longrightarrow \partial(X \setminus \nu(T^2)) = S^1 \times S^1 \times \partial D^2$

sends

$$m_K \rightsquigarrow * \times S^1 \times *$$
$$\ell_K \rightsquigarrow * \times * \times \partial D^2$$
$$S^1 \times * \rightsquigarrow S^1 \times * \times *$$

$$X_K = (X \setminus \nu(T^2)) \bigcup_{T^3} (S^1 \times (S^3 \setminus \nu(K)))$$

Lemma (1) $Q_{X_K} \cong Q_X$

(2) $\pi_1(X \setminus T^2) = 1 \Rightarrow \pi_1(X_K) = 1 \quad \forall K$

proof (1) $H_*(S^3 \setminus \nu(K)) = H_*(S^1 \times D^2)$

(2) $\pi_1(S^1 \times (S^3 \setminus \nu(K)))$

$= \pi_1(S^1) \oplus \pi_1(S^3 \setminus \nu(K))$

normally generated by $S^1 \times *$, m_K, both are null-homotopic in $X \setminus \nu(T^2)$. So $\pi_1(X_K) = 1$

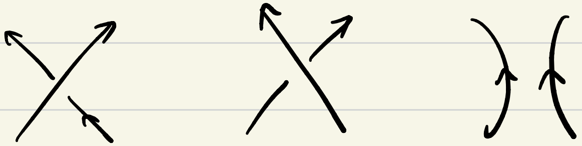
So by Freedman, $X_K \cong_{top} X$ if $\pi_1(X \setminus T^2) = 1$.

Q: How is SW_{X_K} related SW_X and K?

To answer this, we recall two thing:

Given K, the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ is defined via the skein relation:

- $\Delta_{\text{unknot}}(t) = 1$
- $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \Delta_{L_0}(t)$



L_+ $\quad$ L_- $\quad$ L_0

E.g. $H =$ 

$$\Delta_H(t) - \Delta_{O \sqcup O}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \Delta_O(t)$$
$$= t^{\frac{1}{2}} - t^{-\frac{1}{2}}$$

$K =$ 

$\Delta_K(t) - \Delta_O(t) = (t - t^{-1}) \Delta_H(t) = t + t^{-1} - 2$

$\Delta_K(t) = t + t^{-1} - 1$

The symmetized Alexander polynomial: normalize by t^n

s.t $\Delta_K(t) = \Delta_K(t^{-1})$ Also note $\Delta_K(1) = 1$.

Any polynomial $p \in \mathbb{Z}[t, t^{-1}]$ with $p(t) = p(t^{-1})$, $p(1) = 1$

Given X, we define the Seiberg-witten generating function

$$SW_X := \sum_{v \in char(X)} sw_X(v) \cdot e^v$$

Theorem (Fintushel-Stern) Suppose $[T^2] \neq 0 \in H_2(X; \mathbb{R})$

Let X_K be the knot surgery along T^2. Then

$$SW_{X_K} = SW_X \cdot \Delta_K(e^{2[T^2]})$$

(I.e. Suppose $\Delta_K(t) = \sum_{i=-n}^{n} a_i t^i$, then

$$sw_{X_K}(v) = \sum_{i=-n}^{n} a_i \cdot sw_X(v - 2i[T^2]))$$

Corollary: Given any smooth 4-mfd X with

- $b^+(X) \neq 0$, $\pi_1(X) = 1$.
- $SW_X \neq 0$ (e.g. X is symplectic)
- $\exists T^2 \hookrightarrow X$ s.t. $[T^2] \neq 0 \in H_2(X; \mathbb{R})$ $T^2 \cdot T^2 = 0$ $\pi_1(X \setminus T^2) = 1$

Then X has infinity many smooth structure.

Note: all known symplectic 4-mfd with $b^+ > 1$ satisfies this

In particular, elliptic surface satisfies this.

Proof: Any K, L with $\Delta_K \neq \Delta_L$, we have

$$SW_{X_K} = SW_X \cdot \Delta_K(e^{2[T^2]}) \neq SW_{X_L}$$

so $X_K \not\cong_{\text{diff}} X_L$. $\square$

Fintushel-Stern knot surgery conjecture:
Suppose $[T^2] \neq 0 \in H_2(X;\mathbb{R})$, then $X_K \cong_{\text{diff}} X_L$ if and only if $\pi_1(S^3 \setminus K) \cong \pi_1(S^3 \setminus L)$.

In particular, if K, L are prime, then
Conj $\Rightarrow$ $X_K \cong X_L$ iff $K = L$ $K = \overline{L}$.

Open question: Given K, L with $\Delta_K = \Delta_L$, can $X_K \not\cong_{\text{diff}} X_L$? (They have same Seiberg-Witten inv.)