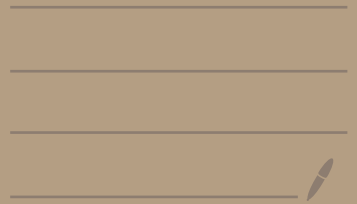


2020-9-25 Kähler geometry



$$(ii) \quad \nabla J = 0 \quad \Leftrightarrow \quad \nabla(JX) = J \nabla X$$

$$\Leftrightarrow \quad \nabla \circ J = J \circ \nabla$$

$$(iii) \quad \rho_{AB}^C = 0 \quad \text{except for } \rho_{ij}^k, \rho_{\bar{i}\bar{j}}^{\bar{k}}$$

$$(iii) \Rightarrow \quad \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^i} = \rho_{ij}^k \frac{\partial}{\partial z^k} + \underbrace{\rho_{\bar{i}\bar{j}}^{\bar{k}}}_{=0} \frac{\partial}{\partial z^{\bar{k}}} = \rho_{ij}^k \frac{\partial}{\partial z^k}$$

$$\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^i} = \underbrace{\rho_{ij}^k}_{=0} \frac{\partial}{\partial z^k} + \underbrace{\rho_{\bar{i}\bar{j}}^{\bar{k}}}_{=0} \frac{\partial}{\partial z^{\bar{k}}} = 0$$

$$\therefore \nabla_X \frac{\partial}{\partial z^i} \in C^\infty(TM) \quad \text{for } X$$

$$J \nabla_X \frac{\partial}{\partial z^i} = \sqrt{-1} \nabla_X \frac{\partial}{\partial \bar{z}^i}$$

On the other hand

$$\nabla_X (J \frac{\partial}{\partial z^i}) = \nabla_X (\sqrt{-1} \frac{\partial}{\partial \bar{z}^i}) = \sqrt{-1} \nabla_X \frac{\partial}{\partial \bar{z}^i}$$

$$\therefore \nabla_X (J \frac{\partial}{\partial z^i}) = J \nabla_X \frac{\partial}{\partial z^i}$$

In the same way

$$\nabla_X (J \frac{\partial}{\partial \bar{z}^i}) = J \nabla_X \frac{\partial}{\partial \bar{z}^i}$$

(1)

$$\therefore \nabla \cdot \vec{J} = \vec{J} \cdot \nabla$$

$$\therefore \nabla \vec{J} = 0 \quad (i)$$

(2)

$$(i) \Rightarrow (ii) \quad \sqrt{2} \frac{\partial}{\partial z^i}$$

$$\nabla_{\frac{\partial}{\partial z^i}} \vec{J} \frac{\partial}{\partial z^i} = \sqrt{2} \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^i} = \sqrt{2} \left(P_{ij}^k \frac{\partial}{\partial z^k} + \underbrace{P_{ij}^{\bar{k}}} \frac{\partial}{\partial \bar{z}^{\bar{k}}} \right)$$

$$\vec{J} \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^i} = \vec{J} \left(P_{ij}^k \frac{\partial}{\partial z^k} + P_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^{\bar{k}}} \right)$$

$$= \sqrt{2} P_{ij}^k \frac{\partial}{\partial z^k} - \sqrt{2} P_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^{\bar{k}}}$$

$$P_{ij}^{\bar{k}} = 0$$

$$\therefore P_{ij}^k = \overline{P_{ij}^{\bar{k}}} = 0$$

$$-\sqrt{2} \frac{\partial}{\partial z^i}$$

$$\nabla_{\frac{\partial}{\partial z^i}} \vec{J} \frac{\partial}{\partial z^i} = -\sqrt{2} \left(P_{ij}^k \frac{\partial}{\partial z^k} + P_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^{\bar{k}}} \right)$$

(ii)

$$\vec{J} \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^i} = \vec{J} \left(P_{ij}^k \frac{\partial}{\partial z^k} + P_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^{\bar{k}}} \right)$$

$$= \sqrt{2} P_{ij}^k \frac{\partial}{\partial z^k} - \sqrt{2} P_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^{\bar{k}}}$$

$$\therefore P_{ij}^k = 0 \quad P_{ij}^{\bar{k}} = 0 = \overline{P_{ij}^k} \quad (iii)$$

$$g(JX, JY) = g(X, Y) \Leftrightarrow g \text{ Hermitian}$$

$$g_{ij} = g_{\bar{i}\bar{j}} = 0$$

Rem $(g_{j\bar{i}}) = (g_{\bar{i}j}) = \overline{(g_{i\bar{j}})}$ $(g_{i\bar{j}})$ Hermitian

$$\frac{\partial}{\partial z^i} (g_{j\bar{k}}) = g(\frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{k}}}) = \Gamma_{ij}^{\bar{k}} (g_{\bar{l}k})$$

$$A = (a_{ij})$$

$$g_{\bar{i}j} = g_{j\bar{i}}$$

$$g^{i\bar{k}} g_{\bar{l}j} = \delta^i_j$$



$$AB = (\sum a_{ik} b_{kj})$$

$$\Gamma_{ij}^{\bar{k}} = g^{k\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^i}$$

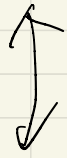
$$\Gamma_{kj}^i = g^{i\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^k}$$

My convention

$$g = (g_{i\bar{j}})$$

$$g^{-1} = (g^{i\bar{j}})$$

$$\left(\Gamma_{kj}^i dz^k \right) = g^{-1} \partial g = \left(g^{i\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^k} dz^k \right)$$



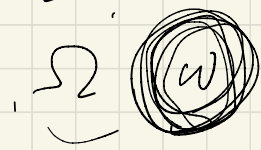
$$\theta^i_j = \Gamma^i_k; dz^k$$

$\theta = (\theta^i_j)$ connection 1-form
connection matrix.

$$= g^{-1} dg \quad \leftarrow \text{Remember}$$

(H) def $= d\theta + \theta \wedge \theta$

curvature form
曲率



$$d = \partial + \bar{\partial}$$

$$= (\partial + \bar{\partial}) \theta + \theta \wedge \theta$$

$$= \bar{\partial} \theta + \partial \theta + \theta \wedge \theta$$

$$= \bar{\partial} \theta - \cancel{g^{-1} dg \cdot g^{-1} dg} + \cancel{g^{-1} dg \wedge g^{-1} dg}$$

$$+ \cancel{g^{-1} dg \wedge g^{-1} dg}$$

$$= \bar{\partial} \theta$$

\leftarrow Remember

$$\begin{aligned} \theta &= g^{-1} dg \\ \theta(g^{-1}) &= -g^{-1} dg \cdot g^{-1} \\ &\uparrow \text{exchange} \end{aligned}$$

Prop

$$e = \left(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^m} \right)$$

(5)

$$\underline{e \oplus (X, Y)} = \left(\underline{\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}} \right) e$$

☹

$$\oplus (X, Y) = \underline{(d\theta + \theta \wedge \theta)} (X, Y)$$

$$= X \theta(Y) - Y \theta(X) - \theta([X, Y])$$

$$+ \theta(X) \theta(Y) - \theta(Y) \theta(X)$$

On the other hand $e \rightarrow e$

$$\text{RHS} = \underline{\nabla_X \nabla_Y e} - \underline{\nabla_Y \nabla_X e} - \nabla_{[X, Y]} e$$

$$= \underline{\nabla_X (e \theta(Y))} - \underline{\nabla_Y (e \theta(X))} - e \cdot \theta([X, Y])$$

$$= \underline{e \theta(X) \cdot \theta(Y)} + \underline{e X \theta(Y)}$$

$$- \underline{e \theta(Y) \theta(X)} - \underline{e Y \theta(X)} - \underline{e \theta([X, Y])}$$

$$= e \oplus (X, Y)$$

☹

If we use 同輩清

(6)

$$x = (x^1 \dots x^n) \begin{pmatrix} e^1 \\ \vdots \\ e^n \end{pmatrix}$$

$\leftarrow e$

$$\nabla e = \theta e$$

~~_____~~

$$\left(\nabla_x \nabla_Y - \nabla_Y \nabla_x - \nabla_{[X, Y]} \right) e$$

$$= \nabla_x (\theta(Y) e) - \nabla_Y (\theta(X) e) - \theta([X, Y]) e$$

$$= X \theta(Y) e - \theta(Y) \theta(X) e$$

$$- Y \theta(X) e - \theta(X) \theta(Y) e - \theta([X, Y]) e$$

$$= (\theta(X, Y)) e - (\theta \wedge \theta)(X, Y) e$$

$$\therefore \textcircled{H} = d\theta - \theta \wedge \theta$$

Shiing-Shen Chern 陳省身

Complex manifolds without potential theory

P principal G -bundle

$$\theta = (\theta^i_j)$$

with left G -action

Western authors

P has right G -action

$$\underline{d\theta + \theta \wedge \theta}$$

$$\textcircled{K} = \textcircled{H}^{\alpha} \beta_{ij} dx^i dx^j$$

vector field

(7)

$$\textcircled{H} = R^k_{ij} dx^i dx^j$$

T.M

Kähler

Riemann

$$\textcircled{9} \quad R_{ij k \bar{l}} = 0$$

$$\textcircled{!} \quad \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^k} = 0$$

$$\left[\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} \right] \leftarrow 0$$

$$R_{ij k \bar{l}} = g \left(\frac{\partial}{\partial z^k}, \left(\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial z^j}} - \nabla_{\frac{\partial}{\partial z^j}} \nabla_{\frac{\partial}{\partial z^i}} \right) \frac{\partial}{\partial z^l} \right)$$

$$= 0 \quad \textcircled{!}$$

Same way $R_{ij \bar{k} l} = 0$

$$\therefore R_{ij \bar{k} l} = -R_{ij l \bar{k}} = 0$$

$$= R_{\bar{i} j k l} = R_{ij k \bar{l}} = 0$$

Then

We only have possible non-zero component
 no $R_{i \bar{j} k \bar{l}}$ or its symmetric.

Thm

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

∴

$$R_{i\bar{j}k\bar{l}} = g \left(\frac{\partial}{\partial z^i} \left(\frac{\partial}{\partial \bar{z}^j} \frac{\partial}{\partial z^k} \frac{\partial}{\partial \bar{z}^l} - \frac{\partial}{\partial \bar{z}^j} \left(\frac{\partial}{\partial z^k} \frac{\partial}{\partial \bar{z}^l} \right) \right) \right)$$

$$= g \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \left(\frac{\partial}{\partial z^k} \frac{\partial}{\partial \bar{z}^l} - \frac{\partial}{\partial \bar{z}^j} \left(\frac{\partial}{\partial z^k} \frac{\partial}{\partial \bar{z}^l} \right) \right) \right) = \left(\frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} \right) g_{k\bar{l}}$$

$$\frac{\partial}{\partial \bar{z}^j} = \frac{\partial}{\partial \bar{z}^j} = g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial \bar{z}^j} = g^{p\bar{q}} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$\therefore R_{i\bar{j}k\bar{l}} = g_{k\bar{l}} \frac{\partial}{\partial z^i} \left(g^{p\bar{q}} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j} \right)$$

$$= \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{q}} g^{r\bar{s}} \frac{\partial g_{k\bar{s}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$= \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

∴

Majority of Kähler parameters

use $R_{i\bar{j}k\bar{l}} = - \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + \dots$

Def (M, g) Riemannian mfd.

(9)

(9)

e_1, \dots, e_n orthonormal frame of TM.
local.

$$\text{Ric}(X, Y) = \sum_{i=1}^n R(X, e_i, Y, e_i)$$

$$\text{Ric}(Y, X) = \text{Ric}(X, Y)$$

Ric is symmetric

$$\text{Ric} = R_{ij} dx^i \otimes dx^j \quad R_{ij} = R_{ji}$$

Ric is called the Ricci curvature
or Ricci tensor.

$$R_{ij} = g^{kl} R_{ikjl}$$

$\frac{2}{220}$

$$= R_{ikj}{}^k = R_{i \cdot j}{}^k{}_{\cdot k}$$

(M, g) Kähler mfd.

$$\cancel{R_{ij}{}^k{}_{\cdot k}} + R_{i \cdot j}{}^k{}_{\cdot k} + \cancel{R_{ikj}{}^k} = 0$$

$$- R_{i \cdot j}{}^k{}_{\cdot k}$$

$$\frac{1}{\sqrt{2}} (e - iJ)e$$

$$\therefore R_{ij}^k{}^k = R_i^k{}_{j\bar{k}} = R_{ij}$$

(11)

Ricci curvature for Kähler

Majority of Kähler geometries use

$$R_{ij} = R_{ij}^k{}^k \quad \text{the same as ours.}$$

Prop $R_{ij} = - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det (g_{k\bar{l}})$

(:) In general for invertible matrix

$$\underline{d(\det A)} = \det A \cdot \underline{tr(A^{-1} dA)}$$

Exercise

So $R_{ij}^k{}^k = g^{k\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z^i}$

$$= \frac{\partial^2}{\partial z^i} \log \det (g_{k\bar{l}})$$

$$\begin{aligned} \therefore R_{ij}^k{}^k &= -g^{k\bar{l}} R_{ij\bar{l}} = -g^{k\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z^i} g_{k\bar{q}} \\ &= -\frac{\partial}{\partial z^i} R_{j\bar{l}}^{\bar{l}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det g \quad (\text{:)}) \end{aligned}$$

$$\textcircled{H} = \underbrace{\bar{\partial}\theta}_{\text{do } \theta \text{ on } \theta} = \left(R^i_{j k \bar{l}} dz^k \wedge d\bar{z}^{\bar{l}} \right) \quad \textcircled{11}$$

curvature form
(2-form)
matrix.

Def $\det \left(I + t \frac{\sqrt{t}}{2\pi} \textcircled{H} \right) \quad m = \dim_{\mathbb{C}} M.$

$$= 1 + t c_1 + t^2 c_2 + \dots + t^m c_m$$

Fact $c_p(g)$ is a real closed 2p-form.
 de Rham class $[c_p(g)]$ is indep of g .

We do it only for $p=1$.

