

A few additional examples

The purpose of this handout is to give a few more examples of the guiding principle mentioned in the talk.

① Let $\mathcal{M}_{3,3} := \{ \text{smooth cubic surfaces } S \subseteq \mathbb{P}^3 \} \cong \mathbb{P}^{19} - \Sigma_{\text{sing}}$

Let $\tilde{\mathcal{M}}_{3,3} := \{ (S, L) : S \in \mathcal{M}_{3,3}, L \subset S \text{ line} \}$
 \uparrow
 $L \cong \mathbb{P}^1$

Cayley, C. Jordan: $\tilde{\mathcal{M}}_{3,3}$
 $\downarrow \pi$ $\pi(S, L) := S$
 $\mathcal{M}_{3,3}$
 "∃ 27 lines on any smooth cubic surface"

is a 27-sheeted cover, with monodromy

$$\mathcal{P}: \pi_1 \mathcal{M}_{3,3} \twoheadrightarrow W(E_6) \subset S_{27}$$

Thm (Huxford 2021, Conj. by Farb): The
 homomorphism \mathcal{P} is unique up to conjugacy.

Let $G(1,3) :=$ Projective Grassmannian of lines in \mathbb{P}^3

$U\text{Conf}_m(X) := \{ \text{unordered } m\text{-tuples of distinct points in } X \}$

The above gives a morphism

$$\Psi: \mathcal{M}_{3,3} \longrightarrow U\text{Conf}_{27}(G(1,3))$$

Guiding Principle gives: $\forall m \geq 1$

If $F: \mathcal{M}_{3,3} \longrightarrow U\text{Conf}_m(G(1,3))$

is "nontrivial" then $m=27$ and $F \sim \Psi$.

If in addition F is holomorphic then $F = \Psi$.

For generic $S \subset \mathcal{M}_{3,3}$,

$$|\{L_i \cap L_j : L_i \neq L_j \text{ lines in } S\}| = 135$$

(only generic since \exists "Eckardt Points")



→ 3 rational section σ of

$$\begin{array}{ccc} U_{\text{conf}_{135}}(S) & \rightarrow & E_{135} \\ & & \pi \downarrow \uparrow \sigma \\ & & M_{3,3} \end{array}$$

Guiding Principle: σ should be only rational n -multisection.

② Guiding Principle #2: Any "exceptional homomorphism" of finite groups is modular: it can be explained by a map of moduli spaces.

Example #1: $S_4 \twoheadrightarrow S_3$ explained by

$$\begin{array}{ccc} \widetilde{\text{Pol}}_4 & \xrightarrow{\text{Fer}} & \widetilde{\text{Pol}}_3 \\ S_4 \downarrow & & \downarrow S_3 \\ \text{Pol}_4 & \xrightarrow{\text{Fer}} & \text{Pol}_3 \end{array}$$

Example #2: $S_6 \cong \text{SP}_4 \mathbb{F}_2$ explained by

$$\begin{array}{ccccc}
 \mathcal{M}_{0,6} & \xrightarrow{\cong} & \mathcal{M}_2[\mathbb{Z}] & \xrightarrow{\cong} & \mathcal{H}_2[\mathbb{Z}] \\
 S_6 \downarrow & & \downarrow & & \downarrow \text{SP}_4 \mathbb{F}_2 \\
 \mathcal{M}_{0,6} / S_6 & \xrightarrow[\mathbb{R}]{\cong} & \mathcal{M}_2 & \xrightarrow[\text{J period mapping}]{\cong} & \mathcal{H}_2
 \end{array}$$

$R(\{P_1, \dots, P_6\}) := 2$ -sheeted cover $X \rightarrow \mathbb{P}^1$ branched over $\{P_1, \dots, P_6\}$

$$\mathcal{M}_2[\mathbb{Z}] = \{ (X, \mathcal{B}) : X \in \mathcal{M}_2, \mathcal{B} \text{ basis for } H_1(X; \mathbb{F}_2) \}$$

\exists many more such examples,

e.g.

$$W(E_6)^+ \cong \text{PSP}_4(\mathbb{F}_3)$$

See Farb-Kisinn-Wolfson, "Modular Functions and Resolvent Problems"

for more examples.

③ A remarkable map:

Given $\bullet S :=$ smooth cubic surface in \mathbb{P}^3
 $\bullet P \in S \setminus \{L_1, \dots, L_27\}$

$\rightsquigarrow \exists$ 2-sheeted cover $B\mathbb{P}^2(S)$ exc. divisor e
 branched over a smooth quartic $\pi \downarrow \mathbb{P}^2 \supset Q$

π is:  $\pi(m) = \begin{cases} \text{line } \overline{pm} & m \notin e \\ [m] & m \in e \end{cases}$

Fact: $\{28 \text{ bitangents to } Q\} = \pi(\{L_1, \dots, L_{27}\} \text{ on } S) \cup \pi(e)$

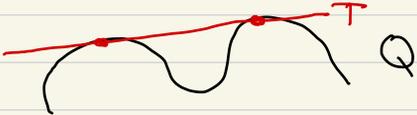
Let $S \rightarrow E_{3,3} := \{(S, P): S \in \mathcal{M}_{3,3}, P \in S\}$
 \downarrow
 $\mathcal{M}_{3,3}$

Let $\mathcal{M}_{4,2} = \{\text{smooth quartic curves in } \mathbb{P}^2\}$

The above gives a dominant rational map

$$\Psi: U_{3,3} \dashrightarrow \{ (Q, T) : Q \in \mathcal{M}_{4,2}, T \text{ is bitangent to } Q \}$$

$\downarrow 28$
 $\mathcal{M}_{4,2}$



Exercise: Apply Guiding Principle to Ψ to give conjectures on holomorphic and topological rigidity.

(4) Many classical moduli spaces \mathcal{M} locally symmetric varieties, i.e.

$$\mathcal{M} = \Gamma \backslash G / K$$

G = real semisimple Lie group of Hermitian type

K = max compact subgroup of G

Γ = arithmetic lattice in G

and the type of rigidity conjectures I've been giving can be proved using Margulis Super-rigidity, Congruence Subgroup Property, etc.