Deligne-Mumford stable curves

For purpose of "counting" curves later, we want to do integration on our moduli space. For this, we compactify it, that is introduce a further compact moduli space $\overline{M}_{g,m} \supset M_{g,m}$.

Consider Riemann surfaces Σ, not necessarily smooth, but whose only singular points are nodes, that is they have a neighbourhood $U \cong \{xy = 0\} \subset \mathbb{C}^2$.

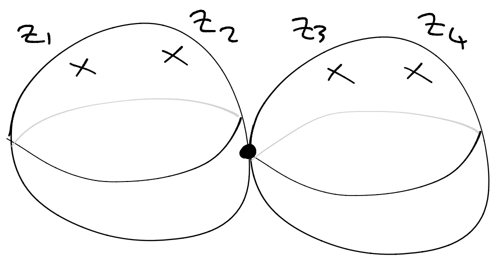
Note For each such Σ we have a family of surfaces Σ_t with a neighbourhood $\{xy = t\}$ for $0 < |t| < \varepsilon$. We call this a smoothing of the node. Similarly, we may define a smoothing of all the nodes on Σ: we define $g(\Sigma)$ as $g(\Sigma_t)$ for this smoothing

We allow marked points $\underline{z} = (z_1 - z_m)$ for smooth $z_i \in \Sigma$, and define stability as before.

<u>Def</u> Let $\overline{\mathcal{M}}_{g,m}$ be the set of isomorphism classes $[\Sigma, \underline{z}]$ of stable pairs $(\Sigma, \underline{z})$ for Σ compact, with singularities only nodes.

<u>Thm</u> $\overline{\mathcal{M}}_{g,m}$ is a compact complex orbifold, of dimension $3g+m-3$, with open set $\mathcal{M}_{g,m}$.

<u>Ex</u> Recall $\mathcal{M}_{0,4} \cong \mathbb{C} - \{0,1\} \subset \mathbb{P}^1$. Under this isomorphism $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. The points $0, 1, \infty \in \mathbb{P}^1$ correspond to nodal curves $\Sigma \cong \mathbb{P}^1 \cup_{pt} \mathbb{P}^1$



This one corresponds to limits where $z_1 \to z_2$ or vice versa or $z_3 \to z_4$ or vice versa.

Moduli of stable maps

Now we return to the setting of a compact symplectic (M,ω) with J a compatible almost complex structure.

Fix a class $\beta \in H_2(M,\mathbb{Z})$.

<u>Notation</u> Write $(\Sigma, \underline{z}, u)$ for $(\Sigma, \underline{z})$ as above (possibly singular) and $u : \Sigma \to M$ J-holomorphic with $u_*[\Sigma] = \beta$.

<u>Def</u> An isomorphism from $(\Sigma, \underline{z}, u)$ to $(\Sigma', \underline{z}', u')$ is a holomorphic map $f : \Sigma \to \Sigma'$ with holomorphic inverse such that $f(z_i) = z_i'$ for all i, and $u' \circ f = u$.

As before we say $(\Sigma, \underline{z}, u)$ is stable if its self-isomorphism group is finite.

Def $\overline{\mathcal{M}}_{g,m}(M,\beta,J)$ is the set of isomorphism classes $[\Sigma, \underline{z}, u]$ of stable $(\Sigma, \underline{z}, u)$.

Def $\mathcal{M}_{g,m}(M,\beta,J)$ is the subset where Σ is non-singular.

Note There are natural evaluation maps

$$ev_i : \overline{\mathcal{M}}_{g,m}(M,\beta,J) \to M, \quad [\Sigma, \underline{z}, u] \mapsto u(z_i)$$

Notice how our notion of isomorphism ($u' \circ f = u$) ensures these are well-defined.

Rem We sometimes drop J, according to context.

Ex $\overline{\mathcal{M}}_{g,m}(M,0) \cong \overline{\mathcal{M}}_{g,m} \times M$ because in this case u maps to a point $m \in M$ (note that $\overline{\mathcal{M}}_{g,m}$ may be empty)

A triple $(\Sigma, \underline{z}, u)$ if <u>stable</u> according to the following

<u>Criterion</u>: Let $\{\Sigma_k\}$ be the components of Σ.

(for instance, if $\Sigma = \mathbb{P}^1 \cup_{pt} \mathbb{P}^1$ then have $\Sigma_1, \Sigma_2 \cong \mathbb{P}^1$)

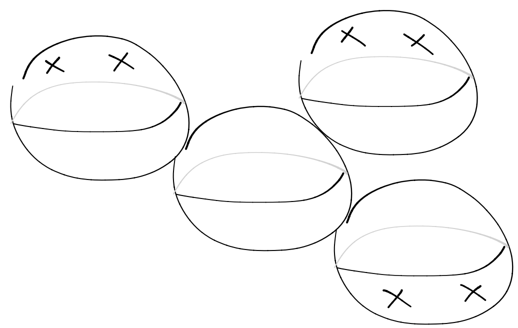
Then if $u(\Sigma_k) \cong pt$ then Σ_k must contain at least 3 (resp. 2) <u>special points</u> if Σ_k has genus 0 (resp 1).

(In particular, if $g(\Sigma_k) \geq 2$ there is no such restriction.)

<u>Note</u>: "special points" means marked points <u>and</u> singular points.

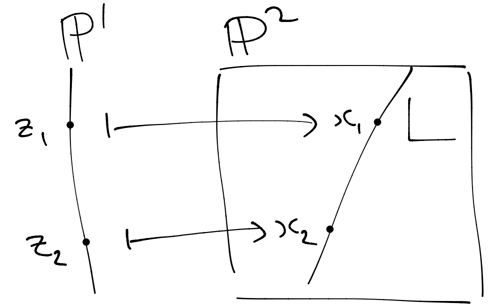
<u>Ex</u> For $(\Sigma, \underline{z}) \in \mathcal{M}_{0,m}$, and $u: \Sigma \to pt$, $(\Sigma, \underline{z}, u)$ is stable if $m \geq 3$

<u>Ex</u> For $(\Sigma, \underline{z}) \in \bar{\mathcal{M}}_{0,6}$ as shown, $(\Sigma, \underline{z}, u)$ is stable for any u.



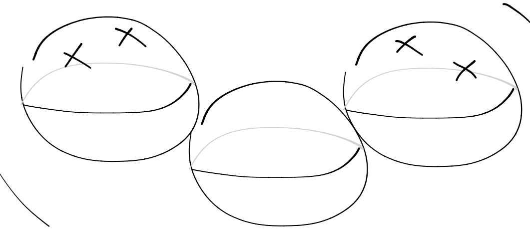
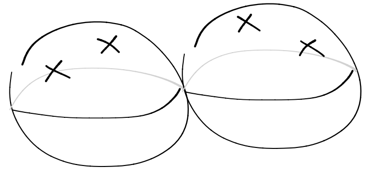
But note, however, that the domain of a stable map u is not necessarily a stable curve:

<u>Ex</u> For $(\Sigma, \underline{z}) \in \mathcal{M}_{0,2}$ with $\Sigma = \mathbb{P}^1$ an embedding $u: \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ (or any other variety) gives a stable map, for example, with $ev_i = x_i \in L \subset \mathbb{P}^2$



We have the following construction for the case $2g+m \geq 3$ (this corresponds to $\overline{\mathcal{M}}_{g,m}$ non-empty)

<u>Def</u> For $(\Sigma, \underline{z}, u) \in \overline{\mathcal{M}}_{g,m}(M, \beta)$ let the <u>stabilization</u> $\mathrm{stab}(\Sigma, \underline{z}) \in \overline{\mathcal{M}}_{g,m}$ be the result of contracting components of $(\Sigma, \underline{z})$ mentioned in the <u>Criterion</u> above until a stable curve is obtained.

Ex stab $\Bigg($  $\Bigg)$ =  $\in \overline{\mathcal{M}}_{0,4}$

2 special points

Def For $2g+m \geq 3$, we have

$$\pi : \overline{\mathcal{M}}_{g,m}(M,\beta) \longrightarrow \overline{\mathcal{M}}_{g,m}$$
$$[\Sigma, z, u] \longmapsto \mathrm{stab}\,[\Sigma, u]$$

Now we explain how close $\overline{\mathcal{M}}_{g,m}(M,\beta)$ is to being like a smooth, oriented manifold.

Thm (Gromov) $\overline{\mathcal{M}}_{g,m}(M,\beta,J)$ is compact and Hausdorff under a certain natural topology ("C^∞ topology")

Furthermore, for J generic, consider the open subset $\mathcal{U}$ of $\overline{\mathcal{M}}_{g,m}(M,\beta,J)$ where (1) Σ is smooth

and (2) u is an embedding

Then $\mathcal{U}$ is a smooth, oriented manifold with

$$\dim = 2d \text{ where } d = \langle c_1(M), \beta \rangle + (n-3)(1-g) + m$$

where $\dim M = 2n$

<u>Note</u> Here $c_1(M)$ is the 1st Chern class, which we pair with the class $\beta \in H_2(M,\mathbb{Z})$.

<u>Virtual class</u>

For a compact <u>oriented</u> manifold N, and more generally an orbifold, of dimension r, there exists a canonical element F of the homology group $H_r(N,\mathbb{Q})$, called the <u>fundamental class</u>.

Note we could take other coefficients here, but use $\mathbb{Q}$ for our application.

Pairing this with a cohomology class $\psi \in H^r(N,\mathbb{Q})$ can be thought of as integration, $\langle F,\psi\rangle = \int_N \psi$

This does not work on $\overline{\mathcal{M}}_{g,m}(M,\beta)$, but we have something to replace it. Note that for $2g+m \geq 3$ we have a map

$$\times_i ev_i \times \pi : \overline{\mathcal{M}}_{g,m}(M,\beta) \to M^m \times \overline{\mathcal{M}}_{g,m}$$

We call this Case I.

For $2g+m<3$ we do not have such a map, but we do have $\times ev_i$ for all $g,m>0$. Call this Case II.

<u>Def</u> (partial) We can define virtual classes

<u>Case I</u> $VC(X_i \xrightarrow{ev_i \times \pi}) \in H_{2d}(M^m \times \bar{\mathcal{M}}_{g,m}, \mathbb{Q})$

<u>Case II</u> $VC(X_i \xrightarrow{ev_i}) \in H_{2d}(M^m, \mathbb{Q})$

<u>Rem</u> Recall that $\dim \bar{\mathcal{M}}_{g,m}(M, \beta) = 2d$. These virtual classes may be thought of as taking the place of the pushforward of the fundamental class along the maps given. There are different approaches to constructing them. See [CK, §7] for a survey. For an approach using Kuranishi structures, see (difficult) work of Fukaya-Ono and Joyce.

The Gromov-Witten "counting" invariants are essentially these virtual classes. To put them into a more convenient form, we use the following.

(A) For N an oriented, closed manifold, dimension r,

$$PD : H^k(N, \mathbb{Q}) \xrightarrow{\sim} H_{r-k}(N, \mathbb{Q}) \quad \text{(Poincaré duality)}$$

$$\psi \longmapsto F \frown \psi \quad \text{where } \frown \text{ is cap product}$$

(B) For N_1, N_2 we have (Künneth formula)

$$H^*(N_1 \times N_2, \mathbb{Q}) \cong H^*(N_1, \mathbb{Q}) \otimes H^*(N_2, \mathbb{Q})$$

that is $H^k = \bigoplus_{i+j=k} H^i \otimes H^j$

<u>Def</u> The Gromov-Witten (GW) <u>invariant</u> is

$$\langle I_{g,m,\beta}\rangle : H^*(M,\mathbb{Q})^{\otimes m} \longrightarrow \mathbb{Q}$$

corresponding to $VC(X_i, ev_i)$ under

$$H_*(M^m,\mathbb{Q}) \cong H^*(M^m,\mathbb{Q})^\vee \cong (H^*(M,\mathbb{Q})^{\otimes m})^\vee$$

<u>Rem</u> The domain of $\langle I_{g,m,\beta}\rangle$ is

$$\bigoplus_{k_1 \dots k_m} H^{k_1}(M,\mathbb{Q}) \otimes \dots \otimes H^{k_m}(M,\mathbb{Q})$$

and $\langle I_{g,m,\beta}\rangle$ is, from the definition, zero on pieces where

$$k_1 + \dots + k_m \neq 2d$$

<u>Rem</u> Taking $N_1 \dots N_m \subset M$ and $\psi_1 \dots \psi_m$ such that $PD\,\psi_i = [N_i]$ in homology, we may think of $\langle I_{g,m,b}\rangle$ (heuristically) as counting curves passing through the N_i.

Similarly we have

Def In Case I, we define the GW class

$$I_{g,m,\beta} : H^*(M,\mathbb{Q})^{\otimes m} \longrightarrow H^*(\overline{\mathcal{M}}_{g,m},\mathbb{Q})$$

corresponding to $VC(\times_i ev_i \times \pi)$ under

$$H_*(M^m \times \overline{\mathcal{M}}_{g,m},\mathbb{Q}) \cong (H^*(M^m,\mathbb{Q})^{\otimes m})^\vee \otimes H^*(\overline{\mathcal{M}}_{g,m},\mathbb{Q})$$

where we use (A) and (B).

Rem As a map to $H^\ell(\overline{\mathcal{M}}_{g,m},\mathbb{Q})$, $I_{g,m,\beta}$ is zero if $k_1 + \_ + k_m + \underbrace{2(3g+m-3)}_{\dim_{\mathbb{R}} \overline{\mathcal{M}}_{g,m}} - \ell \neq 2d$

Rem We have a relation $\langle I_{g,m,\beta} \rangle = \int_{\overline{\mathcal{M}}_{g,m}} I_{g,m,\beta}$