2020-9-25 Kähler geometry

(ii) $\nabla J = 0 \iff \nabla(JX) = J\nabla X$

$\iff \nabla \circ J = J \circ \nabla$

(iii) $\Gamma^{C}_{AB} = 0$ except for Γ^{k}_{ij}, $\Gamma^{\bar k}_{\bar i \bar j}$

(iii) $\Longrightarrow$ $$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \Gamma^{k}_{ij} \frac{\partial}{\partial z^k} + \underbrace{\Gamma^{\bar k}_{ij}}_{0} \frac{\partial}{\partial \bar z^k} = \Gamma^{k}_{ij} \frac{\partial}{\partial z^k}$$

$$\nabla_{\frac{\partial}{\partial \bar z^i}} \frac{\partial}{\partial z^j} = \underbrace{\Gamma^{k}_{\bar i j}}_{0} \frac{\partial}{\partial z^k} + \underbrace{\Gamma^{\bar k}_{\bar i j}}_{0} \frac{\partial}{\partial \bar z^k} = 0$$

$$\therefore \nabla_X \frac{\partial}{\partial z^i} \in C^\infty(T'M) \quad \text{for } \forall X$$

$$J \nabla_X \frac{\partial}{\partial z^i} = \sqrt{-1}\, \nabla_X \frac{\partial}{\partial z^i}$$

On the other hand

$$\nabla_X \left(J \frac{\partial}{\partial z^i}\right) = \nabla_X \left(\sqrt{-1} \frac{\partial}{\partial z^i}\right) = \sqrt{-1}\, \nabla_X \frac{\partial}{\partial z^i}$$

$$\therefore \nabla_X \left(J \frac{\partial}{\partial z^i}\right) = J \nabla_X \frac{\partial}{\partial z^i}$$

In the same way

$$\nabla_X \left(J \frac{\partial}{\partial \bar z^i}\right) = J \nabla_X \frac{\partial}{\partial \bar z^i}$$

$\therefore\ \nabla \circ J = J \circ \nabla \qquad \therefore\ \nabla J = 0.$

(ii)

(ii) $\Longrightarrow$ (iii)

$$\nabla_{\frac{\partial}{\partial z^i}} \overbrace{J \frac{\partial}{\partial z^j}}^{\sqrt{-1}\frac{\partial}{\partial z^j}} = \sqrt{-1}\, \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \sqrt{-1}\left(\Gamma^k_{ij} \frac{\partial}{\partial z^k} + \Gamma^{\bar k}_{ij} \frac{\partial}{\partial \bar z^k}\right)$$

$\|$ (ii)

$$J \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = J\left(\Gamma^k_{ij} \frac{\partial}{\partial z^k} + \Gamma^{\bar k}_{ij} \frac{\partial}{\partial \bar z^k}\right)$$

$$= \sqrt{-1}\, \Gamma^k_{ij} \frac{\partial}{\partial z^k} - \sqrt{-1}\, \Gamma^{\bar k}_{ij} \frac{\partial}{\partial \bar z^k}$$

$$\Gamma^{\bar k}_{ij} = 0 \qquad \therefore\ \Gamma^k_{i\bar j} = \overline{\Gamma^{\bar k}_{ij}} = 0$$

$$\nabla_{\frac{\partial}{\partial z^i}} \overbrace{J \frac{\partial}{\partial \bar z^j}}^{-\sqrt{-1}\frac{\partial}{\partial \bar z^j}} = -\sqrt{-1}\left(\Gamma^k_{i\bar j} \frac{\partial}{\partial z^k} + \Gamma^{\bar k}_{i\bar j} \frac{\partial}{\partial \bar z^k}\right)$$

$\|$ (ii)

$$J \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar z^j} = J\left(\Gamma^k_{i\bar j} \frac{\partial}{\partial z^k} + \Gamma^{\bar k}_{i\bar j} \frac{\partial}{\partial \bar z^k}\right)$$

$$= \sqrt{-1}\, \Gamma^k_{i\bar j} \frac{\partial}{\partial z^k} - \sqrt{-1}\, \Gamma^{\bar k}_{i\bar j} \frac{\partial}{\partial \bar z^k}$$

$$\therefore\ \Gamma^k_{i\bar j} = 0 \qquad \Gamma^{\bar k}_{i\bar j} = 0 = \Gamma^k_{\bar j i}$$

$g(JX, JY) = g(X, Y) \iff g$ Hermitian

$g_{ij} = g_{\bar{i}\bar{j}} = 0$

<u>Rem</u> $(g_{j\bar{i}}) = (g_{\bar{i}j}) = \overline{(g_{i\bar{j}})}$ $\quad (g_{\bar{i}j})$ Hermitian

$$\frac{\partial}{\partial z^i} g_{j\bar{k}} = g\left(\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{k}}}\right) = \Gamma^l_{ij}\, g_{l\bar{k}}$$

$g_{\bar{k}j}$ $\qquad g_{\bar{k}l}$

$$A = (a_{ij})$$

$$g_{\bar{i}j} = g_{j\bar{i}} \qquad g^{i\bar{k}} g_{\bar{k}j} = \delta^i_j$$

$$AB = \left(\sum a_{i\underline{k}} b_{\underline{k}j}\right)$$

$$\Gamma^k_{ij} = g^{k\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^i} \qquad \Gamma^i_{kj} = g^{i\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^k}$$

My convention $\qquad g = (g_{\bar{i}j})$

$$g^{-1} = (g^{i\bar{j}})$$

$$\left(\Gamma^i_{kj}\, dz^k\right) = g^{-1} \partial g = \left(g^{i\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^k} dz^k\right)$$

$\theta^i{}_j = \Gamma_k{}^i{}_j \, dz^k$

$\theta = (\theta^i{}_j)$ connection 1-form
connection matrix.

$= \underline{g^{-1} \partial g}$ ⟵ Remember

$\Theta \overset{def}{=} \underline{d\theta + \theta \wedge \theta}$ Ω ω

curvature form

曲率

$d = \partial + \bar\partial$

$\theta = g^{-1} \partial g$

$\partial(g^{-1})$

$= -g^{-1} \partial g \cdot g^{-1}$

exchange.

$= (\partial + \bar\partial)\theta + \theta \wedge \theta$

$= \bar\partial\theta + \partial\theta + \underline{\theta \wedge \theta}$

$= \bar\partial\theta - \cancel{g^{-1} \partial g \cdot g^{-1} \wedge \partial g}$

$+ \cancel{g^{-1} \partial g \wedge g^{-1} \partial g}$

$= \underline{\bar\partial\theta}$ ⟵ Remember

Prop $\quad e = \left(\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^m}\right)$

$$e\,\Theta(X,Y) = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) e$$

$\because$

$$\Theta(X,Y) = (d\theta + \theta\wedge\theta)(X,Y)$$
$$= X\theta(Y) - Y\theta(X) - \theta([X,Y])$$
$$+ \theta(X)\theta(Y) - \theta(Y)\theta(X)$$

On the other hand $\qquad e \to e$

$$\text{RHS} = \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e$$
$$= \nabla_X (e\,\theta(Y)) - \nabla_Y (e\,\theta(X)) - e\cdot\theta([X,Y])$$
$$= e\,\theta(X)\cdot\theta(Y) + e\,X\theta(Y)$$
$$- e\,\theta(Y)\theta(X) - e\,Y\theta(X) - e\,\theta([X,Y])$$
$$= e\,\Theta(X,Y)$$

$\because$

If we use 周華清 $\longleftarrow$

$$X = (x^1 \cdots x^n)\begin{pmatrix} e^1 \\ \vdots \\ e^n \end{pmatrix} \qquad \nabla e = \theta e$$

$\longleftarrow e$

$$\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) e$$

$$= \nabla_X(\theta(Y)e) - \nabla_Y(\theta(X)e) - \theta([X,Y])e$$

$$= X\theta(Y)e + \theta(Y)\theta(X)e$$

$$- Y\theta(X) - \theta(X)\theta(Y)e - \theta([X,Y])e$$

$$= (d\theta(X,Y))e - (\theta\wedge\theta)(X,Y)e.$$

$$\therefore \Theta = d\theta - \theta\wedge\theta$$

Shiing-Shen Chern 陳省身

Complex manifolds without potential theory

P principal G-bundle $\qquad \theta = (\theta^i_j)$

with left G-action

Western author P has right G-action $\quad \underline{d\theta + \theta\wedge\theta}$

$\Theta = \Theta^{\alpha}{}_{\beta i \bar{j}}\, dz^i \wedge d\bar{z}^j$ vector bdle

$\Theta = R^k{}_{l i \bar{j}}\, dz^i \wedge d\bar{z}^j$ $\quad T'M$

Kähler

Riemann

$\textcircled{\scriptsize\theta}$ $\underline{R_{\bar{i}\bar{j} k \bar{l}} = 0}$

$\because$ $\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^l} = 0.$ $\quad \nabla_{\left[\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right]} \leftarrow 0$

$$R_{\bar{i}\bar{j} k \bar{l}} = g\left(\frac{\partial}{\partial z^k}, \left(\nabla_{\frac{\partial}{\partial \bar{z}^i}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} - \nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial \bar{z}^i}} \right) \frac{\partial}{\partial \bar{z}^l} \right)$$

$= 0$ $\quad \because$

Same way $\underline{R_{ij \bar{k} \bar{l}} = 0}$

$\therefore$ $R_{\bar{i}\bar{j} \bar{l} k} = -R_{\bar{i}\bar{j} k \bar{l}} = 0$

$\therefore$ $R_{\bar{i}\bar{j} \bar{k} l} = R_{\bar{i}\bar{j} k \bar{l}} = 0$

Thm

We only have possible non-zero component as $\underline{R_{i\bar{j} k \bar{l}}}$ or its symmetries.

$\boxed{\text{Thm}}$

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

:)

$$R_{i\bar{j}k\bar{l}} = g\left(\frac{\partial}{\partial z^k}, \left(\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} - \nabla_{\frac{\partial}{\partial \bar{z}^j}} \nabla_{\frac{\partial}{\partial z^i}}\right) \frac{\partial}{\partial \bar{z}^l}\right)$$

$$= g\left(\frac{\partial}{\partial z^k}, \nabla_{\frac{\partial}{\partial z^i}} \left(\Gamma^{\bar{q}}_{\bar{j}\bar{l}} \frac{\partial}{\partial \bar{z}^q}\right)\right) = \left(\frac{\partial}{\partial z^i} \Gamma^{\bar{q}}_{\bar{j}\bar{l}}\right) g_{k\bar{q}}$$

$$\Gamma^{\bar{q}}_{\bar{j}\bar{l}} = \overline{\Gamma^{q}_{jl}} = \overline{g^{q\bar{p}} \frac{\partial g_{l\bar{p}}}{\partial z^j}} = g^{p\bar{q}} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$\therefore R_{i\bar{j}k\bar{l}} = g_{k\bar{q}} \frac{\partial}{\partial z^i} \left(g^{p\bar{q}} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j} \right)$$

$$= \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g_{k\bar{q}} g^{p\bar{s}} g^{t\bar{q}} \frac{\partial g_{t\bar{s}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$= \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{s}} \frac{\partial g_{k\bar{s}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

:(

Majority of Kähler geometers use

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + \cdots$$

⑨ ⑨

Def (M, g) Riemannian mfd.

$e_1, \dots, e_n$ orthonormal frame of TM.
local.

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i)$$

$$\mathrm{Ric}(Y, X) = \mathrm{Ric}(X, Y)$$

Ric is symmetric

$$\mathrm{Ric} = R_{ij}\, dx^i \otimes dx^j \qquad R_{ij} = R_{ji}$$

Ric is called the Ricci curvature or Ricci tensor.

$$\frac{\partial}{\partial x^i}$$

$$R_{ij} = g^{kl} R_{ikjl}$$

$$= R_{ikj}{}^{k} = R_{i}{}^{k}{}_{jk}.$$

(M, g) Kähler mfd.

$$R_{i\bar{j}}{}^{k}{}_{k} + R_{i}{}^{k}{}_{k\bar{j}} + \overbrace{R_{ik\bar{j}}{}^{k}}^{\text{"} 0} = 0$$

$$\text{"} \quad -R_{i}{}^{k}{}_{\bar{j}k}.$$

$$\frac{1}{\sqrt{2}}(e - iJe)$$

$$\therefore R_{i\bar{j}}{}^{k}{}_{k} = R_{i}{}^{k}{}_{\bar{j}k} = R_{i\bar{j}}$$

Ricci curvature for Kähler

Majority of Kähler geometers use

$$R_{i\bar{j}} = R_{i\bar{j}k}{}^{k}$$ the same as ours.

<u>Prop</u> $R_{i\bar{j}} = -\dfrac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det (g_{k\bar{l}})$

$\because$ In general for invertible matrix

$$d(\det A) = \det A \cdot \operatorname{tr}(A^{-1} dA)$$

Exercise

So $\Gamma^{j}_{ij} = g^{j\bar{l}} \dfrac{\partial g_{j\bar{l}}}{\partial z^i}$

$$= \frac{\partial}{\partial z^i} \log \det (g_{k\bar{l}})$$

$$\therefore R_{i\bar{j}}{}^{k}{}_{k} = -g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -g^{k\bar{l}} \frac{\partial \Gamma^{\bar{q}}_{\bar{j}\bar{l}}}{\partial z^i} g_{k\bar{q}}$$

$$= -\frac{\partial}{\partial z^i} \Gamma^{\bar{l}}_{\bar{j}\bar{l}} = -\frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^j} \log \det g \quad ☺$$

$$\Theta = \underbrace{\bar\partial \theta}_{d\theta + \theta\wedge\theta} = \underbrace{\left(R^i{}_{jk\bar l} \; dz^k \wedge d\bar z^l \right)}_{\text{curvature form matrix}} \quad (2\text{-form})$$

<u>Def</u> $\det\left(I + t \frac{\sqrt{-1}}{2\pi} \Theta \right)$ $m = \dim_{\mathbb{C}} M$.

$$= 1 + t c_1 + t^2 c_2 + \cdots + t^m c_m$$

<u>Fact</u> $c_p(g)$ is a real closed $2p$-form.

de Rham class $[c_p(g)]$ is indep of J.

We do it only for $p = 1$.